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On Strong Convergence of a Halpern-Mann's Type Iteration with Perturbations for Common Fixed Point and Generalized Equilibrium Problems

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Abstract : We establish strong convergence of a sequence generated by Halpern-Mann's type iteration with perturbation for approximating a common element of the set of fixed points of a countable family of quasi-nonexpansive mappings and the set of solutions of a generalized equilibrium problem in a real Hilbert space. With an appropriate setting, some results for solving the minimum-norm problems are also included. Finally, we consider the modified viscosity method of a countable family of nonexpansive mappings. The results presented in this paper extend and improvement of previously known results in this research area.

Keywords : common minimum-norm; fixed point; generalized equilibrium problem; perturbation; quasi-nonexpansive mapping.
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1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let φ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and a nonlinear mapping $A : C \to H$. The generalized equilibrium problem [1] is to find $x \in C$ such that

$$\varphi(x, y) + \langle Ax, y - x \rangle \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of such solutions x is denoted by $EP(\varphi, A)$, i.e.,

$$EP(\varphi, A) = \{x \in C : \varphi(x, y) + \langle Ax, y - x \rangle \geq 0 \quad \text{for all } y \in C\}.$$

In the case of $A \equiv 0$, then this problem coincides with the equilibrium problem and $EP(\varphi, A)$ is denoted by $EP(\varphi)$. In the case of $\varphi \equiv 0$, this problem coincides with the variational inequality problem and $EP(\varphi, A)$ is denoted by $VI(C, A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games (see [2,3]). Note that many researchers studied iterative methods for finding the equilibrium problems. For example, one can see [3].

Let $T : C \to H$ be a mapping. We denote by $F(T) = \{x \in C : x = Tx\}$ the set of fixed points of T. A mapping $T : C \to H$ is said to be

- *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C;$$

- *quasi-nonexpansive* if $F(T) \neq \varnothing$ and

$$\|Tx - y\| \leq \|x - y\| \quad \text{for all } x \in C \text{ and } y \in F(T).$$

Several articles have appeared providing methods for approximating fixed points of (quasi-)nonexpansive mappings. Two classical iterative schemes are Mann iteration [4] and Halpern iteration [5]. It is known that under appropriate conditions the Mann iteration converges only weakly to a fixed point of T but the Halpern iteration converges strongly to a fixed point of T.

In 2007, Aoyama et al. [6] extended Mann iteration to obtain weak convergence to a common fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_n x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. On the other hand, Aoyama et al. [7] extended Halpern iteration to obtain strong convergence to a common fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ by the following iteration:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. In the literature, the schemes (1.2) and (1.3) have been widely studied and extended in [8–13] and references therein. Iterative methods for finding a common element of the set of solutions for an (a generalized) equilibrium problem and the set of fixed points of a nonexpansive mapping are studied. For instance, Takahashi and Takahashi [1] introduced an iterative method for a generalized equilibrium problem in the following way:

$$\begin{cases} x_1 = x \in C, \\ \varphi(u_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, u_n - x_n \rangle \geq 0 \quad \text{for all } y \in C, \\ y_n = \alpha_n x + (1-\alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Ty_n \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$. It is known that under appropriate conditions the iterative method (1.4) converges strongly to an element of $F(T) \cap EP(\varphi, A)$. Recently, there are many authors introduced and studied the iterative methods for finding a common element of the set of solutions for a generalized equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings (see [14–18]).

We know that there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. In 2011, Yao and Shahzad [19] defined a iteration with perturbation for a nonexpansive mapping as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n P_C(\alpha_n w_n + (1-\alpha_n)Tx_n) \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{w_n\}$ is a small perturbation in H. Recently, Chuang et al. [20] defined an iteration with perturbation for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points for a quasi-nonexpansive mapping as follows:

$$\begin{cases} x_1 \in H, \\ \varphi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n w_n + (1-\alpha_n)(\beta_n u_n + (1-\beta_n)Tu_n) \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{r_n\}$ is a sequence in $(0,\infty)$ and $\{w_n\}$ is a perturbation in H. They proved a strong convergence theorem for such iterations under some appropriate assumptions.

In this paper, motivated by above works, we introduce a new iterative scheme for finding a common element of a generalized equilibrium problem and a fixed point problem for a countable family of quasi-nonexpansive mappings with perturbation and then establish strong convergence theorems of the scheme by using the NST-condition introduced by Nakajo et al. [9]. Our results extend and improve the main results of Chuang et al. [20] and Wang [21]. Some results for solving the minimum-norm problems are also included. Using the technique in [22], we obtain the strong convergence theorems by the modified viscosity method for finding the common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space which extend and improve Nilsrakoo and Saejung [23], Duan and He [24], and many others.

2 Preliminaries

We present several definitions and preliminaries which are needed in this paper. Let H be a real Hilbert space. It is well-known that

$$\|x+y\|^2 = \|x\|^2 - \|y\|^2 + 2\langle x+y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and all $\lambda \in (0,1)$. The identity (2.2) implies that the following inequality holds

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 \tag{2.3}$$

for all $x, y, z \in H$ and all $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma = 1$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x.

Let C be a subset of H and $\alpha > 0$. A mapping $T : C \to H$ is said to be

- *firmly nonexpansive* if
$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - \|(x-y) - (Tx - Ty)\|^2 \quad \text{for all } x, y \in C;$$
- *α-Lipschitzian* if
$$\|Tx - Ty\| \leq \alpha\|x - y\| \quad \text{for all } x, y \in C;$$
in particular, if $\alpha \in [0,1)$, then T is called *α-contraction*;
- *monotone* if
$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \text{for all } x, y \in C;$$
- *α-strongly monotone* if
$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x-y\|^2 \quad \text{for all } x, y \in C;$$
- *α-inverse-strongly monotone* if
$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2 \quad \text{for all } x, y \in C.$$

Remark 2.1. *It follows directly from the definitions above that:*

- *Every firmly nonexpansive mapping is nonexpansive and every nonexpansive mapping is* 1*-Lipschitzian.*
- *If T is α-inverse-strongly monotone, then T is $1/\alpha$-Lipschitzian.*
- *If T is α-strongly monotone and L-Lipschitzian, then T is α/L^2-inverse-strongly monotone.*

If C is nonempty closed and convex, then for every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

Such a P_C is called the *metric projection* from H onto C. We know that P_C is a firmly nonexpansive mapping from H onto C. Furthermore, for any $x \in H$ and $z \in C,\ z = P_C x$ if and only if

$$\langle x - z,\ z - y\rangle \ge 0 \quad \text{for all } y \in C.$$

For solving the equilibrium problem for a bifunction $\varphi : C \times C \to \mathbb{R}$, let us assume that φ satisfies the following conditions (see [1–3]):

(A1) $\varphi(x, x) = 0$ for all $x \in C$;

(A2) φ is monotone, i.e., $\varphi(x, y) + \varphi(y, x) \le 0$ for all $x, y \in C$;

(A3) $\lim\limits_{t \to 0} \varphi(tz + (1 - t)x, y) \le \varphi(x, y)$ for all $x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto \varphi(x, y)$ is convex and lower semicontinuous.

