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Certain Regular Semigroups of Infinite Matrices

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Abstract : Let F be a field and $\mathbb{N}$ the set of natural numbers. It is known that the multiplicative semigroup of all bounded $\mathbb{N} \times \mathbb{N}$ matrices over F is a regular semigroup. Our purpose is to consider the multiplicative semigroup $U^*(F)$ of all column bounded upper triangular $\mathbb{N} \times \mathbb{N}$ matrices A over F with for some $k \in \mathbb{N}$, $A_{ii} \neq 0$ for $i \in \{1, \ldots, k\}$ and $A_{ij} = 0$ for $i > k$ and all $j \in \mathbb{N}$. In this paper, we show that $U^*(F)$ is a regular semigroup which is a disjoint union of right simple regular semigroups, and its idempotents are also determined.

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1 Introduction

An *idempotent* of a semigroup S is an element $a \in S$ with $a^2 = a$. The set of all idempotents of S is denoted by $E(S)$. A semigroup S is called a *regular semigroup* if for every $x \in S, x = xyx$ for some $y \in S$, and S is called an *inverse semigroup* if for every $x \in S$, there is a unique element $x^{-1} \in S$ such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. It is well-known that a semigroup S is inverse if and only if S is regular and any two idempotents commute with each other ([1], page 28). A semigroup S is called *right* [*left*] *simple* if S itself is the only right [left] ideal of S. It follows that S is right [left] simple if and only if $aS = S$ [$Sa = S$] for all $a \in S$ ([1], page 7).

Let $\mathbb{N}$ be the set of natural numbers (positive integers), $n \in \mathbb{N}$, F a field and $M_n(F)$ the multiplicative semigroup of all $n \times n$ matrices over F. It is well-known that $M_n(F)$ is a regular semigroup ([3], page 114) with identity I_n, the identity $n \times n$ matrix over F. Let $U_n(F)$ be the set of upper triangular matrices $A \in M_n(F)$. Then $U_n(F)$ is a subsemigroup of $M_n(F)$ but it is clearly seen that $U_n(F)$ is not regular if $n > 1$. For $i, j \in \{1, \ldots, n\}$, the (i, j)-entry of $A \in M_n(F)$ is denoted by A_{ij}. Let

$$\widetilde{U}(F) = \{A \in U_n(F) \mid A \text{ is invertible in } M_n(F)\}.$$

Then

$$\widetilde{U}(F) = \{A \in U_n(F) \mid A_{ii} \neq 0 \text{ for all } i \in \{1, \dots, n\}\}$$

which is a subgroup of $U_n(F)$ ([2], page 410).

By an $\mathbb{N} \times \mathbb{N}$ *matrix over* F we mean an infinite matrix over F of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ & \vdots & & \end{bmatrix}.$$

For an $\mathbb{N} \times \mathbb{N}$ matrix A over F, its (i, j)-entry is also denoted by A_{ij}. Upper triangular $\mathbb{N} \times \mathbb{N}$ matrices over F are defined naturally. Following [4], an $\mathbb{N} \times \mathbb{N}$ matrix A over F is called *column* [*row*] *bounded* if there is a positive integer N such that $A_{ij} = 0$ if $i > N$ [$j > N$], and A is called *bounded* if A is both column bounded and row bounded. Hence a column [row] bounded $\mathbb{N} \times \mathbb{N}$ matrix over F is an $\mathbb{N} \times \mathbb{N}$ matrix over F with only finitely many nonzero rows [columns]. Let $BM(F)$ be the multiplicative semigroup of all bounded $\mathbb{N} \times \mathbb{N}$ matrices over F. It follows from [4] that $BM(F)$ is also a regular semigroup. For $k \in \mathbb{N}$, let $I(k)$ be an $\mathbb{N} \times \mathbb{N}$ matrix over F defined by

$$I_{ij}(k) = \begin{cases} 1 & \text{if } i = j \in \{1, \dots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $I(k)I(l) = I(k)$ if $k \leq l$. Let $U^*(F)$ be the set of all column bounded upper triangular $\mathbb{N} \times \mathbb{N}$ matrices A over F of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} & A_{1,k+1} & \dots \\ 0 & A_{22} & \dots & A_{2k} & A_{2,k+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{kk} & A_{k,k+1} & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ & & & \vdots & & \end{bmatrix} \quad \text{where } A_{ii} \neq 0 \text{ for } i \in \{1, \dots, k\}. \tag{1.1}$$

Then $I(k) \in U^*(F)$ for all $k \in \mathbb{N}$. It is clearly seen that $U^*(F)$ is a semigroup under matrix multiplication. If $A \in U^*(F)$ and $k \in \mathbb{N}$ are such that $A_{ii} \neq 0$ for all $i \in \{1, \dots, k\}$ and $A_{ij} = 0$ for $i > k$ and $j \in \mathbb{N}$ (see (1)), then

$$I(k)A = A$$

and

$$AI(k) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} & 0 & 0 & \dots \\ 0 & A_{22} & \dots & A_{2k} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{kk} & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ & & & \vdots & & & \end{bmatrix}. \tag{1.2}$$

Notice that $BM(F)$ and $U^*(F)$ are not subsets of each other and both are semigroups without identity.

