Thai Journal of Mathematics
Volume 14 (2016) Number 2 : 491–495
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Smith Multiples of a Class of Primes with Small Digital Sum

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Abstract : Using prime numbers whose digits are zeros and ones, we demonstrate how to construct integers m for which mP is a Smith number for any prime P with a fixed, small digital sum. Conversely, using numbers with small digital sums, we can obtain Smith multiples of a given prime whose digits are zeros and ones. Our approach relies on numbers with small digital sum in order that every multiplication process is free from carries.

Keywords : Smith numbers; sum of digits.
2010 Mathematics Subject Classification : 11A63; 11A41.

1 Introduction

A Smith number n is a composite whose digital sum $S(n)$ equals $S_p(n)$, the digital sum of all the prime factors of n, counting multiplicity. For example, 728 is a Smith number because the factorization $728 = 2^3 \times 7 \times 13$ gives us $S_p(728) = 2 + 2 + 2 + 7 + 1 + 3 = 17$, matching $S(728) = 7 + 2 + 8 = 17$.

It is known that Smith numbers are infinitely many [1]. And over thirty years since Smith numbers were first introduced [2], we have now quite a variety of methods for constructing Smith numbers [3–7]. In particular, it was shown [6, Theorem 2.1] that for any prime number P with $S(P) = 5$, the product $21P$ is a Smith number. The simple proof is based on the fact that the multiplication of the two small-digit numbers are free from carries, so that $S(21P) = S(21)S(P) = 3 \times 5 = 15$, which equals $S_p(21P) = 3 + 7 + 5 = 15$.

This present article is in part a response to the challenge to find similar results using other primes of a small digital sum. Still relying on carry-free multiplication, we shall deal with primes P for which $S(P) \leq 8$ and construct the Smith multiples mP with the help of prime numbers whose digits are composed of zeros and ones.

Along this line, we acknowledge an early work [3] in which Smith multiples of a prime repunit are studied. (Such multipliers are listed on page A104167 at OEIS, the On-Line Encyclopedia of Integer Sequences.) Nevertheless, our results do not overlap with such a case since we are to find one Smith multiplier that works for a whole class of primes given by their digital sum.

2 Main Results

Throughout our discussion we will mostly deal with the product of a zero-one integer and a number of digital sum at most nine. Hence, we will settle first the claim that such multiplication is free from carries.

Theorem 2.1. *Let n be a number with $S(n) \leq 9$ and m be a number whose digits are only zeros and ones. Then $S(mn) = S(m)S(n)$.*

Proof. Let $r = S(n)$ and write $n = 10^{e_1} + \cdots + 10^{e_r}$, where the exponents $e_1, \ldots, e_r$ are not assumed distinct. Then mn is the sum of r numbers that are composed of zeros and ones, each of which has digital sum $S(m)$. Since $r \leq 9$, this summation does not involve carries. Thus $S(mn) = S(m)r$. □

Definition 2.2. For convenience, we shall employ the notation P_N to denote any prime number whose digits are only zeros and ones, and with exactly N ones. In other words, P_N stands for a prime number which can be expressed as the sum of N distinct powers of ten.

Theorem 2.3. *Fix a prime number $P > 3$ with $S(P) = a \leq 7$. (Note that $a \in \{2, 4, 5, 7\}$.) Let a prime P_N be given and let $b = N \bmod 7$. If $(a, b) \in \{(2, 2), (4, 6), (5, 3), (7, 0)\}$, then there exists an integer $k \geq 0$, determined solely by the choice of a and N, such that the product $P \times P_N \times 10^k$ is a Smith number.*

Proof. Writing $N = 7t + b$ with some integer $t \geq 0$, we have, by Theorem 2.1,

$$\begin{aligned} S(P \times P_N \times 10^k) - S_p(P \times P_N \times 10^k) &= (S(P) \times N) - (S(P) + N + 7k) \\ &= a(7t + b) - (a + 7t + b + 7k) \\ &= 7t(a - 1) + ab - a - b - 7k. \end{aligned}$$

We have Smith number when this displayed quantity equals zero, i.e., if and only if

$$k = t(a - 1) + \frac{ab - a - b}{7}.$$

Now note that for $(a, b) = (2, 2), (4, 6), (5, 3), (7, 0)$, we have a nonnegative integer value for k: respectively, $k = t, 3t + 2, 4t + 1, 6t - 1$. (The value of $6t - 1$ is nonnegative since $b = 0$ implies that $t \geq 1$ for this case.) □

Theorem 2.3 is practical only if we can find at least one prime P_N for each of the specified residue classes of N modulo 7. One may turn to page A020449 at OEIS to see the first thousand terms of the sequence given by the primes P_N. In particular, we have the seven primes

$$11, 10111, 101111, 11110111, 101111111, 110111111101, 101101111111111,$$

representing the least prime number P_N in their respective class of N modulo 7 (The last of these seven is the 966th term in the P_N sequence!).

Example 2.4. To illustrate, we consider the prime $P = 4021$, where $S(P) = 7$. According to Theorem 2.3, we need a prime P_N with N a multiple of 7, e.g., $P_7 = 11110111$, for which $t = 1$ and $k = 5$ as explained in the above proof. The resulting Smith product is

$$4021 \times 11110111 \times 10^5 = 4,467,375,633,100,000,$$

of digital sum 49.