The following lemma gives a characterization of a solution of an equilibrium problem proved by [2, Corollary 1] and [3, Lemma 2.12].

Lemma 2.2. *Let C be a closed convex subset of a real Hilbert space H, let φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let $r > 0$. Define a mapping $T_r : H \to C$ by $T_r(x) = x^*$ where x^* is the unique element in C such that*

$$\varphi(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \ge 0 \quad \text{for all } y \in C.$$

Such a mapping T_r is called the resolvent of φ for r. Then, the followings hold:

(i) T_r is firmly nonexpansive, that is,

$$\|T_r x - T_r y\|^2 \le \|x - y\|^2 - \|(x - y) - (T_r x - T_r y)\|^2 \quad \text{for all } x, y \in C;$$

(ii) $F(T_r) = EP(\varphi)$;

(iii) $EP(\varphi)$ is closed and convex.

Remark 2.3. *Some well-known examples of resolvents of bifunctions satisfying the conditions* (A1)-(A4) *are presented in [3, Lemma 2.15].*

Lemma 2.4. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of C into H. Let $\{x_n\}$ and $\{u_n\}$ be bounded sequences in C such that $u_n = T_{r_n}(x_n - r_n A x_n)$ for all $n \in \mathbb{N}$ where T_{r_n} is the resolvent of φ for r_n with $r_n \in [a, b] \subset (0, 2\alpha)$. Then we have the following assertions.*

(i) $\|u_n - p\|^2 \le \|x_n - p\|^2 - a(2\alpha - b)\|Ax_n - Ap\|^2$ *for each* $p \in EP(\varphi, A)$.

(ii) If $x_n - u_n \to 0$ *and* $u_n \rightharpoonup u$, *then* $u \in EP(\varphi, A)$.

Proof. (i) Since T_{r_n} is firmly nonexpansive, we obtain

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\|^2 \\
&\le \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2r_n\alpha\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - a(2\alpha - b)\|Ax_n - Ap\|^2.
\end{aligned}$$

(ii) Since A is $1/\alpha$-Lipschitzian and $x_n - u_n \to 0$, we obtain $Ax_n - Au_n \to 0$. Since

$$\varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \ge 0$$

and by the condition (A2), we obtain

$$\langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \ge \varphi(y, u_n) \tag{2.4}$$

for all $y \in C$. Put $z_t = ty + (1-t)u$ for all $t \in (0, 1]$. Then, we have $z_t \in C$. So, from (2.4) and monotonicity of A we have

$$\begin{aligned}
\langle z_t - u_n, Az_t\rangle &\ge \langle z_t - u_n, Az_t\rangle - \langle z_t - u_n, Ax_n\rangle - \left\langle z_t - u_n, \frac{u_n - x_n}{r_n}\right\rangle + \varphi(z_t, u_n) \\
&= \langle z_t - u_n, Az_t - Au_n\rangle + \langle z_t - u_n, Au_n - Ax_n\rangle \\
&\quad - \left\langle z_t - u_n, \frac{u_n - x_n}{r_n}\right\rangle + \varphi(z_t, u_n) \\
&\ge \langle z_t - u_n, Au_n - Ax_n\rangle - \left\langle z_t - u_n, \frac{u_n - x_n}{r_n}\right\rangle + \varphi(z_t, u_n).
\end{aligned}$$

From the condition (A4) and let $n \to \infty$ we get

$$\langle z_t - u, Az_t\rangle \ge \varphi(z_t, u). \tag{2.5}$$

From the conditions (A1), (A4) and (2.5), we also have

$$\begin{aligned}
0 = \varphi(z_t, z_t) &\le t\varphi(z_t, y) + (1-t)\varphi(z_t, u) \\
&\le t\varphi(z_t, y) + (1-t)\langle z_t - u, Az_t\rangle \\
&= t\varphi(z_t, y) + t(1-t)\langle y - u, Az_t\rangle
\end{aligned}$$

and hence

$$0 \le \varphi(z_t, y) + (1-t)\langle y - u, Az_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \varphi(u, y) + \langle y - u, Au \rangle.$$

This implies $u \in EP(\varphi, A)$. □

The following lemma is needed for proving the main results.

Lemma 2.5 ([25, Lemma 2.6] and [26, Lemma 2.1])**.** *Let $\{a_n\}$ be a sequence in $[0, \infty)$, $\{\zeta_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \zeta_n = \infty$, and $\{\delta_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \zeta_n) a_n + \zeta_n \delta_n \quad \textit{for all } n \in \mathbb{N},$$

and one of the following holds:

(i) $\sum_{n=1}^{\infty} \zeta_n \delta_n < \infty$*;*

(ii) $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Halpern-Mann's Type Iterations with Perturbations

In this section, motivated by [20, Thoerem 3.1], we give strong convergence theorems for finding the common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a countable family of quasi-nonexpansive mappings by iterations with perturbations in a real Hilbert space. To this end, the following condition of a family of mappings is needed in this paper. A family $\{T_n\}$ of mappings of C into H is said to satisfy the *NST-condition* [9] if every weak cluster point of $\{x_n\}$ belong to $\cap_{n=1}^{\infty} F(T_n)$ whenever $\{x_n\}$ is a bounded sequence in C. We know several examples of families of quasi-nonexpansive mappings satisfying the *NST-condition*; see [7–13, 15]).

