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Generalizations of n-Absorbing Ideals of Commutative Semirings

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Abstract : Let R be a commutative semiring with nonzero identity and ϕ a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\emptyset\}$ where $\mathscr{I}(R)$ is the set of ideals of R. Let n be a positive integer. In this paper, we introduce the concept of ϕ-n-absorbing ideals which are a generalization of n-absorbing ideals. A proper ideal I of R is called a ϕ-n-absorbing ideal if whenever $x_1x_2\cdots x_{n+1} \in I - \phi(I)$ for $x_1, x_2, \ldots, x_{n+1} \in R$, then $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. A number of results concerning relationships between ϕ-n-absorbing ideals and n-absorbing ideals as well as examples of n-absorbing ideals are given. Moreover, ϕ-n-absorbing ideals of decomposable semirings, of quotient semirings and of semirings of fractions are investigated.

Keywords : semirings; k-ideals; n-absorbing ideals; ϕ-n-absorbing ideals.
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1 Introduction

Throughout this paper, all rings and semirings are assumed to be commutative rings with nonzero identity and commutative semirings with nonzero identity, respectively. Moreover, the notation ϕ is assumed to be a function from

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$\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\emptyset\}$ in which $\mathscr{I}(R)$ is the set of ideals of a semiring R (ring R). Furthermore, if R is a semiring (ring) and ϕ is a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\emptyset\}$, then R is called a *semiring with* ϕ (*ring with* ϕ). Let n and m be positive integers, we denote $\hat{x}_{i,n+1}$ the element of R obtained by eliminating x_i from the product $x_1x_2\cdots x_{n+1}$ where $x_1, x_2, \ldots, x_{n+1} \in R$; in addition, we denote $\hat{x}_{\{i_1,i_2,\ldots,i_m\},n+1}$ the element of R obtained by eliminating $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ from the product $x_1x_2\cdots x_{n+1}$ where $x_1, x_2, \ldots, x_{n+1} \in R$ and $\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n+1\}$.

The concept of 2-absorbing ideals of rings was introduced and investigated by A. Badawi in 2007 [1]. He defined a 2-*absorbing ideal* I of a ring R to be a proper ideal and if whenever $a, b, c \in R$, $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. In 2011, D. F. Anderson and A. Badawi [2] generalized the concept of 2-absorbing ideals of rings to n-absorbing ideals of rings. A proper ideal I of a ring R is called an n-*absorbing ideal* if for $x_1, x_2, \ldots, x_{n+1} \in R$, $x_1x_2\cdots x_{n+1} \in I$ implies $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. From the definition of n-absorbing ideals, it is easy to see that if n and n' are positive integers such that $n \leq n'$ and I is an n-absorbing ideal, then I is an n'-absorbing ideal. Moreover, if $n = 1$, then 1-absorbing ideals are just prime ideals.

In 2008, D. D. Anderson and M. Bataineh [3] generalized the concept of prime ideals, weakly prime ideals, almost prime ideals, n-almost prime ideals and ω-prime ideals of rings to ϕ-prime ideals of rings with ϕ. They defined a ϕ-*prime ideal* I of a ring R with ϕ to be a proper ideal and if for $a, b \in R$, $ab \in I - \phi(I)$ implies $a \in I$ or $b \in I$. After that, in 2012, M. Ebrahimpour and R. Nekooei [4] introduced the concept of $(n-1, n)$-ϕ-prime ideals ($n \geq 2$) of rings with ϕ which are a generalization of n-absorbing ideals of rings and ϕ-prime ideals of rings with ϕ. They defined an $(n-1, n)$-ϕ-*prime ideal* I of a ring R with ϕ to be a proper ideal and if whenever $x_1, x_2, \ldots, x_n \in R$ and $x_1x_2\cdots x_n \in I - \phi(I)$, then $\hat{x}_{i,n} \in I$ for some $i \in \{1, 2, \ldots, n\}$. Then $(n-1, n)$-ϕ-prime ideals are just $(n-1)$-absorbing ideals if $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ is a function with $\phi[\mathscr{I}(R)] = \{\emptyset\}$.

In this paper, we extend notions of n-absorbing ideals and $(n-1, n)$-ϕ-prime ideals of rings to n-absorbing ideals and ϕ-n-absorbing ideals of semirings. We define n-absorbing ideals of semirings in the same manner as the definition of n-absorbing ideals of rings. Besides, we define a ϕ-n-*absorbing ideal* I of a semiring R with ϕ to be a proper ideal and if whenever $x_1x_2\cdots x_{n+1} \in I - \phi(I)$ for $x_1, x_2, \ldots, x_{n+1} \in R$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Hence, if I is an n-absorbing ideal, then I is, obviously, a ϕ-n-absorbing ideal for any ϕ.

As a result, we give an equivalent definition of ϕ-n-absorbing ideals. In addition, relationships between ϕ-n-absorbing ideals (weakly n-absorbing ideals) and n-absorbing ideals are investigated in decomposable semirings. Moreover, if ϕ-n-absorbing ideals of semirings are given, then we can construct ϕ-n-absorbing ideals of quotient semirings and of semirings of fractions. We also show that , for a semiring R and its Q-ideal I, if P/I is a ϕ-n-absorbing ideal of R/I, then P is a ϕ-n-absorbing ideal of R.

2 ϕ-n-Absorbing Ideals

In this section, we investigate ϕ-n-absorbing ideals of semirings with ϕ. For the sake of completeness, we state some definitions in the same fashion as found in [2] and [3] which are used throughout this paper.

Definition 2.1. Let R be a semiring and n a positive integer.

A proper ideal I of R is said to be *n-absorbing* if $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I$ implies $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \dots, n+1\}$.

