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Fixed Point and Best Proximity Point Results for Generalised Cyclic Coupled Mappings

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Abstract : In this paper, by introducing the new concept called cyclic coupled proximal mappings we explore the existence of strong coupled best proximity point in metric spaces that generalizes the results of [1]. Further, we also proved the existence of strong coupled fixed point for multi-valued cyclic coupled mapping under suitable conditions.

Keywords : cyclic coupled contraction; best proximity point; multivalued mapping; fixed point.
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1 Introduction and Preliminaries

Initially, in 1922 Banach proved the existance and uniqueness of fixed point for contraction mapping. Later, among several interesting results given by various authors; [2] introduced a kind of mapping which has its own significance, as it also admits fixed point on discontinuous maps. Inspire of many authors proving the existence of fixed point on self mappings, it has been proved by [3] that fixed points do exist on a special kind of maps called *cyclic maps*.
Let A and B be two non-empty subsets of metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be cyclic if $T(A) \subset B$ and $T(B) \subset A$.
Meanwhile, another class of mappings called *coupled maps* were introduced by

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[4] to find coupled fixed point which has wide range of applications to partial differential equations and boundary value problems.

Definition 1.1. An element $(x, y) \in X \times X$ in a non-empty set X is said to be *coupled fixed point* for a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

These kind of maps were later generalized by [5] finding out coupled best proximity points for coupled proximal maps with respect to A and B as non-empty closed subsets of metric space (X, d) with $A \cap B = \emptyset$. Very recently [1] extended concept of cyclic maps by introducing cyclic coupled Kannan-type contraction as follows.

Definition 1.2. Let A and B be two non-empty subsets of a metric space (X, d). A mapping $F : X \times X \to X$ is called *cyclic coupled Kannan-type* mapping if F is cyclic with respect to A and B satisfying, for some $k \in (0, \frac{1}{2})$, the inequality

$$d(F(x, y), F(u, v)) \le k[d(x, F(x, y)) + d(u, F(u, v))].$$

where $x, v \in A$ and $y, u \in B$.

Definition 1.3. Let X be a non-empty set. An element $(x, x) \in X \times X$ is said to be *strong coupled fixed point* if $F(x, x) = x$.

The following theorem was proved by [1].

Theorem 1.4. *Let A and B be two non-empty closed subsets of a complete metric space (X, d) with $A \cap B \neq \emptyset$ and $F : X \times X \to X$ be a cyclic coupled Kannan-type mapping with respect to A and B with $A \cap B \neq \emptyset$.Then F has a strong coupled fixed point on $A \cap B$.*

Immediately, [6] extended the result of [1] using Ciric-type contractions.
The existence and convergence of best proximity points is an interesting topic on optimization theory on which several interesting results were published [7–10]. Such results may sometime assume a sequential property on metric spaces called *UC-property.*

Definition 1.5. Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the *UC-property* if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $lim_{n\to\infty} d(z_n, y_n) = d(A, B)$, then $lim_{n\to\infty} d(x_n, z_n) = 0$.

In this paper, we define a new concept called cyclic coupled mappings and prove the existance of proximity point for such mappings which reduces to strong coupled Fixed point on particular case that $A \cap B \neq \emptyset$. We also find strong coupled fixed point for multi-valued cyclic coupled mappings.

2 Best Proximity Points for Cyclic Coupled Mapping

In this part we introduce cyclic coupled proximal maps and proved the existence of proximity points for those maps under suitable conditions.

Definition 2.1. Let A and B be two non-empty subsets of a metric space (X, d) with $A \cap B = \emptyset$. A mapping $F : X \times X \to X$ is called *cyclic coupled proximal mapping of type I* if F is cyclic with respect to A and B satisfying the inequality

$$d(F(x, y), F(u, v)) \leq k \max[d(x, F(x, y)), d(u, F(u, v))] + (1 - k)d(A, B)$$

where $x, v \in A$ and $y, u \in B$ for some $k \in (0, 1)$.

Definition 2.2. Let A and B be two non-empty subsets of a metric space (X, d) with $A \cap B = \emptyset$. A mapping $F : X \times X \to X$ is called *cyclic coupled proximal mapping of type II* if F is cyclic with respect to A and B satisfying the inequality

$$d(F(x, y), F(u, v)) \leq k[d(x, F(x, y)) + d(u, F(u, v))] + (1 - 2k)d(A, B)$$

where $x, v \in A$ and $y, u \in B$ for some $k \in (0, \frac{1}{2})$.