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of H and let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into H satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let $\{w_n\}$ be a sequence in H and $\{r_n\}$ be a sequence in $[a, b] \subset (0, 2\alpha)$. For a given $x_1 \in H$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \textit{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1 - \lambda_n) u_n), \\ x_{n+1} = \alpha_n w_n + \beta_n y_n + \gamma_n T_n y_n \quad \textit{for all } n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$. Assume that *(i)* $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty}\beta_n\gamma_n > 0$;

(iii) $\limsup_{n\to\infty}\lambda_n < 1$;

(iv) $\lim_{n\to\infty} w_n = w$ *for some* $w \in H$.

Then $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{\mathfrak{F}}w$.

Proof. Since $\mathfrak{F}$ is nonempty closed convex, for convenience, we put $\bar{x} = P_{\mathfrak{F}}w$. Note that u_n can be rewritten as $u_n = T_{r_n}(x_n - r_nAx_n)$ for each $n \in \mathbb{N}$.

We prove the theorem in the following steps.

Step 1: We show that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are bounded. From Lemma 2.4(i), we obtain

$$\|u_n - \bar{x}\| \le \|x_n - \bar{x}\|. \tag{3.2}$$

By the definition of y_n, we have

$$\begin{aligned}\|y_n - \bar{x}\| &\le \|\lambda_n(x_n - \bar{x}) + (1-\lambda_n)(u_n - \bar{x})\| \\ &\le \lambda_n\|x_n - \bar{x}\| + (1-\lambda_n)\|u_n - \bar{x}\| &(3.3)\\ &\le \|x_n - \bar{x}\|. &(3.4)\end{aligned}$$

So, we obtain

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &= \|\alpha_n w_n + \beta_n y_n + \gamma_n T_n y_n - \bar{x}\| \\ &\le \alpha_n\|w_n - \bar{x}\| + \beta_n\|y_n - \bar{x}\| + \gamma_n\|T_n y_n - \bar{x}\| \\ &\le \alpha_n\|w_n - \bar{x}\| + \beta_n\|y_n - \bar{x}\| + \gamma_n\|y_n - \bar{x}\| \\ &= \alpha_n\|w_n - \bar{x}\| + (1-\alpha_n)\|y_n - \bar{x}\| &(3.5)\\ &\le \alpha_n\|w_n - \bar{x}\| + (1-\alpha_n)\|x_n - \bar{x}\|. &(3.6)\end{aligned}$$

Since $\lim_{n\to\infty} w_n = w$ for some $w \in H$, we obtain $\{w_n\}$ is a bounded sequence. Then there exists $M > 0$ such that $\sup_{n\in\mathbb{N}}\|w_n - \bar{x}\| \le M$. From (3.6) and by induction, we obtain

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\le \alpha_n\|w_n - \bar{x}\| + (1-\alpha_n)\|x_n - \bar{x}\| \\ &\le \alpha_n K + (1-\alpha_n)K \\ &= K,\end{aligned}$$

where $K = \max\{\|x_1 - \bar{x}\|, M\}$. Hence, the sequence $\{x_n\}$ is bounded and so $\{y_n\}$ and $\{u_n\}$ are also bounded sequences.

Step 2: We show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|) \ge 0,$$

then

(a) $\lim_{k\to\infty} \|x_{n_k} - u_{n_k}\| = 0$;

(b) $\lim_{k\to\infty} \|T_{n_k} y_{n_k} - y_{n_k}\| = 0$;

(c) $\lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| = 0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|) \geq 0.$$

(a) By (3.5), (3.3), (3.2) and $\alpha_n \to 0$, we obtain

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty} \big(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) \\
&\leq \liminf_{k\to\infty} \big(\alpha_{n_k}\|w_{n_k} - \bar{x}\| + (1-\alpha_{n_k})\|y_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) \\
&= \liminf_{k\to\infty}(\|y_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|) \\
&\leq \liminf_{k\to\infty} \big(\lambda_{n_k}\|x_{n_k} - \bar{x}\| + (1-\lambda_{n_k})\|u_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) \\
&= \liminf_{k\to\infty}(1-\lambda_{n_k})\big(\|u_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) \\
&\leq \limsup_{k\to\infty}(1-\lambda_{n_k})\big(\|u_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) \\
&\leq \limsup_{k\to\infty}(1-\lambda_{n_k})\big(\|x_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) \\
&= 0.
\end{aligned}$$

Thus

$$\lim_{k\to\infty}(1-\lambda_{n_k})\big(\|u_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) = 0.$$

Since $\limsup_{n\to\infty} \lambda_n < 1$, we obtain

$$\lim_{k\to\infty} \big(\|u_{n_k} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\big) = 0. \tag{3.7}$$

This together with Lemma 2.4(i) gives

$$a(2\alpha - b)\|Ax_{n_k} - A\bar{x}\|^2 \leq \|x_{n_k} - \bar{x}\|^2 - \|u_{n_k} - \bar{x}\|^2 \to 0.$$

This implies that

$$\|Ax_{n_k} - A\bar{x}\| \to 0. \tag{3.8}$$

Using Lemma 2.2 (i), we obtain

$$\begin{aligned}
\|u_n - \bar{x}\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(\bar{x} - r_nA\bar{x})\|^2 \\
&\leq \|(x_n - r_nAx_n) - (\bar{x} - r_nA\bar{x})\|^2 - \|u_n - x_n\|^2 \\
&= \|(x_n - \bar{x}) - r_n(Ax_n - A\bar{x})\|^2 - \|u_n - x_n\|^2 \\
&= \|x_n - \bar{x}\|^2 - 2r_n\langle x_n - \bar{x}, Ax_n - A\bar{x}\rangle + r_n^2\|Ax_n - A\bar{x}\|^2 - \|x_n - u_n\|^2.
\end{aligned}$$

Thus

$$\|x_n - u_n\|^2 \le \|x_n - \bar{x}\|^2 - \|u_n - \bar{x}\|^2 - 2r_n\langle x_n - \bar{x}, Ax_n - A\bar{x}\rangle + r_n^2\|Ax_n - A\bar{x}\|^2.$$

This together with (3.7) and (3.8) gives

$$\lim_{k\to\infty} \|x_{n_k} - u_{n_k}\| = 0$$

(b) By (3.6) and $\alpha_n \to 0$, we obtain

$$\|x_{n+1} - \bar{x}\| - \|x_n - \bar{x}\| \le \alpha_n(\|w_n - \bar{x}\| - \|x_n - \bar{x}\|) \to 0.$$