A proper ideal I of R is said to be *weakly n-absorbing* if $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - \{0\}$ implies $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \dots, n+1\}$.

A proper ideal I of R is said to be *almost n-absorbing* if $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - I^2$ implies $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \dots, n+1\}$.

A proper ideal I of R is said to be *m-almost n-absorbing* ($m \geq 2$) if $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - I^m$ implies $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \dots, n+1\}$.

A proper ideal I of R is said to be *ω-n-absorbing* if $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - \cap_{l=1}^{\infty} I^l$ implies $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \dots, n+1\}$.

Example 2.2. Let n be a positive integer with $n \geq 2$ and $p_1, p_2, \dots, p_n$ prime numbers (not necessary distinct). Then $p_1p_2\cdots p_n\mathbb{Z}_0^+$ is an n-absorbing ideal but not an $(n-1)$-absorbing ideal of the semiring $\mathbb{Z}_0^+$ under usual addition and usual multiplication.

In the following, we define ϕ-n-absorbing ideals of semirings. These ideals generalize n-absorbing ideals, weakly n-absorbing ideals, almost n-absorbing ideals, m-almost n-absorbing ideals and ω-n-absorbing ideals of semirings.

Definition 2.3. A proper ideal I of a semiring R is said to be *ϕ-n-absorbing* if whenever $x_1, x_2, \dots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - \phi(I)$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \dots, n+1\}$.

Hence, for a semiring R, if we define $\phi_\emptyset : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ by $\phi_\emptyset(I) = \emptyset$ for all $I \in \mathscr{I}(R)$, then a $\phi_\emptyset$-n-absorbing ideal is just an n-absorbing ideal. Similarly, if we define $\phi_0 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ by $\phi_0(I) = \{0\}$ for all $I \in \mathscr{I}(R)$, then a ϕ_0-n-absorbing ideal is a weakly n-absorbing ideal. In the same way, if we define the functions $\phi_\alpha : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ such that $\phi_2(I) = I^2, \phi_m(I) = I^m$ where $m \in \mathbb{N}$ with $m \geq 2$ and $\phi_\omega(I) = \cap_{l=1}^{\infty} I^l$ for all $I \in \mathscr{I}(R)$, then a ϕ_2-n-absorbing (ϕ_m-n-absorbing, ϕ_ω-n-absorbing) ideal is an almost n-absorbing (m-almost n-absorbing, ω-n-absorbing) ideal, respectively. These functions are defined analogously to those (for the ring-case) found in [3].

Recall that a *k-ideal* (*subtractive ideal*) of a semiring R is an ideal I of R such that if for $x, y \in R$ and $x, x + y \in I$, then $y \in I$. It is easy to see that if A, B are k-ideals of a semiring R and $I = A \cup B$ is an ideal of R, then $I = A$ or $I = B$.

Let R be a semiring. Given two functions $\varphi_1, \varphi_2 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$, we define $\varphi_1 \leq \varphi_2$ if $\varphi_1(I) \subseteq \varphi_2(I)$ for each $I \in \mathscr{I}(R)$ in the same manner as given in [3].

Proposition 2.4. *Let R be a semiring, I a proper ideal of R and $\varphi_1 \leq \varphi_2$ where φ_1 and φ_2 are functions from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\emptyset\}$. If I is a φ_1-n-absorbing ideal, then I is a φ_2-n-absorbing ideal.*

Proof. The proof is straightforward. □

Corollary 2.5. *Let I be a proper ideal of a semiring and $n, m \in \mathbb{N}$ with $m \geq 2$. Consider the following statements:*

(1) I is an n-absorbing ideal.

(2) I is a weakly n-absorbing ideal.

(3) I is an ω-n-absorbing ideal.

(4) I is an $(m+1)$-almost n-absorbing ideal.

(5) I is an m-almost n-absorbing ideal.

(6) I is an almost n-absorbing ideal.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

In the following result, we give an equivalent definition of ϕ-n-absorbing ideals.

Theorem 2.6. *Let R be a semiring with ϕ, I a proper ideal of R and n, n' positive integers with $n < n'$. Then I is a ϕ-n-absorbing ideal if and only if whenever $x_1x_2\cdots x_{n'} \in I - \phi(I)$ for any $x_1, x_2, \ldots, x_{n'} \in R$, then $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.*

Proof. First, assume that I is a ϕ-n-absorbing ideal of R. Let $x_1, x_2, \ldots, x_{n'} \in R$ be such that $x_1x_2\cdots x_n(x_{n+1}x_{n+2}\cdots x_{n'}) = x_1x_2\cdots x_{n'} \in I - \phi(I)$. Since I is a ϕ-n-absorbing ideal, $x_1x_2\cdots x_n \in I$ or $\hat{x}_{i,n}(x_{n+1}x_{n+2}\cdots x_{n'}) \in I$ for some $i \in \{1, 2, \ldots, n\}$. If $x_1x_2\cdots x_n \in I$, then we are done. So we suppose that $\hat{x}_{i,n}x_{n+1}x_{n+2}\cdots x_{n'} \in I$. Since $x_1x_2\cdots x_{n'} \notin \phi(I)$, we obtain $\hat{x}_{i,n}x_{n+1}x_{n+2}\cdots x_{n'} = \hat{x}_{i,n}x_{n+1}(x_{n+2}\cdots x_{n'}) \in I - \phi(I)$. Because I is a ϕ-n-absorbing ideal, it follows that $\hat{x}_{i,n}x_{n+1} \in I$ or $\hat{x}_{\{i,j\},n+1}(x_{n+2}\cdots x_{n'}) \in I$ for some $j \in \{1, 2, \ldots, n+1\} - \{i\}$. If $\hat{x}_{i,n}x_{n+1} \in I$, then we are done. If not, we continue this process, and hence we obtain $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.