Definition 2.3. Let (X, d) be a metric space. An element $(x, y) \in X \times X$ is said to be *strong coupled proximal point* if $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$ with $d(x, y) = d(A, B)$.

Theorem 2.4. *Let (X, d) be a complete metric space and A, B are two non-empty closed subsets of X such that $A \cap B = \emptyset$. Let $F : X \times X \to X$ be cyclic coupled proximal mapping of type I. Then F has strong coupled proximal point if (A, B) satisfies UC-property.*

Proof. Let $x_0 \in A, y_0 \in B$ be any two arbitrary elements of X. Let $\{x_n\}$ and $\{y_n\}$ are two sequences defined as $F(x_n, y_n) = y_{n+1}$ and $F(y_n, x_n) = x_{n+1}$. Then, for n=1,

$$\begin{aligned} d(x_1, y_2) &= d(F(y_0, x_0), F(x_1, y_1)) \\ &\leq k \max[d(y_0, F(y_0, x_0)), d(x_1, F(x_1, y_1))] + (1 - k)d(A, B) \\ &\leq k \max[d(y_0, x_1)), d(x_1, y_2)] + (1 - k)d(A, B) \end{aligned}$$

and

$$\begin{aligned} d(y_1, x_2) &= d(F(x_0, y_0), F(y_1, x_1)) \\ &\leq k \max[d(x_0, F(x_0, y_0)), d(y_1, F(y_1, x_1))] + (1 - k)d(A, B) \\ &\leq k \max[d(x_0, y_1)), d(y_1, x_2)] + (1 - k)d(A, B). \end{aligned}$$

Similarly,for $n = 2$,

$$\begin{aligned} d(x_2, y_3) &\leq k \max[d(y_1, x_2)), d(x_2, y_3)] + (1 - k)d(A, B), \\ d(y_2, x_3) &\leq k \max[d(x_1, y_2)), d(y_2, x_3)] + (1 - k)d(A, B). \end{aligned}$$

In general,

$$d(x_n, y_{n+1}) \leq k \max[d(y_{n-1}, x_n)), d(x_n, y_{n+1})] + (1-k)d(A, B), \tag{2.1}$$

$$d(y_n, x_{n+1}) \leq k \max[d(x_{n-1}, y_n)), d(y_n, x_{n+1})] + (1-k)d(A, B). \tag{2.2}$$

If $d(x_n, y_{n+1}) \leq kd(x_n, y_{n+1}) + (1-k)d(A, B)$, the inequality reduces to $d(x_n, y_{n+1}) \leq d(A, B)$ which gives $d(x_n, y_{n+1}) = d(A, B)$. Knowing the fact that, $d(y_{n-1}, x_n) \geq d(A, B)$ equation (2.1) reduces to

$$d(x_n, y_{n+1}) \leq kd(y_{n-1}, x_n) + (1-k)d(A, B), \forall n \in \mathbb{N}.$$

Similarly, equation (2.2) reduces as

$$d(y_n, x_{n+1}) \leq kd(x_{n-1}, y_n) + (1-k)d(A, B), \forall n \in \mathbb{N}.$$

Thus we conclude,

$$\begin{aligned}
d(x_m, y_{m+1}) &\leq kd(y_{m-1}, x_m) + (1-k)d(A, B) \\
&\leq k[kd(x_{m-2}, y_{m-1}) + (1-k)d(A, B)] + (1-k)d(A, B) \\
&\leq k^2 d(x_{m-2}, y_{m-1}) + k(1-k)d(A, B) + (1-k)d(A, B) \\
&\leq k^2 d(x_{m-2}, y_{m-1}) + \left[\sum_{i=0}^{1} k^i\right](1-k)d(A, B)] \\
&\vdots \\
&\leq \begin{cases} k^m d(x_0, y_1) + \left[\sum_{i=0}^{m-1} k^i\right](1-k)d(A, B)] & \text{if } m \text{ is even,} \\ k^m d(y_0, x_1) + \left[\sum_{i=0}^{m-1} k^i\right](1-k)d(A, B)] & \text{if } m \text{ is odd.} \end{cases}
\end{aligned}$$