Thus

$$\limsup_{n\to\infty}(\|x_{n+1} - \bar{x}\| - \|x_n - \bar{x}\|) \le 0.$$

This implies that

$$\lim_{k\to\infty}(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|) = 0. \tag{3.9}$$

By (2.3) and (3.4), we obtain

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \|\alpha_n w_n + \beta_n y_n + \gamma_n T_n y_n - \bar{x}\|^2 \\ &\le \alpha_n\|w_n - \bar{x}\|^2 + \beta_n\|y_n - \bar{x}\|^2 + \gamma_n\|T_n y_n - \bar{x}\|^2 - \beta_n\gamma_n\|T_n y_n - y_n\|^2 \\ &\le \alpha_n\|w_n - \bar{x}\|^2 + (\beta_n + \gamma_n)\|y_n - \bar{x}\|^2 - \beta_n\gamma_n\|T_n y_n - y_n\|^2 \\ &\le \alpha_n\|w_n - \bar{x}\|^2 + (1 - \alpha_n)\|x_n - \bar{x}\|^2 - \beta_n\gamma_n\|T_n y_n - y_n\|^2.\end{aligned}$$

Thus

$$\beta_n\gamma_n\|T_n y_n - y_n\|^2 \le \alpha_n(\|w_n - \bar{x}\|^2 - \|x_n - \bar{x}\|^2) + \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2.$$

By $\alpha_n \to 0$ and (3.9), we obtain

$$\beta_{n_k}\gamma_{n_k}\|T_{n_k} y_{n_k} - y_{n_k}\| \to 0.$$

It follows from the condition (ii) that

$$\|T_{n_k} y_{n_k} - y_{n_k}\| \to 0.$$

(c) It follows from (a), (b) and $\alpha_n \to 0$ that

$$\begin{aligned}\|x_{n_k+1} - u_{n_k}\| &\le \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - u_{n_k}\| \\ &\le \|\alpha_{n_k}(w_{n_k} - y_{n_k}) + \gamma_{n_k}(T_{n_k} y_{n_k} - y_{n_k})\| \\ &\quad + \|P_C(\lambda_{n_k} x_{n_k} + (1 - \lambda_{n_k})u_{n_k}) - P_C u_{n_k}\| \\ &\le \alpha_{n_k}\|w_{n_k} - y_{n_k}\| + \gamma_{n_k}\|T_{n_k} y_{n_k} - y_{n_k}\| + \lambda_{n_k}\|x_{n_k} - u_{n_k}\| \to 0.\end{aligned}$$

Step 3: We show that

$$\|x_{n+1} - \bar{x}\|^2 \leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle x_{n+1} - \bar{x}, w_n - \bar{x}\rangle \quad \text{for all} \quad n \in \mathbb{N}.$$

To do this, by using (2.1), (2.3) and (3.4), we obtain

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq \|\alpha_n(w_n - \bar{x}) + \beta_n(y_n - \bar{x}) + \gamma_n(T_n y_n - \bar{x})\|^2 \\
&\leq \|\beta_n(y_n - \bar{x}) + \gamma_n(T_n y_n - \bar{x})\|^2 + 2\alpha_n\langle x_{n+1} - \bar{x}, w_n - \bar{x}\rangle \\
&\leq \beta_n\|y_n - \bar{x}\|^2 + \gamma_n\|T_n y_n - \bar{x}\|^2 + 2\alpha_n\langle x_{n+1} - \bar{x}, w_n - \bar{x}\rangle \\
&\leq (\beta_n + \gamma_n)\|y_n - \bar{x}\|^2 + 2\alpha_n\langle x_{n+1} - \bar{x}, w_n - \bar{x}\rangle \\
&\leq (1 - \alpha_n)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle x_{n+1} - \bar{x}, w_n - \bar{x}\rangle.
\end{aligned}$$

Step 4: We show that $x_n \to \overline{x}$, $y_n \to \overline{x}$ and $u_n \to \overline{x}$. From Step 3 with setting $a_n = \|x_n - \bar{x}\|^2$, we obtain

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n,$$

where $\delta_n = 2\langle x_{n+1} - \bar{x}, w_n - \bar{x}\rangle$ for all $n \in \mathbb{N}$. From Lemma 2.5(ii) and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we only show that $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$. To show this inequality, take a subsequence $\{n_i\}$ of $\{n_k\}$ such that

$$\limsup_{k\to\infty}\langle x_{n_k+1} - \bar{x}, w_{n_k} - \bar{x}\rangle = \lim_{i\to\infty}\langle x_{n_i+1} - \bar{x}, w_{n_i} - \bar{x}\rangle. \tag{3.10}$$

Since $\{u_n\}$ is bounded, without loss of generality, we may assume that $u_{n_i} \rightharpoonup u$. Since

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) = \liminf_{k\to\infty}(\|x_{n_k+1} - \bar{x}\|^2 - \|x_{n_k} - \bar{x}\|^2) \geq 0,$$

we get

$$\liminf_{k\to\infty}(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|) \geq 0.$$

From Step 2(b) and $\{T_n\}$ satisfies the NST-condition, we get $u \in \cap_{n=1}^{\infty} F(T_n)$. Furthermore, from Step 2(a) and Lemma 2.4(ii), we obtain $u \in EP(\varphi, A)$. This means that $u \in \mathfrak{F}$. From Step 2(c), we have $x_{n_i+1} \to u$. Since $w_n \to w$ and the property of metric projection with $\bar{x} = P_{\mathfrak{F}}w$, it follows from (3.10) that

$$\limsup_{k\to\infty}\langle x_{n_k+1} - \bar{x}, w_{n_k} - \bar{x}\rangle = \lim_{i\to\infty}\langle x_{n_i+1} - \bar{x}, w_{n_i} - \bar{x}\rangle = \langle u - \bar{x}, w - \bar{x}\rangle \leq 0.$$

Thus $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ and then $x_n \to \overline{x}$. It follows from (3.2) and (3.4) that $u_n \to \overline{x}$ and $y_n \to \overline{x}$. This completes the proof. □

If we put $T_n \equiv T$ and use Theorem 3.1, then we get the following result.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of H and let T be a quasi-nonexpansive mapping of C into H satisfying $I-T$ is demiclosed at 0 and $F(T)\cap EP(\varphi, A) \neq \varnothing$. Let $\{w_n\}$ be a sequence in H and $\{r_n\}$ be a sequence in $[a, b] \subset (0, 2\alpha)$. For a given $x_1 \in H$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1 - \lambda_n)u_n), \\ x_{n+1} = \alpha_n w_n + \beta_n y_n + \gamma_n T y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$;

(iv) $\lim_{n\to\infty} w_n = w$ *for some* $w \in H$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to $P_{F(T)\cap EP(\varphi,A)}w$.