Conversely, the proof is clear by choosing $n' = n + 1$. □

Corollary 2.7. *Let R be a semiring, I a proper ideal of R and n, n' positive integers with $n < n'$. Then I is an n-absorbing ideal if and only if whenever $x_1x_2\cdots x_{n'} \in I$ for $x_1, x_2, \ldots, x_{n'} \in R$, then $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.*

It is easy to see that n-absorbing ideals imply n'-absorbing ideals for any $n, n' \in \mathbb{N}$ with $n \leq n'$; moreover, this statement is also true for ϕ-n-absorbing ideals as shown in the next proposition.

Proposition 2.8. *Let R be a semiring with ϕ, I a proper ideal of R and n a positive integer. If I is a ϕ-n-absorbing ideal, then I is a ϕ-n'-absorbing ideal for all $n' \in \mathbb{N}$ with $n \leq n'$.*

Proof. Assume that I is a ϕ-n-absorbing ideal of R. Let $n' \in \mathbb{N}$ be such that $n \leq n'$. Note that, if $n' = n$, then there is nothing to do. So we assume that $n < n'$. Let $x_1, x_2, \ldots, x_{n'+1} \in R$ be such that $x_1 x_2 \cdots x_{n'+1} \in I - \phi(I)$. We obtain from Theorem 2.6 that $x_{i_1} x_{i_2} \cdots x_{i_n} \in I$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'+1\}$. By choosing all distinct

$$i_{n+1}, i_{n+2}, \ldots, i_{n'} \in \{1, 2, \ldots, n'+1\} - \{i_1, i_2, \ldots, i_n\}$$

and by multiplying, $x_{i_1} x_{i_2} \cdots x_{i_{n'}} = (x_{i_1} x_{i_2} \cdots x_{i_n}) \left(x_{i_{n+1}} x_{i_{n+2}} \cdots x_{n'}\right) \in I$. Hence I is a ϕ-n'-absorbing ideal of R. Therefore, I is a ϕ-n'-absorbing ideal for all $n \leq n'$. □

Since the empty set is a subset of all sets, n-absorbing ideals imply ϕ-n-absorbing ideals for any ϕ by Proposition 2.4. The converse of this statement is not true. Nevertheless, in 2015, M. K. Dubey and P. Sarohe [5] gave the conditions for ϕ-n-absorbing ideals to be n-absorbing ideals.

Proposition 2.9 ([5])**.** *Let R be a semiring with ϕ, n a positive integer and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal. If I is a ϕ-n-absorbing ideal with $I^{n+1} \nsubseteq \phi(I)$, then I is an n-absorbing ideal.*

From Corollary 2.5, we know that every n-absorbing ideal is a weakly n-absorbing ideal. Nonetheless, the converse of this statement is not true. In 2015, M. K. Dubey and P. Sarohe [5] also gave some characters of ideals which are weakly n-absorbing k-ideal but are not n-absorbing ideals as follows.

Corollary 2.10 ([5])**.** *Let R be a semiring and n a positive integer. If I is a weakly n-absorbing k-ideal but is not an n-absorbing ideal, then $I^{n+1} = \{0\}$.*

We would like to point out here that Corollary 2.10 is used to prove several results in the next section.

3 On Decomposable Semirings

In this section, we examine n-absorbing ideals, weakly n-absorbing ideals and ϕ-n-absorbing ideals of decomposable semirings.

For a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$ ($m \in \mathbb{N}$ with $m \geq 2$) such that R_i is a semiring with φ_i for all $i \in \{1, 2, \ldots, m\}$ and an ideal $I_1 \times I_2 \times \cdots \times I_m$ of R, it follows that $\varphi_1(I_1) \times \varphi_2(I_2) \times \cdots \times \varphi_m(I_m)$ is an ideal of R or the empty set. Hence there is a function $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\emptyset\}$ such that $\phi(I_1 \times I_2 \times \cdots \times I_m) = \varphi_1(I_1) \times \varphi_2(I_2) \times \cdots \times \varphi_m(I_m)$ for all $I_1 \times I_2 \times \cdots \times I_m \in \mathscr{I}(R)$; in addition, we denote the function ϕ which is defined as the previous by $\phi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_m$.

Moreover, for an ideal $I = I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring

$R = R_1 \times R_2 \times \cdots \times R_m$, it is easy to see that I is a k-ideal of R if and only if I_i is a k-ideal of R_i for all $i \in \{1, 2, \ldots, m\}$.

First, we would like to show that, for $m, n \in \mathbb{N}$ with $m \geq n + 1$, a nonzero weakly n-absorbing ideal $I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring $R_1 \times R_2 \times \cdots \times R_m$ has at least one $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$.

Proposition 3.1. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ where $m, n \in \mathbb{N}$ with $m \geq n+1$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper ideal of R. If I is a weakly n-absorbing ideal, then $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$.*

Proof. Assume that I is a weakly n-absorbing ideal. Since I is a nonzero ideal, there is $(x_1, x_2, \ldots, x_m) \in I$ such that $(x_1, x_2, \ldots, x_m) \neq (0, 0, \ldots, 0)$. Then

$$\begin{aligned}(0, 0, \ldots, 0) &\neq (x_1, x_2, \ldots, x_m)\\ &= (x_1, 1, \ldots, 1)(1, x_2, 1, \ldots, 1) \cdots (1, \ldots, 1, x_{n+1}, \ldots, x_m) \in I.\end{aligned}$$

Thus $(x_1, x_2, \ldots, x_n, 1, \ldots, 1) \in I$ or $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1}, \ldots, x_m) \in I$ for some $i \in \{1, 2, \ldots, n\}$ because I is a weakly n-absorbing ideal. Hence $1 \in I_i$ for some $i \in \{1, 2, \ldots, m\}$. Therefore, $I_i = R_i$. □

We know that n-absorbing ideals imply weakly n-absorbing ideals but not vice versa in general. However, in decomposable semirings, the converse of this statement is true if we assume those ideals are nonzero proper k-ideals.