Note that the case $S(P) = 8$ is not covered in Theorem 2.3. In fact, the form $P \times P_N \times 10^k$ is never a Smith number if $S(P) = 8$, for we would have $S(P \times P_N \times 10^k) = 8N \equiv N \pmod 7$, whereas $S_p(P \times P_N \times 10^k) = 8 + N + 7k \equiv N + 1 \pmod 7$; hence the two quantities would not coincide. For this reason, we shall now treat the case $S(P) = 8$ separately.

Theorem 2.5. *Let P be a prime number with $S(P) = 8$. Then $1011P$ is a Smith number.*

Proof. Theorem 2.1 applies, hence $S(1011P) = 3 \times 8 = 24$. Since $1011 = 3 \times 337$, we have also $S_p(1011P) = 16 + 8 = 24$. □

We leave it to the reader to verify that Smith multiples also occur when the multiplier 1011 in Theorem 2.5 is substituted by 11010111, 11011101, 11100001, 11110100, or 11111100.

For the sake of completeness, we proceed to answer the next challenge. Given a prime P_N, can we find Smith multiples of P_N using small-digit multipliers and carry-free multiplication? Of course, Theorem 2.3 already suffices for the cases where $N \bmod 7 = 0, 2, 3, 6$. The next proposition provides another part of the answer.

Theorem 2.6. *Fix a prime number P_N. Let $b = N \bmod 7$, and let $c \in \{3, 6, 9\}$. If $(b, c) \in \{(1, 6), (5, 3), (6, 9)\}$, then there exists an integer $k \geq 0$ such that the product $c \times P_N \times 10^k$ is a Smith number.*

Proof. Again, we note that $c \times P_N$ is a carry-free multiplication. Writing $N = 7t + b$ for some integer $t \geq 0$, we have

$$\begin{aligned} S(c \times P_N \times 10^k) - S_p(c \times P_N \times 10^k) &= (S(c) \times N) - (S_p(c) + N + 7k) \\ &= c(7t + b) - (S_p(c) + 7t + b + 7k) \\ &= 7t(c - 1) + bc - b - S_p(c) - 7k. \end{aligned}$$

This time, we seek for a nonnegative integer solution for

$$k = t(c-1) + \frac{bc - b - S_p(c)}{7}.$$

By inspection, if $(b,c) = (1,6), (5,3), (6,9)$, then $k = 5t, 2t+1, 8t+6$, respectively. □

Theorems 2.3 and 2.6 together leave one case still undealt with: the primes P_N with $N \bmod 7 = 4$. Since there seems to be no similar approach that works specifically for this last case, we resort to finding a multiplier that applies to P_N in general, regardless of the residue class of N modulo 7.

Theorem 2.7. *Let the prime P_N be given with $N \geq 16$. Then the product $n = 4P_N \times (10^{26}+1) \times 10^{N-16}$ is a Smith number.*

Proof. Observe that $S(4(10^{26}+1)) = 8$, hence Theorem 2.1 gives us $S(n) = 8N$. On the other hand, we have the factorization

$$10^{26} + 1 = 101 \times 521 \times 1900381976777332243781,$$

so that $S_p(10^{26}+1) = 108$ and $S_p(n) = 4 + N + 108 + 7(N-16) = 8N$. Thus $S(n) = S_p(n)$. □

We remark that the choice of $10^{26}+1$ in the proof is suitable merely because $S_p(10^{26}+1) \bmod 7 = 3$. In fact, one may generate similar Smith numbers of the form $4P_N \times (10^e+1) \times 10^k$ provided that $S_p(10^e+1) \bmod 7 = 3$, e.g., with $e = 23$ or 24, and with the exponent k for the power of ten adjusted accordingly.

Collecting our results thus far, we are down to the treatment of the primes P_N for which $N \leq 15$ and $N \bmod 7 = 4$. Equivalently, these are the two subcases $N = 4$ and $N = 11$, with which we now conclude. For the P_4 case, we may as well include all prime numbers in the class of digital sum four—this would therefore overlap with Theorem 2.3 and so to avoid repetition, we will now include the digit 2 in choosing the suitable multiplier.

Theorem 2.8. *Let P denote a prime number with $S(P) = 4$. Then $1220P$ is a Smith number.*

Proof. Since the digits in 1220 are at most two, multiplying 1220 by any number of digital sum four does not involve carries. Hence, $S(1220P) = 5 \times 4 = 20$. With $1220 = 2^2 \times 5 \times 61$, we see that $S_p(1220P) = 16 + 4 = 20$. □

Among the five-digit numbers, the multipliers 12012, 12020, 12220, and 21020 can also be used as a substitute for 1220 in Theorem 2.8—for the reader to verify.

Theorem 2.9. *Let P_{11} stand for a prime number composed of only zeros and eleven ones. Then both $10011P_{11}$ and $20001P_{11}$ are Smith numbers.*

Proof. Carry-free multiplication still applies. For $m = 10011$ or 20001, we have $S(mP_{11}) = 3 \times 11 = 33$ and $S_p(mP_{11}) = S_p(m) + 11$. It suffices to check that $S_p(m) = 22$, and this is true as $10011 = 3 \times 47 \times 71$ and $20001 = 3 \times 59 \times 113$. □

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(Received 5 September 2013)
(Accepted 23 March 2015)