Remark 3.3. *(1) Corollary 3.2 extends and improves [20, Theorem 3.1] by setting $A \equiv 0$ and $\lambda_n \equiv 0$.*

(2) Corollary 3.2 is still true if T is one of generalized hybrid mappings, hybrid mappings, λ-hybrid mappings, nonspreading mappings, nonexpansive mappings, mappings with condition (B) or (C). Indeed, these mappings are quasi-nonexpansive mapping satisfying $I - T$ is demiclosed at 0. For more details, see [20].

If we put $\varphi \equiv 0$, $A \equiv 0$, $r_n \equiv 1$ and use Theorem 3.1, then $T_{r_n} \equiv P_C$ and we get the following result.

Corollary 3.4. *Let C be a closed convex subset of a real Hilbert space H and let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into H satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $\{w_n\}$ be a sequence in H. For a given $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = P_C(\alpha_n w_n + \beta_n x_n + \gamma_n T_n x_n) \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$*;*

(ii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$*;*

(iii) $\lim_{n\to\infty} w_n = w$ *for some* $w \in H$*;*

Then $\{x_n\}$ converges strongly to $P_{\mathfrak{F}}w$.

Remark 3.5. *If $T_n \equiv T$, then Corollary 3.4 reduces to [20, Theorem 4.1].*

Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself and let $\lambda_1^n, \lambda_2^n, \lambda_3^n$ be real numbers for all $n \in \mathbb{N}$ such that $0 < \lambda_i^n \leq 1$ for $i = 1, 2, 3$ with $\lambda_1^n + \lambda_2^n + \lambda_3^n = 1$ and $\lambda_3^n \geq a > 0$ for all $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, we define a mapping S_n of C into itself [16] as follow:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_1^n T_n U_{n,n+1} + \lambda_2^n U_{n,n+1} + \lambda_3^n I,\\
U_{n,n-1} &= \lambda_1^{n-1} T_{n-1} U_{n,n} + \lambda_2^{n-1} U_{n,n} + \lambda_3^{n-1} I,\\
&\vdots\\
U_{n,k} &= \lambda_1^k T_k U_{n,k+1} + \lambda_2^k U_{n,k+1} + \lambda_3^k I,\\
&\vdots\\
U_{n,2} &= \lambda_1^2 T_2 U_{n,3} + \lambda_2^2 U_{n,3} + \lambda_3^2 I,\\
S_n = U_{n,1} &= \lambda_1^1 T_1 U_{n,2} + \lambda_2^1 U_{n,2} + \lambda_3^1 I. \qquad (3.11)
\end{aligned}$$

Such a mapping S_n is called the *S-mapping* generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1^n, \lambda_2^n, \lambda_3^n$. It is obvious that S_n and $U_{n,k}$ are nonexpansive for every $n \geq k$. The following lemma is proved in [15, Lemma 3.4].

Lemma 3.6. *Let C be a closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself with $\cap_{i=1}^\infty F(T_i)$ is nonempty. Let $\{\lambda_i^n : i = 1, 2, 3\}$ be real numbers in $[0, 1]$ such that $\lambda_1^n + \lambda_2^n + \lambda_3^n = 1$ and $\lambda_3^n \geq a > 0$ for all $n \in \mathbb{N}$. Then the sequence $\{S_n\}$ of S-mapping defined by (3.11) satisfies the NST-condition.*

Using Theorem 3.1 and Lemma 3.6, we get the following result.

Theorem 3.7. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself with $\mathfrak{F} := \cap_{n=1}^\infty F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let $\{w_n\}$ be a sequence in H and $\{r_n\}$ be a sequence in $[a, b] \subset (0, 2\alpha)$. For a given $x_1 \in H$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$\begin{cases}
\varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C,\\
y_n = P_C(\lambda_n x_n + (1 - \lambda_n)u_n),\\
x_{n+1} = \alpha_n w_n + \beta_n y_n + \gamma_n S_n y_n \quad \text{for all } n \in \mathbb{N},
\end{cases}$$

where S_n is the S-mapping defined by (3.11) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$;

(iv) $\lim_{n\to\infty} w_n = w$ *for some* $w \in H$.

Then $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{\mathfrak{F}}w$.

The common minimum-norm problem of a subset D of a real Hilbert space H is finding $x^* \in D$ such that

$$\|x^*\| = \min\{\|x\| : x \in D\}.$$

In other words, x^* is the metric projection of the origin (0) on D. In many practical problems, such as optimization problems, finding the minimum norm fixed point of mappings is quite important (see [15, 19, 27–30]). For finding the common minimum-norm element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a countable family of quasi-nonexpansive mappings in a real Hilbert space, let us put $w_n \equiv 0$ and use Theorems 3.1. We get the following result.

Theorem 3.8. *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $\varphi : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1)-(A4). *Let* A *be an* α*-inverse-strongly monotone mapping of* H *and let* $\{T_n\}$ *be a family of quasi-nonexpansive mappings of* C *into* H *satisfying the NST-condition and* $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. *Let* $\{r_n\}$ *be a sequence in* $[a, b] \subset (0, 2\alpha)$. *For a given* $x_1 \in H$, *define the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \textit{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1 - \lambda_n)u_n), \\ x_{n+1} = \beta_n y_n + (1 - \alpha_n - \beta_n)T_n y_n \quad \textit{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$. *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$.

(iii) $\limsup_{n\to\infty} \lambda_n < 1$;

Then $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{\mathfrak{F}}0$ *which is the common minimum-norm element of* $\mathfrak{F}$.

Proof. Let $w_n \equiv 0$ and $\gamma_n \equiv 1 - \alpha_n - \beta_n$. Then $\lim_{n\to\infty} w_n = 0$, $\alpha_n + \beta_n + \gamma_n \equiv 1$ and $\liminf_{n\to\infty} \beta_n\gamma_n > 0$. Using Theorem 3.1, we obtain the desired result. □

If T_n is the S-mapping S_n in Theorem 3.8, then we get the following result.