Proposition 3.2. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ where $m, n \in \mathbb{N}$ with $m \geq n+1$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper k-ideal of R. Then I is a weakly n-absorbing ideal if and only if I is an n-absorbing ideal.*

Proof. Assume that I is a weakly n-absorbing ideal of R. Then $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$ by Proposition 3.1. Thus $I^{n+1} \neq \{0\}$. Therefore, I is an n-absorbing ideal by Corollary 2.10. The converse is clear by Corollary 2.5. □

From Proposition 3.2, we can conclude that weakly n-absorbing ideals and n-absorbing ideals are coincide if we provide that they are nonzero proper k-ideals of decomposable semirings with m components where $m \geq n + 1$. In the following theorem, we assume the condition that "there is at least one $I_i = R_i$ where $i \in \{1, 2, \ldots, m\}$" holds while the condition that "I is a nonzero ideal and $m \geq n+1$" can be omitted. We still obtain the same result; moreover, we get that any proper components of I are n-absorbing ideals.

Theorem 3.3. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with at least one $I_i = R_i$ where $i \in \{1, 2, \ldots, m\}$. Consider the following statements:*

(1) I is a weakly n-absorbing ideal of R.

(2) I is an n-absorbing ideal of R.

(3) If $I_j \neq R_j$ where $j \in \{1, 2, \ldots, m\}$, then I_j is an n-absorbing ideal of R_j.

Then (1) and (2) are equivalent and (2) implies (3).

Proof. The proof for (1) $\Leftrightarrow$ (2) is clear by Corollary 2.5 and Corollary 2.10.

To show (2) $\Rightarrow$ (3), assume that I is an n-absorbing ideal of R and $I_j \neq R_j$ for some $j \in \{1, 2, \dots, m\}$. Let $x_1, x_2, \dots, x_{n+1} \in R_j$ be such that $x_1 x_2 \cdots x_{n+1} \in I_j$. We obtain $(0, \dots, 0, x_1, 0, \dots, 0)(0, \dots, 0, x_2, 0, \dots, 0) \cdots (0, \dots, 0, x_{n+1}, 0, \dots, 0) = (0, \dots, 0, x_1 x_2 \cdots x_{n+1}, 0, \dots, 0) \in I$. Since I is an n-absorbing ideal, it follows that $(0, \dots, 0, \hat{x}_{l,n+1}, 0, \dots, 0) \in I$ for some $l \in \{1, 2, \dots, m\}$. Hence $\hat{x}_{l,n+1} \in I_j$. Therefore, I_j is an n-absorbing ideal of R_j. $\square$

From Theorem 3.3, we can conclude that if $I_1 \times I_2 \times \cdots \times I_m$ is an n-absorbing ideal (weakly n-absorbing ideal) of $R_1 \times R_2 \times \cdots \times R_m$, then I_j with $I_j \neq R_j$ is an n-absorbing ideal of R_j where $j \in \{1, 2, \dots, m\}$. Nevertheless, the converse of this statement is not true in general as we show in the following example.

Example 3.4. Let $R = R_1 \times R_2 \times \cdots \times R_m = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$ and n a positive integer. Let $I_1 = p_1 p_2 \cdots p_n \mathbb{Z}_0^+$ and $I_2 = q_1 q_2 \cdots q_n \mathbb{Z}_0^+$ where $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n$ are positive primes. Thus I_1 and I_2 are n-absorbing ideals of $\mathbb{Z}_0^+$. Since $(p_1, 1, 1, \dots, 1)(p_2, q_1, 1, \dots, 1) \cdots (p_n, q_{n-1}, 1, \dots, 1)(1, q_n, 1, \dots, 1) = (p_1 p_2 \cdots p_n, q_1 q_2 \cdots q_n, 1, 1, \dots, 1) \in I_1 \times I_2 \times R_3 \times \cdots \times R_m$ but $\hat{p}_{i,n} \notin I_1$ and $\hat{q}_{j,n} \notin I_2$ for all $i, j \in \{1, 2, \dots, n\}$, the ideal $I_1 \times I_2 \times R_3 \times \cdots \times R_m$ is not an n-absorbing ideal.

In the next theorem, we assume a stronger condition than conditions given in Theorem 3.3 in order to make (1), (2) and (3) be equivalent.

Theorem 3.5. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \dots, m\}$. The following statements are equivalent.*

(1) I is a weakly n-absorbing ideal of R.

(2) I is an n-absorbing ideal of R.

(3) I_i is an n-absorbing ideal of R_i.