Since $\left[\sum_{i=0}^{\infty} k^i\right] = \frac{1}{1-k}$ for all $k \in (0, 1)$, the above inequality become $d(x_m, y_{m+1}) \to d(A, B)$ as letting $m \to \infty$ and similarly it can be shown that $d(y_m, x_{m+1}) \to d(A, B)$ as $m \to \infty$. The results implies that $\lim_{m\to\infty} d(x_m, y_m) = d(A, B)$.
Claim: $\{x_n\}$ is a Cauchy sequence.
Let $m < n$

$$\begin{aligned}
d(x_n, y_m) &= d(F(y_{n-1}, x_{n-1}), F(x_{m-1}, y_{m-1})) \\
&\leq k \max[d(y_{n-1}, x_n), d(x_{m-1}, y_m)] + (1-k)d(A, B).
\end{aligned}$$

Therefore, $\lim_{m\to\infty} d(x_n, y_m) = d(A, B)$. Therefore by using UC-property we get, $\lim_{n\to\infty} d(x_n, x_m) = 0$(*i.e.*, for given $\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall m > n > n_0$, $d(x_n, x_m) < \epsilon$). Hence, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. Similarly, it can be proved that $\{y_n\}$ is a Cauchy sequence and converges to some point $y \in B$. Since, $\lim_{m\to\infty}(x_m, y_m) = d(A, B)$ and d is uniformly continuous, we get $d(x, y) = d(A, B)$.
Now,

$$\begin{aligned}
d(x, F(x, y)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) \\
&= d(x, x_{n+1}) + d(F(y_n, x_n), F(x, y)) \\
&\leq d(x, x_{n+1}) + k \max[d(y_n, x_{n+1}), d(x, F(x, y))] + (1-k)d(A, B),
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, as $n \to \infty$, $d(x, F(x,y)) \leq d(A,B)$. *i.e.,* $d(x, F(x,y)) = d(A,B)$. Similarly, we can prove that $d(y, F(y,x)) = d(A,B)$ which concludes (x,y) is the strong coupled proximal point of F. □

Example 2.5. Consider $A = \{(0,a) \mid a \in [0,1]\}$ and $B = \{(1,b) \mid b \in [-1,0]\}$ on $\mathbb{R}^2$ under $1-$norm with $d(A,B) = 1$. Also the sets satisfies UC-property.
Define

$$F(a',b') = \begin{cases} (1, \frac{ab}{4+a}) \text{ if } (a',b') \in A \times B, \text{where } a' = (0,a) \text{and } b' = (1,b); \\ (0, \frac{-ab}{4-a}) \text{ if } (a',b') \in B \times A, \text{where } a' = (1,a) \text{and } b' = (0,b). \end{cases}$$

Let $x' = (0,x)$, $v' = (0,v)$ be any two elements of A and $y' = (1,y)$, $u' = (1,u)$ be any two elements of B.
Now,

$$\begin{aligned} d(F(x',y'), F(u',v')) &= d((1, \frac{xy}{4+x}), (0, \frac{-uv}{4-u}) \\ &= \mid 1 \mid + \mid \frac{xy}{4+x} + \frac{uv}{4-u} \mid \\ &\leq \mid 1 \mid + \mid \frac{xy}{4+x} \mid + \mid \frac{uv}{4-u} \mid \\ &\leq \mid 1 \mid + \frac{1}{3} [\mid x \mid + \mid -u \mid] \\ &\leq \mid 1 \mid + \frac{1}{3} \left[\mid x - \frac{xy}{4+x} \mid + \mid -u - \frac{uv}{4-u} \mid \right] \\ &= \mid 1 \mid -2 \times \frac{1}{3} + \frac{1}{3} [d(x', F(x',y')) + d(u', F(u',v'))] \\ &= (1 - \frac{2}{3}) d(A,B) + \frac{1}{3} [d(x', F(x',y')) + d(u', F(u',v'))] \\ &= (1 - \frac{2}{3}) d(A,B) + \frac{2}{3} [\frac{d(x', F(x',y')) + d(u', F(u',v'))}{2}] \\ &\leq (1 - \frac{2}{3}) d(A,B) + \frac{2}{3} \max[d(x', F(x',y')), d(u', F(u',v'))]. \end{aligned}$$

Thus, F satisfies all conditions of theorem(2.4). The strong coupled proximal points of F on A and B are $(0,0)$ and $(1,0)$.