Corollary 3.9. *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $\varphi : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1)-(A4). *Let* A *be an* α*-inverse-strongly monotone mapping of* H *and let* $\{T_n\}$ *be a family of nonexpansive mappings of* C *into itself with* $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. *Let* $\{r_n\}$ *be a* *sequence in* $[a,b] \subset (0, 2\alpha)$. *For a given* $x_1 \in H$, *define the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1-\lambda_n)u_n), \\ x_{n+1} = \beta_n y_n + (1 - \alpha_n - \beta_n)S_n y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where S_n *is the S-mapping defined by* (3.11), *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$. *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{\mathfrak{F}}0$ *which is the common minimum-norm element of* $\mathfrak{F}$.

If $\varphi \equiv 0$, $\lambda_n \equiv 0$ and use Corollary 3.9, then we get the following corollary by the iterative method like Yao et al.'s iteration [28] with a new control parameter which is complementary to Yao's result [28, Theorem 3.1] and Wang's result [21, Theorem 3.4]. More precisely, it provides a new convergence theorem for a wider class of the operator A (see Remark 2.1) and $S_n \equiv W_n$ which is called the W-mapping defined by Takahashi [31] if setting $\lambda_2^n \equiv 0$ in (3.11).

Corollary 3.10. *Let* H *be a real Hilbert space and* A *be an* α*-inverse-strongly monotone mapping of* H *and let* $\{T_n\}$ *be a family of nonexpansive mappings of* H *with* $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap VI(H, A) \neq \varnothing$. *Let* $\{r_n\}$ *be a sequence in* $[a,b] \subset (0, 2\alpha)$. *For a given* $x_1 \in H$, *define the sequences* $\{x_n\}$, *and* $\{y_n\}$ *by*

$$\begin{cases} y_n = x_n - r_n A x_n, \\ x_{n+1} = \beta_n y_n + (1 - \alpha_n - \beta_n)S_n y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where S_n *is the S-mapping defined by* (3.11), *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$. *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $P_{\mathfrak{F}}0$ *which is the common minimum-norm element of* $\mathfrak{F}$.

We present a strong convergence of the iteration (3.1) with another control condition of perturbations which extends and improves [19, Theorem 3.7] as follows.

Theorem 3.11. *Let* C *be a closed convex subset of a real Hilbert space* H *and let* $\varphi : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1)-(A4). *Let* A *be* *an α-inverse-strongly monotone mapping of C into H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let $\{w_n\}$ be a sequence in H and $\{r_n\}$ be a sequence in $[a, b] \subset (0, 2\alpha)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated iteratively by* (3.1). *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$;

(iv) $\sum_{n=1}^{\infty} \alpha_n\|w_n\| < \infty$ *or* $\lim_{n\to\infty} w_n = 0$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to $P_{\mathfrak{F}}0$ which is the common minimum-norm element of $\mathfrak{F}$.

Proof. Let $\widehat{w}_n \equiv 0$ and let $\{\widehat{x}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{u}_n\}$ be sequences defined as follows:

$$\begin{cases} \widehat{x}_1 = x_1 \in C, \\ \varphi(\widehat{u}_n, y) + \langle A\widehat{x}_n, y - \widehat{u}_n\rangle + \frac{1}{r_n}\langle y - \widehat{u}_n, \widehat{u}_n - \widehat{x}_n\rangle \geq 0 \quad \text{for all } y \in C, \\ \widehat{y}_n = P_C(\lambda_n\widehat{x}_n + (1-\lambda_n)\widehat{u}_n), \\ \widehat{x}_{n+1} = \alpha_n\widehat{w}_n + \beta_n\widehat{y}_n + \gamma_n T_n\widehat{y}_n \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

Using Theorem 3.1, we get that $\{\widehat{x}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{u}_n\}$ converge strongly to $P_{\mathfrak{F}}0$. Notice that $u_n \equiv T_{r_n}(I - r_nA)x_n$ and $\widehat{u}_n \equiv T_{r_n}(I - r_nA)\widehat{x}_n$. Then

$$\|\widehat{u}_n - u_n\| \leq \|\widehat{x}_n - x_n\|,$$

$$\begin{aligned} \|\widehat{y}_n - y_n\| &\leq \|\lambda_n(\widehat{x}_n - x_n) + (1-\lambda_n)(\widehat{u}_n - u_n)\| \\ &\leq \lambda_n\|\widehat{x}_n - x_n\| + (1-\lambda_n)\|\widehat{u}_n - u_n\| \\ &\leq \|\widehat{x}_n - x_n\| \end{aligned}$$

and so

$$\begin{aligned} \|\widehat{x}_{n+1} - x_{n+1}\| &\leq \alpha_n\|w_n\| + \beta_n\|\widehat{y}_n - y_n\| + \gamma_n\|T_n\widehat{y}_n - T_ny_n\| \\ &\leq (1-\alpha_n)\|\widehat{y}_n - y_n\| + \alpha_n\|w_n\| \\ &\leq (1-\alpha_n)\|\widehat{x}_n - x_n\| + \alpha_n\|w_n\|. \end{aligned}$$

By Lemma 2.5, we get $\widehat{x}_n - x_n \to 0$. It follows that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to $P_{\mathfrak{F}}0$ as desired. ☐

Remark 3.12. *Our results can be applied to solve the problem of finding common zeros of maximal monotone operators, the common minimizer problem, the multiple-sets split feasibility problem. For more details, see [15, 32].*

4 Modified Viscosity Methods

Using Theorem 3.1 and the technique in [22], we obtain the following strong convergence theorem by the modified viscosity method for finding the common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into H satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let f is a κ-contraction of H with $\kappa \in [0,1)$. Let $\{r_n\}$ be a sequence in $[a,b] \subset (0, 2\alpha)$. For a given $x_1 \in H$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1-\lambda_n)u_n), \\ x_{n+1} = \alpha_n f(T_n x_n) + \beta_n y_n + \gamma_n T_n y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n \gamma_n > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathfrak{F}$ where $z = P_{\mathfrak{F}} f(z)$.