Proof. It remains to show (3) $\Rightarrow$ (2). Assume I_i is an n-absorbing ideal of R_i. Let $(x_{11}, \dots, x_{1m}), (x_{21}, \dots, x_{2m}), \dots, (x_{(n+1)1}, \dots, x_{(n+1)m}) \in R$ be such that

$$(x_{11}, \dots, x_{1m})(x_{21}, \dots, x_{2m}) \cdots (x_{(n+1)1}, \dots, x_{(n+1)m}) \in I.$$

Note that $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$. Thus

$$(x_{11} x_{21} \cdots x_{(n+1)1}, \dots, x_{1i} x_{2i} \cdots x_{(n+1)i}, \dots, x_{1m} x_{2m} \cdots x_{(n+1)m}) \in I.$$

Since I_i is an n-absorbing ideal of R_i, we obtain $\hat{x}_{ji,(n+1)i} \in I_i$ for some $j \in \{1, 2, \dots, n+1\}$. Thus $(x_{11}, \dots, x_{1m}) \cdots (x_{(j-1)1}, \dots, x_{(j-1)m})(x_{(j+1)1}, \dots, x_{(j+1)m}) \cdots (x_{(n+1)1}, \dots, x_{(n+1)m}) \in I$. Therefore, I is an n-absorbing ideal of R. $\square$

Corollary 3.6. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring with ϕ, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. If I_i is an n-absorbing ideal of R_i, then I is a ϕ-n-absorbing ideal of R.*

Besides, we investigate that if I_i is an n-absorbing ideal of a semiring R_i, then $I = R_1 \times R_2 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$ is a ϕ-n-absorbing ideal of the decomposable semiring $R_1 \times R_2 \times \cdots \times R_m$ for any ϕ. We also study in case of I_i is a weakly n-absorbing ideal of R_i as follows.

Theorem 3.7. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. If I_i is a weakly n-absorbing ideal of R_i, then I is a ϕ-n-absorbing ideal of R for all $\phi_\omega \leq \phi$.*

Proof. In fact, $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$ for some $i \in \{1, 2, \ldots, m\}$. Without loss of generality, we assume that $i = 1$. Assume that I_1 is a weakly n-absorbing ideal of R_1. Since I is a k-ideal, I_1 is a k-ideal. If I_1 is an n-absorbing ideal of R_1, then I is an n-absorbing ideal of R by Theorem 3.5, and so I is a ϕ_ω-n-absorbing ideal of R. Assume that I_1 is not an n-absorbing ideal of R_1. Thus $I_1^{n+1} = \{0\}$ by Corollary 2.10. Consider the element $(x_1, \ldots, x_m) \in \phi_\omega(I) = \cap_{l=1}^{\infty} I^l \subseteq I^{n+1} = (I_1 \times R_2 \times \cdots \times R_m)^{n+1} \subseteq I_1^{n+1} \times R_2 \times \cdots \times R_m = \{0\} \times R_2 \times \cdots \times R_m$. Let $(x_{11}, \ldots, x_{1m}), (x_{21}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, \ldots, x_{(n+1)m}) \in R$ be such that $(x_{11}x_{21} \cdots x_{(n+1)1}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I - \phi_\omega(I)$. Then $x_{11}x_{21} \cdots x_{(n+1)1} \in I_1 - \{0\}$. Since I_1 is a weakly n-absorbing ideal, we obtain $\hat{x}_{j1,(n+1)1} \in I_1$ for some $j \in \{1, 2, \ldots, n+1\}$. Hence $(\hat{x}_{j1,(n+1)1}, \hat{x}_{j2,(n+1)2}, \ldots, \hat{x}_{jm,(n+1)m}) \in I$. Thus I is a ϕ_ω-n-absorbing ideal. Therefore, in any cases, I is a ϕ_ω-n-absorbing ideal, and so I is a ϕ-n-absorbing ideal for all $\phi_\omega \leq \phi$. ☐

Next, we are interested in case of $I = I_1 \times I_2 \times \cdots \times I_m$ is a weakly n-absorbing k-ideal which every component $I_i \neq \{0\}$.

Theorem 3.8. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer with $n \geq 2$ and $I = I_1 \times I_2 \times \cdots \times I_m$ where $I_i \neq \{0\}$ for all $i \in \{1, 2, \ldots, m\}$ is a weakly n-absorbing k-ideal. Then I is an n-absorbing ideal of R or I_i is an $(n-1)$-absorbing ideal of R_i for all $i \in \{1, 2, \ldots, m\}$.*

Proof. If I is an n-absorbing ideal of R, then we are done. Suppose that I is not an n-absorbing ideal of R. Then $I^{n+1} = \{0\}$ by Corollary 2.10. Hence $I_j \neq R_j$ for all $j \in \{1, 2, \ldots, m\}$. Let $i, j \in \{1, 2, \ldots, m\}$. Without loss of generality, we assume that $j < i$. We show that I_j is an $(n-1)$-absorbing ideal of R_j. Let $x_1, x_2, \ldots, x_n \in R_j$ be such that $x_1x_2 \cdots x_n \in I_j$. Since $I_i \neq \{0\}$, there exists $0 \neq y_i \in I_i$. So $(0, 0, \ldots, 0) \neq (0, \ldots, 0, x_1x_2 \cdots x_n, 0, \ldots, 0, y_i, 0, \ldots, 0) \in I$. Thus

$$(0, 0, \ldots, 0) \neq (0, \ldots, 0, x_1, 0, \ldots, 0, 1, 0, \ldots, 0)(0, \ldots, 0, x_2, 0, \ldots, 0, 1, 0, \ldots, 0) \cdots (0, \ldots, 0, x_n, 0, \ldots, 0, 1, 0, \ldots, 0)(0, \ldots, 0, 1, 0, \ldots, 0, y_i, 0, \ldots, 0) \in I.$$

Since I is weakly n-absorbing, $1 \in I_i$ or $\hat{x}_{l,n} \in I_j$ for some $l \in \{1, 2, \ldots, n\}$. Since $I_i \neq R_i$, we obtain $1 \notin I_i$, and hence $\hat{x}_{l,n} \in I_j$. Therefore, I_j is an $(n-1)$-absorbing ideal of R_j. ☐

Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. From Theorem 3.5, if I_i is an n-absorbing ideal of R_i, then I is an n-absorbing ideal of R. In the next result, we consider in case of every component I_i of I is an n_i-absorbing ideal of R_i, then we obtain an interesting result which is I must be an n-absorbing ideal where $n = n_1 + n_2 + \cdots + n_m$; in addition, in this theorem n_i can be zero. In case $n_i = 0$, we denote 0-absorbing ideal the ideal R.