The purpose of this paper is to prove the following facts.

(i) $U^*(F)$ is a regular semigroup which is a disjoint union of right simple regular semigroups.

(ii) $E(U^*(F))$ consists of all $A \in U^*(F)$ with

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & A_{1,k+1} & \dots \\ 0 & 1 & \dots & 0 & A_{2,k+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & A_{k,k+1} & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ & & & \vdots & & \end{bmatrix} \tag{1.3}$$

where $k \in \mathbb{N}$.

2 Main Results

To prove (i) of our purpose, the following lemma is needed.

Lemma 2.1. *Let $k \in \mathbb{N}$ and let S_k consist of all $A \in U^*(F)$ with*

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} & A_{1,k+1} & \dots \\ 0 & A_{22} & \dots & A_{2k} & A_{2,k+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{kk} & A_{k,k+1} & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ & & & \vdots & & \end{bmatrix} \quad \text{where } A_{ii} \neq 0 \text{ for } i \in \{1, \dots, k\}. \tag{2.1}$$

Then S_k is a right simple regular subsemigroup of $U^*(F)$.

Proof. It is clear that S_k is a subsemigroup of $U^*(F)$. Let $A \in S_k$ and define $\widetilde{A} \in \widetilde{U}(F)$ by $\widetilde{A}_{ij} = A_{ij}$ for all $i, j \in \{1, \dots, k\}$. Since $\widetilde{U}(F)$ is a group with identity I_k, it follows that $\widetilde{A}B = I_k$ for some $B \in \widetilde{U}(F)$. Define $B^* \in S_k$ by

$$B^*_{ij} = \begin{cases} B_{ij} & \text{if } i, j \in \{1, \dots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $AB^* = I(k)$. Consequently,

$$AB^*A = I(k)A = A$$

and

$$S_k \supseteq AS_k \supseteq A(B^*S_k) = (AB^*)S_k = I(k)S_k = S_k,$$

so $AS_k = S_k$.

This proves that S_k is regular and right simple. □

Theorem 2.2. *Th semigroup $U^*(F)$ is a regular semigroup which is a disjoint union of right simple regular semigroups.*

Proof. For each $k \in \mathbb{N}$, define S_k as in Lemma 2.1. By Lemma 2.1, S_k is a regular right simple semigroup. It is clear that

$$U^*(F) = \bigcup_{k \in \mathbb{N}} S_k \quad \text{and} \quad S_k \cap S_l = \emptyset \ \text{ if } \ k \neq l.$$

Hence $U^*(F)$ is a regular semigroup, so the theorem is proved. □

Remark 2.3. It is clearly seen that for every $N \in \mathbb{N}$, $\bigcup_{k=1}^{N} S_k$ is a right ideal of $U^*(F)$. It follows that $U^*(F)$ contains infinitely many right ideals.

Theorem 2.4. *Let $A \in U^*(F)$ be written as in* (2.1), *$A \in E(U^*(F))$ if and only if*

$$A_{ij} = \begin{cases} 1 & \textit{if } \ i = j \in \{1, \ldots, k\}, \\ 0 & \textit{if } \ i, j \in \{1, \ldots, k\} \textit{ with } i \neq j, \end{cases}$$

that is, A is written as in (1.3).

Proof. Assume that $A \in E(U^*(F))$. Then $AA = A$. Define $\widetilde{A} \in \widetilde{U}(F)$ as in the proof of Lemma 2.1 and let $B \in \widetilde{U}(F)$ be such that $\widetilde{A}B = I_k$. Also, define $B^* \in U^*(F)$ as in the proof of Lemma 2.1. Then

$$I(k) = AB^* = AAB^* = AI(k) = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1k} & 0 & 0 & \ldots \\ 0 & A_{22} & \ldots & A_{2k} & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_{kk} & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ & & & \vdots & & & \end{bmatrix}$$

which implies that

$$A_{ij} = \begin{cases} 1 & \text{if } \ i = j \in \{1, \ldots, k\}, \\ 0 & \text{if } \ i, j \in \{1, \ldots, k\} \text{ with } i \neq j. \end{cases}$$

By direct multiplication, the converse holds. □

Corollary 2.5. *The semigroup $U^*(F)$ is not an inverse semigroup.*

Proof. Recall that any two idempotents of an inverse semigroup commute. To prove the corollary, it suffices to show that there are $A, B \in E(U^*(F))$ such that $AB \neq BA$. Let

$$A = I(1) \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ & & \vdots & & \end{bmatrix}.$$

By Theorem 2.4, $A, B \in E(U^*(F))$. Since $AB = I(1)B = B$ and $BA = BI(1) = I(1)$, we have $AB \neq BA$. □

We give a note that the duals of the given results are obtained when we consider the multiplicative semigroup $L^*(F)$ of all row bounded lower triangular $\mathbb{N} \times \mathbb{N}$ matrices A over F with for some $k \in \mathbb{N}$, $A_{jj} \neq 0$ for $j \in \{1, \dots, k\}$ and $A_{ij} = 0$ for $j > k$ and all $i \in \mathbb{N}$.

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