Proof. Since $P_{\mathfrak{F}} f$ is contraction, there is $z \in \mathfrak{F}$ such that $z = P_{\mathfrak{F}} f(z)$. Let $\widehat{w}_n \equiv f(z)$ and let $\{\widehat{x}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{u}_n\}$ be sequences defined as follows:

$$\begin{cases} \widehat{x}_1 = x_1 \in C, \\ \varphi(\widehat{u}_n, y) + \langle A\widehat{x}_n, y - \widehat{u}_n\rangle + \frac{1}{r_n}\langle y - \widehat{u}_n, \widehat{u}_n - \widehat{x}_n\rangle \geq 0 \quad \text{for all } y \in C, \\ \widehat{y}_n = P_C(\lambda_n \widehat{x}_n + (1-\lambda_n)\widehat{u}_n), \\ \widehat{x}_{n+1} = \alpha_n \widehat{w}_n + \beta_n \widehat{y}_n + \gamma_n T_n \widehat{y}_n \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

Using Theorem 3.1, we get that $\{\widehat{x}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{u}_n\}$ converge strongly to $z = P_{\mathfrak{F}} f(z)$. We also obtain

$$\|\widehat{u}_n - u_n\| \leq \|\widehat{x}_n - x_n\|, \quad \|\widehat{y}_n - y_n\| \leq \|\widehat{x}_n - x_n\|$$

and

$$\begin{aligned}
&\|\widehat{x}_{n+1} - x_{n+1}\| \\
&\leq \alpha_n \|f(z) - f(T_n x_n)\| + \beta_n \|\widehat{y}_n - y_n\| + \gamma_n \|T_n \widehat{y}_n - T_n y_n\| \\
&\leq \alpha_n \kappa \|z - T_n x_n\| + (1-\alpha_n)\|\widehat{y}_n - y_n\| \\
&\leq \alpha_n \kappa \|z - x_n\| + (1-\alpha_n)\|\widehat{x}_n - x_n\| \\
&\leq \alpha_n \kappa \|\widehat{x}_n - x_n\| + \alpha_n \kappa \|z - \widehat{x}_n\| + (1-\alpha_n)\|\widehat{x}_n - x_n\| \\
&= (1 - \alpha_n(1-\kappa))\|\widehat{x}_n - x_n\| + \alpha_n(1-\kappa)\frac{\kappa\|z - \widehat{x}_n\|}{1-\kappa}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have $\sum_{n=1}^{\infty} \alpha_n(1-\kappa) = \infty$. By Lemma 2.5 and $\widehat{x}_n \to z$, we get $\widehat{x}_n - x_n \to 0$. It follows that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to z as desired. □

Similarly to the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into H satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. Let f is a κ-contraction of H with $\kappa \in [0,1)$ and let $\{r_n\}$ be a sequence in $[a,b] \subset (0, 2\alpha)$. For a given $x_1 \in C$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1-\lambda_n)u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n y_n + \gamma_n T_n y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n \gamma_n > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathfrak{F}$ where $z = P_{\mathfrak{F}} f(z)$.

Next we consider a more generalized iterative like the viscosity iteration with a family of contraction mappings and we obtain strong convergence theorem for finding the common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space.

Theorem 4.3. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)-(A4). *Let A be an α-inverse-strongly monotone mapping of H and let $\{T_n\}$ be a family of nonexpansive mappings of C into H satisfying the NST-condition and $\mathfrak{F} :=$*

$\cap_{n=1}^{\infty} F(T_n) \cap EP(\varphi, A) \neq \varnothing$. *Let* f_n *is a family of* κ_n*-contractions of* H *with* $\kappa_n \in [0,1)$ *and* $\limsup_{n\to\infty} \kappa_n < 1$. *Let* $\{r_n\}$ *be a sequence in* $[a,b] \subset (0, 2\alpha)$. *For a given* $x_1 \in C$, *define the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *by*

$$\begin{cases} \varphi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \\ y_n = P_C(\lambda_n x_n + (1-\lambda_n)u_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n y_n + \gamma_n T_n y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0,1)$. *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$;

(iii) $\limsup_{n\to\infty} \lambda_n < 1$.

If there is a contraction f *such that* $f(x) = \lim_{n\to\infty} f_n(x)$ *for each* $x \in \mathfrak{F}$, *then* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in \mathfrak{F}$ *where* $z = P_{\mathfrak{F}} f(z)$.

Proof. Let $z = P_{\mathfrak{F}} x_1 \in \mathfrak{F}$. There is a contraction f such that $f(z) = \lim_{n\to\infty} f_n(z)$. Let $\widehat{w}_n \equiv f_n(z)$ and let $\{\widehat{x}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{u}_n\}$ be sequences of C defined as follows:

$$\begin{cases} \widehat{x}_1 = x_1 \in C, \\ \varphi(\widehat{u}_n, y) + \langle A\widehat{x}_n, y - \widehat{u}_n\rangle + \frac{1}{r_n}\langle y - \widehat{u}_n, \widehat{u}_n - \widehat{x}_n\rangle \geq 0 \quad \text{for all } y \in C, \\ \widehat{y}_n = P_C(\lambda_n \widehat{x}_n + (1-\lambda_n)\widehat{u}_n), \\ \widehat{x}_{n+1} = \alpha_n \widehat{w}_n + \beta_n \widehat{y}_n + \gamma_n T_n \widehat{y}_n \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

Using Theorem 3.1, we get that $\{\widehat{x}_n\}$, $\{\widehat{y}_n\}$ and $\{\widehat{u}_n\}$ converge strongly to $P_{\mathfrak{F}} f(z)$. Since $z \in \mathfrak{F}$, we get $z = P_{\mathfrak{F}} f(z)$. Moreover, we have

$$\|\widehat{u}_n - u_n\| \leq \|\widehat{x}_n - x_n\|, \quad \|\widehat{y}_n - y_n\| \leq \|\widehat{x}_n - x_n\|$$

and

$$\begin{aligned} &\|\widehat{x}_{n+1} - x_{n+1}\| \\ &\leq \alpha_n \|f_n(z) - f_n(x_n)\| + \beta_n \|\widehat{y}_n - y_n\| + \gamma_n \|T_n\widehat{y}_n - T_n y_n\| \\ &\leq \alpha_n\kappa_n \|z - x_n\| + (1-\alpha_n)\|\widehat{y}_n - y_n\| \\ &\leq \alpha_n\kappa_n \|\widehat{x}_n - x_n\| + \alpha_n\kappa_n\|z - \widehat{x}_n\| + (1-\alpha_n)\|\widehat{x}_n - x_n\| \\ &= (1 - \alpha_n(1-\kappa_n))\|\widehat{x}_n - x_n\| + \alpha_n(1-\kappa_n)\frac{\kappa_n\|z - \widehat{x}_n\|}{1-\kappa_n}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_{n\to\infty} \kappa_n < 1$ and $\widehat{x}_n \to z$, we have $\sum_{n=1}^{\infty} \alpha_n(1-\kappa_n) = \infty$ and $\frac{\kappa_n\|z-\widehat{x}_n\|}{1-\kappa_n} \to 0$. By Lemma 2.5, we get $\widehat{x}_n - x_n \to 0$. It follows that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to z as desired. □