Theorem 3.9. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ an ideal of R. If I_i is an n_i-absorbing ideal of R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$, then I is an n-absorbing ideal of R where $n = n_1 + n_2 + \cdots + n_m$, so that I is a ϕ-n-absorbing ideal of R.*

Proof. Assume that I_i is an n_i-absorbing ideal of R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$. Let $n = n_1 + n_2 + \cdots + n_m$. Let $(x_{11}, x_{12}, \ldots, x_{1m}), (x_{21}, x_{22}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in R$ be such that

$$(x_{11}, x_{12}, \ldots, x_{1m})(x_{21}, x_{22}, \ldots, x_{2m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in I.$$

We obtain $(x_{11}x_{21} \cdots x_{(n+1)1}, x_{12}x_{22} \cdots x_{(n+1)2}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I$. Since I_i is an n_i-absorbing ideal, $x_{1i}x_{2i} \cdots x_{(n+1)i} \in I_i$ and $n_i < n+1$, we obtain $x_{j_1 i}x_{j_2 i} \cdots x_{j_{n_i} i} \in I_i$ for some distinct $j_1, j_2, \ldots, j_{n_i} \in \{1, 2, \ldots, n+1\}$ by Corollary 2.7. Suppose that $\cup_{i=1}^{m}\{j_1, j_2, \ldots, j_{n_i}\} = \{j'_1, j'_2, \ldots, j'_h\}$. Thus $\{j'_1, j'_2, \ldots, j'_h\} \subseteq \{1, 2, \ldots, n+1\}$ and $h \leq n$ since $n_1 + n_2 + \cdots + n_m = n$. Since $\{j_1, j_2, \ldots, j_{n_i}\} \subseteq \{j'_1, j'_2, \ldots, j'_h\}$ and $x_{j_1 i}x_{j_2 i} \cdots x_{j_{n_i} i} \in I_i$ for all $i \in \{1, 2, \ldots, m\}$, we obtain

$$x_{j'_1 i}x_{j'_2 i} \cdots x_{j'_h i} \in I_i.$$

By choosing all distinct $j'_{h+1}, j'_{h+2}, \ldots, j'_n \in \{1, 2, \ldots, n+1\} - \{j'_1, j'_2, \ldots, j'_h\}$, hence

$$x_{j'_1 i}x_{j'_2 i} \cdots x_{j'_n i} = (x_{j'_1 i}x_{j'_2 i} \cdots x_{j'_h i})(x_{j'_{h+1} i}x_{j'_{h+2} i} \cdots x_{j'_n i}) \in I_i.$$

Then we obtain

$$(x_{j'_1 1}, x_{j'_1 2}, \ldots, x_{j'_1 m})(x_{j'_2 1}, x_{j'_2 2}, \ldots, x_{j'_2 m}) \cdots (x_{j'_n 1}, x_{j'_n 2}, \ldots, x_{j'_n m})$$
$$= (x_{j'_1 1}x_{j'_2 1} \cdots x_{j'_n 1}, x_{j'_1 2}x_{j'_2 2} \cdots x_{j'_n 2}, \ldots, x_{j'_1 m}x_{j'_2 m} \cdots x_{j'_n m}) \in I.$$

Therefore, I is an n-absorbing ideal of R, and hence I is a ϕ-n-absorbing ideal of R. □

Example 3.10. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

(1) Then $2\mathbb{Z}_0^+ \times 6\mathbb{Z}_0^+ \times 30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ is a 6-absorbing ideal of R because $2\mathbb{Z}_0^+$ is a 1-absorbing ideal, $6\mathbb{Z}_0^+$ is a 2-absorbing ideal, $30\mathbb{Z}_0^+$ is a 3-absorbing ideal and $\mathbb{Z}_0^+$ is a 0-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

(2) Then $2^2\mathbb{Z}_0^+ \times 2^3\mathbb{Z}_0^+ \times 2^4\mathbb{Z}_0^+ \times 2^5\mathbb{Z}_0^+$ is a 14-absorbing ideal of R because $2^l\mathbb{Z}_0^+$ is an l-absorbing ideal of the semiring $\mathbb{Z}_0^+$ for all $l \in \mathbb{N}$.

From Theorem 3.9, we can conclude that, for an ideal $I = I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$, if every component of I is a prime ideal of its semiring, then I is an m-absorbing ideal of R.

4 On Quotient Semirings and Semirings of Fractions

In this final section, we concern with ϕ-n-absorbing ideals of quotient semirings and ϕ-n-absorbing ideals of semirings of fractions.

An ideal I of a semiring R is called a Q-*ideal* (*partitioning ideal*) if there exists a subset Q of R such that $R = \cup\{q + I \mid q \in Q\}$ and $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$ for $q_1, q_2 \in Q$.

Let I be a Q-ideal of a semiring R and $R/I = \{q + I \mid q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$$(q_1 + I) \oplus (q_2 + I) = q_3 + I \qquad \text{and} \qquad (q_1 + I) \odot (q_2 + I) = q_4 + I$$

where $q_3, q_4 \in Q$ are the unique elements such that $q_1 + q_2 + I \subseteq q_3 + I$ and $q_1 q_2 + I \subseteq q_4 + I$. This semiring R/I is called the *quotient semiring of* R *by* I. In addition, since R is a commutative semiring with nonzero identity, R/I is a commutative semiring with nonzero identity, see [6].

Next, we would like to give the notion of a subtractive extension of an ideal which was introduced by D. R. Bonde and J. N. Chuadhari in 2014 [7].

Definition 4.1 ([7]). Let I be an ideal of a semiring R. An ideal P of R containing I is said to be *subtractive extension of* I if whenever $x, y \in R$ and $x \in I, x + y \in P$, then $y \in P$.

Note that, every k-ideal of a semiring R containing an ideal I of R is a subtractive extension of I; nevertheless, the converse of this statement is not true as shown in the next example.

Example 4.2. Let $I = 4\mathbb{Z}_0^+ \times \{0\}$ and $P = 2\mathbb{Z}_0^+ \times (\mathbb{Z}_0^+ - \{1\})$. Then I and P are ideals of the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ such that $I \subseteq P$. Since $(4,2), (4,2) + (2,1) = (6,3) \in P$ but $(2,1) \notin P$, the ideal P is not a k-ideal of R. Let $x \in I$ and $x + y \in P$. Thus $x = (4n, 0)$ for some $n \in \mathbb{Z}_0^+$ and $x + y = (2m, l)$ for some $m \in \mathbb{Z}_0^+$ and for some $l \in \mathbb{Z}_0^+ - \{1\}$. Let $y = (a, b)$ for some $a, b \in \mathbb{Z}_0^+$. Then $(2m, l) = x + y = (4n, 0) + (a, b) = (4n + a, b)$. Hence $4n + a = 2m$ and $b = l$, and so we obtain $a \in 2\mathbb{Z}_0^+$ and $b \in \mathbb{Z}_0^+ - \{1\}$. That is $y = (a, b) \in P$. Therefore, P is a subtractive extension of I.

Let R be a semiring and I a Q-ideal of R. Then L is an ideal of R/I if and only if there exists an ideal P of R such that P is a subtractive extension of I and $P/I = \{q + I : q \in Q \cap P\} = L$ as shown by D. R. Bonde and J. N. Chaudhari in 2014, see [7].

Moreover, if I is a Q-ideal of a semiring R and P is a k-ideal containing I, then I is an $(P \cap Q)$-ideal of the semiring P and $P/I = \{q + I : q \in P \cap Q\}$ is a k-ideal of R/I as given by S. E. Atani in 2007, see [6].

Let R be a semiring and I a Q-ideal of R. Moreover, let ϕ be a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\emptyset\}$ such that $\phi(L)$ is a subtractive extension of I for all ideal L of R where L is a subtractive extension of I. We define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ by $\phi_I(J/I) = (\phi(J))/I$ for each ideal J of R where J is a subtractive extension of I.

We call R a *semiring with ϕ satisfying the property* $(*)$ if R is a semiring with ϕ, I is a Q-ideal of R and ϕ_I is a function from $\mathscr{I}(R/I)$ into $\mathscr{I}(R/I) \cup \{\emptyset\}$ where ϕ and ϕ_I are defined in the previous paragraph.

Example 4.3. Consider the semiring $\mathbb{Z}_0^+$ and its Q-ideal $I = 12\mathbb{Z}_0^+$. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\emptyset\}$ by $\phi(J) = 3\mathbb{Z}_0^+$ if J is a subtractive extension of I and $\phi(J) = \{0\}$ otherwise for all $J \in \mathscr{I}(\mathbb{Z}_0^+)$. Certainly, $\phi(L) = 3\mathbb{Z}_0^+$ is a subtractive extension of $I = 12\mathbb{Z}_0^+$ for all $L \in \mathscr{I}(R)$ where L is a subtractive extension of I. Define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\emptyset\}$ by $\phi_I(J/I) = (3\mathbb{Z}_0^+)/I$ for each ideal J of R where J is a subtractive extension of I. Thus $\mathbb{Z}_0^+$ is the semiring with ϕ satisfying the property $(*)$.

Theorem 4.4. *Let R be a semiring with ϕ satisfying the property $(*)$, n a positive integer, I a Q-ideal of R and P a subtractive extension of I. Then P is a ϕ-n-absorbing ideal of R if and only if P/I is a ϕ_I-n-absorbing ideal of R/I.*

Proof. First, assume that P is a ϕ-n-absorbing ideal of R. Then P/I is an ideal of R/I because P is a subtractive extension of I. Let $q_1+I, q_2+I, \ldots, q_{n+1}+I \in R/I$ be such that $(q_1 + I)(q_2 + I) \cdots (q_{n+1} + I) \in P/I - \phi_I(P/I)$. Thus $q_1q_2 \cdots q_{n+1} \in P-\phi(P)$. Since P is a ϕ-n-absorbing ideal, $\hat{q}_{i,n+1} \in P$ for some $i \in \{1, 2, \ldots, n+1\}$. Hence $(q_1 + I) \cdots (q_{i-1} + I)(q_{i+1} + I) \cdots (q_{n+1} + I) \in P/I$. Therefore, P/I is a ϕ_I-n-absorbing k-ideal of R/I.