If we put $\varphi \equiv 0$, $A \equiv 0$, $r_n \equiv 1$ and use Theorem 4.3, then we get the following result.

Corollary 4.4. *Let C be a closed convex subset of a real Hilbert space H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself satisfying the NST-condition and $\mathfrak{F} := \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $\{f_n\}$ is a family of κ_n-contractions of H with $\kappa_n \in [0,1)$ and $\limsup_{n\to\infty} \kappa_n < 1$. For a given $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = P_C(\alpha_n f_n(x_n) + \beta_n x_n + \gamma_n T_n x_n) \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\liminf_{n\to\infty} \beta_n \gamma_n > 0$.

If there is a contraction f such that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathfrak{F}$, then $\{x_n\}$ converges strongly to $z \in \mathfrak{F}$ where $z = P_{\mathfrak{F}} f(z)$.

Remark 4.5. *If $T_n \equiv T \circ T_{r_n}$ and $f_n \equiv f$, then T_n satisfies the NST-condition (see [10, Theorem 4.10]) and hence Corollary 4.2 extends and improves [23, Theorem 5]. More precisely, the condition $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$ is removed.*

Finally, we show that our results not only include [24, Theorem 3.1] as special cases but also give a simple proof.

Theorem 4.6. *Let C be a closed convex subset of a real Hilbert space H and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself. Let $\{f_n\}$ is a family of κ_n-contractions of C into itself with $\kappa_n \in [0,1)$ and $\limsup_{n\to\infty} \kappa_n < 1$. For a given $x_1 \in C$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) T_n x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$ *or* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Let $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C and T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If there is a contraction f of C into itself such that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in F(T)$, then $\{x_n\}$ converges strongly to $z \in F(T)$ where $z = P_{F(T)} f(z)$.

Proof. Let $z = P_F x_1 \in \mathfrak{F}$. There is a contraction f of C into itself such that $f(z) = \lim_{n\to\infty} f_n(z)$. Define a sequence $\{z_n\}$ by $z_1 = x_1$ and

$$z_{n+1} = \alpha_n f(z) + (1 - \alpha_n) T_n z_n \quad \text{for all } n \in \mathbb{N}.$$

As in [7, Theorem 3.1], we can show that $\{z_n\}$ converges strongly to $z = P_{F(T)}f(z)$. We consider

$$\begin{aligned}
&\|z_{n+1} - x_{n+1}\| \\
&\leq \alpha_n\|f(z) - f_n(x_n)\| + (1-\alpha_n)\|T_n z_n - T_n x_n\| \\
&\leq \alpha_n\|f(z) - f_n(z)\| + \alpha_n\|f_n(z) - f_n(z_n)\| + \alpha_n\|f_n(z_n) - f_n(x_n)\| \\
&\quad + (1-\alpha_n)\|z_n - x_n\| \\
&\leq (1-\alpha_n(1-\kappa_n)\|z_n - x_n\| + \alpha_n(1-\kappa_n)\frac{\|f(z) - f_n(z)\| + \kappa_n\|z_n - z\|}{1-\kappa_n}.
\end{aligned}$$

By Lemma 2.5, we get $z_n - x_n \to 0$. It follows that $\{x_n\}$ converges strongly to z as desired. □

Remark 4.7. *1. In the presence of the stronger family than a family of nonexpansive mappings such as a family of firmly type nonexpansive mappings or a family of strongly nonexpansive mappings, using [11, Theorem 10] in the proof of Theorem 4.6, we can weaken and remove the restriction on the sequence $\{\alpha_n\}$. More precisely, condition (ii) is removed.*

2. *The condition that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C and T is a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ can be replaced by a more general assumption, e.g. for any bounded subset B of C, there exists a nonexpansive mapping T of C into itself such that $\lim_{n\to\infty} \sup\{\|T_n z - T(T_n z)\| : z \in B\} \to 0$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Moreover, we are able to weaken the restriction on $\{\alpha_n\}$ when we use [12, Theorem 3.1] or [13, Theorem 5.1] in the proof of Theorem 4.6,*

3. *By using the same ideas and techniques, we can also discuss the strong convergence in a wider space which is a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable.*

If we put $T_n \equiv T$ and use Theorem 4.6, then we get the following result.

Corollary 4.8. *Let C be a closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself. Let $\{f_n\}$ is a family of κ_n-contractions of C with $\kappa_n \in [0,1)$ and $\limsup_{n\to\infty} \kappa_n < 1$. For a given $x_1 \in C$, define the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ by*

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n)Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$ *or* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

If there is a contraction f such that $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in F(T)$, then $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)}f(z)$.

Proof. Let $T_n \equiv T$. Then $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} = 0 < \infty$ for any bounded subset B of C, $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Using Theorem 4.6, we obtain the desired result. □

Remark 4.9. *Corollary 4.8 extends and improves [24, Thoerem 3.1] in the following ways:*

(i) The conditions $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *are not comparable in general.*

(ii) The restriction that $\{f_n(x)\}$ *is uniformly convergent for each* $x \in D$, *where* D *is any bounded subset of* C *is weakened and replaced by there is a contraction* f *such that* $f(x) = \lim_{n\to\infty} f_n(x)$ *for each* $x \in F(T)$.

(iii) The condition $\liminf_{n\to\infty} \kappa_n > 0$ *is removed.*

(iv) We can give a simple proof of [24, Thoerem 3.1] by using this technique and [26, Theorem 2.3].

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