Conversely, assume that P/I is a ϕ_I-n-absorbing ideal of R/I. We show that P is a ϕ-n-absorbing ideal of R. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2 \cdots x_{n+1} \in P - \phi(P)$. Then there exist $q_1, q_2 \ldots, q_{n+1} \in Q$ such that $x_i \in q_i + I$ for all $i \in \{1, 2, \ldots, n + 1\}$. So there is $y_i \in I$ such that $x_i = q_i + y_i$ for all $i \in \{1, 2, \ldots, n + 1\}$. Hence we obtain $(q_1 + y_1)(q_2 + y_2) \cdots (q_{n+1} + y_{n+1}) \in P - \phi(P)$. Then $q_1q_2 \cdots q_{n+1} \in P - \phi(P)$ because P and $\phi(P)$ are subtractive extensions of I. Thus $(q_1 + I)(q_2 + I) \cdots (q_{n+1} + I) \in P/I - \phi_I(P/I)$. Hence $(q_1 + I) \cdots (q_{i-1} + I)(q_{i+1} + I) \cdots (q_{n+1} + I) \in P/I$ for some $i \in \{1, 2, \ldots, n + 1\}$ since P/I is a ϕ_I-n-absorbing ideal. Then $\hat{q}_{i,n+1} \in P$. Thus $\hat{x}_{i,n+1} = (q_1 + y_1) \cdots (q_{i-1} + y_{i-1})(q_{i+1} + y_{i+1}) \cdots (q_{n+1} + y_{n+1}) \in P$. Therefore, P is a ϕ-n-absorbing ideal of R. □

Corollary 4.5. *Let R be a semiring with ϕ satisfying the property $(*)$, n a positive integer and I a Q-ideal of R. Then I is a ϕ-n-absorbing ideal of R if and only if the zero ideal of R/I is a ϕ_I-n-absorbing ideal.*

Let R be a semiring and S the set of all multiplicatively cancellable elements of R. Define a relation $\sim$ on $R \times S$ as follows :

$$(a,s) \sim (b,t) \quad \text{if and only if} \quad at = bs$$

for all $(a,s),(b,t) \in R \times S$. Then $\sim$ is an equivalence relation on $R \times S$.

For $(a,s) \in R \times S$, denote the equivalence class of $\sim$ containing (a,s) by $\frac{a}{s}$, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S forms a semiring under operations

$$\frac{a}{s} + \frac{b}{t} = \frac{at+sb}{st} \qquad \text{and} \qquad \left(\frac{a}{s}\right)\left(\frac{b}{t}\right) = \frac{ab}{st}$$

for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the *semiring of fractions of R with respect to S*, see [8].

Let I be an ideal of R. The ideal generated by I of R_S, that is the set of all finite sums $a_1s_1 + a_2s_2 + \cdots + a_ns_n$ where $a_i \in I$ and $s_i \in R_S$, is called the *extension of I to R_S*, and is denoted by IR_S. Let J be an ideal of R_S. Then the *contraction of J in R* is $J \cap R = \left\{r \in R : \frac{r}{1} \in J\right\}$, which is an ideal of R. Moreover, $x \in IR_S$ if and only if it can be written in from $x = \frac{a}{c}$ for some $a \in I$ and $c \in S$, see [8].

Let R be a semiring with ϕ. We define $\phi_S : \mathscr{I}(R_S) \to \mathscr{I}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = \phi(J \cap R)R_S$ if $\phi(J \cap R) \in \mathscr{I}(R)$ and $\phi_S(J) = \emptyset$ if $\phi(J \cap R) = \emptyset$ for all $J \in \mathscr{I}(R_S)$.

In the last theorem, we would like to show that if I is a ϕ-n-absorbing ideal of R under some conditions, then IR_S is a ϕ_S-n-absorbing ideal of R_S.

Theorem 4.6. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I an ideal of R with $I \cap S = \emptyset$ and $\phi(I)R_S \subseteq \phi_S(IR_S)$. If I is a ϕ-n-absorbing ideal of R, then IR_S is a ϕ_S-n-absorbing ideal of R_S.*

Proof. Assume that I is a ϕ-n-absorbing ideal of R. Since $I \cap S = \emptyset$, it follows that IR_S is a proper ideal of R_S. Let $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots, \frac{x_{n+1}}{s_{n+1}} \in R_S$ be such that $\frac{x_1x_2\cdots x_{n+1}}{s_1s_2\cdots s_{n+1}} \in IR_S - \phi_S(IR_S)$. Then $\frac{x_1x_2\cdots x_{n+1}}{s_1s_2\cdots s_{n+1}} \in IR_S - \phi(I)R_S$ because $\phi(I)R_S \subseteq \phi_S(IR_S)$. Thus there exist $a \in I$ and $v \in S$ such that $\frac{x_1x_2\cdots x_{n+1}}{s_1s_2\cdots s_{n+1}} = \frac{a}{v}$. Hence $x_1x_2\cdots x_{n+1}v = s_1s_2\cdots s_{n+1}a \in I$. If $x_1x_2\cdots x_{n+1}v \in \phi(I)$, then $\frac{x_1x_2\cdots x_{n+1}}{s_1s_2\cdots s_{n+1}} = \frac{x_1x_2\cdots x_{n+1}v}{s_1s_2\cdots s_{n+1}v} \in \phi(I)R_S$ which is a contradiction. Then $x_1x_2\cdots x_{n+1}v \in I - \phi(I)$. Since I is ϕ-n-absorbing, $x_1x_2\cdots x_n \in I$ or $\hat{x}_{i,n}x_{n+1}v \in I$ for some $i \in \{1,2,\ldots,n\}$. Thus $\frac{x_1x_2\cdots x_n}{s_1s_2\cdots s_n} \in IR_S$ or $\frac{\hat{x}_{i,n}x_{n+1}v}{\hat{s}_{i,n}s_{n+1}v} \in IR_S$. Hence $\frac{\hat{x}_{j,n+1}}{\hat{s}_{j,n+1}} \in IR_S$ for some $j \in \{1,2,\ldots,n+1\}$. Therefore, IR_S is a ϕ_S-n-absorbing ideal of R_S. □

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