Remarks 2.1. *If $d(A,B) = 0$ in the above theorem, it reduces to fixed point result which holds even if UC-property is neglected. In that case it is an generalization to result given by [1].*

Theorem 2.6. *Let (X,d) be a complete metric space and A, B be two non-empty closed subsets of X such that $A \cap B = \emptyset$. Let $F : X \times X \to X$ be cyclic coupled proximal mapping of type II. Then F has strong coupled proximal point if it satisfies,*

$$d(u,v) + d(A,B) \leq d(u, F(a,b))$$

whenever $v = F(b, a)$ *and* $\{u, v\}$ *belongs to set* A *or* B.

Proof. Let $x_0 \in A, y_0 \in B$ be any two arbitrary elements of X. Define $F(x_n, y_n) = y_{n+1}$ and $F(y_n, x_n) = x_{n+1}$.
Then, for $n = 1$,

$$\begin{aligned} d(x_1, y_2) &= d(F(y_0, x_0), F(x_1, y_1)) \\ &\leq k[d(y_0, F(y_0, x_0)) + d(x_1, F(x_1, y_1))] + (1 - 2k)d(A, B) \\ &\leq k[d(y_0, x_1) + d(x_1, y_2)] + (1 - 2k)d(A, B). \end{aligned}$$

Hence $d(x_1, y_2) \leq \left(\frac{k}{1-k}\right) d(y_0, x_1) + \left(\frac{1-2k}{1-k}\right) d(A, B)$.
Similarly, $d(y_1, x_2) \leq \left(\frac{k}{1-k}\right) d(x_0, y_1) + \left(\frac{1-2k}{1-k}\right) d(A, B)$.
For $n = 2$,

$$\begin{aligned} d(x_2, y_3) &= d(F(y_1, x_1), F(x_2, y_2)) \\ &\leq k[d(y_1, F(y_1, x_1)) + d(x_1, F(x_1, y_1)] + (1 - 2k)d(A, B) \\ &\leq k[d(y_1, x_2) + d(x_2, y_3)] + (1 - 2k)d(A, B) \end{aligned}$$

and hence,

$$\begin{aligned} d(x_2, y_3) &\leq \left(\frac{k}{1-k}\right) d(y_1, x_2) + \left(\frac{1-2k}{1-k}\right) d(A, B) \\ &\leq \left(\frac{k}{1-k}\right)^2 d(x_0, y_1) + \left(\frac{1-2k}{1-k}\right) \left(1 + \frac{k}{1-k}\right) d(A, B) \\ &\leq \left(\frac{k}{1-k}\right)^2 d(x_0, y_1) + \left(\frac{1-2k}{1-k}\right) \left[\sum_{i=0}^{1} \left(\frac{k}{1-k}\right)^i\right] d(A, B). \end{aligned}$$

Similarly,

$$d(y_2, x_3) \leq \left(\frac{k}{1-k}\right)^2 d(y_0, x_1) + \left(\frac{1-2k}{1-k}\right) \left[\sum_{i=0}^{1} \left(\frac{k}{1-k}\right)^i\right] d(A, B).$$

For $n = 3$ it is easy to obtained that,

$$d(x_3, y_4) \leq \left(\frac{k}{1-k}\right)^3 d(y_0, x_1) + \left(\frac{1-2k}{1-k}\right) \left[\sum_{i=0}^{2} \left(\frac{k}{1-k}\right)^i\right] d(A, B).$$

$$d(y_3, x_4) \leq \left(\frac{k}{1-k}\right)^3 d(x_0, y_1) + \left(\frac{1-2k}{1-k}\right) \left[\sum_{i=0}^{2} \left(\frac{k}{1-k}\right)^i\right] d(A, B).$$

In general it can be concluded that,

$$d(x_n, y_{n+1}) \leq \begin{cases} \left(\frac{k}{1-k}\right)^n d(y_0, x_1) + \left(\frac{1-2k}{1-k}\right) \left[\sum_{i=0}^{n-1} \left(\frac{k}{1-k}\right)^i\right] d(A, B) & \text{if } n \text{ is odd;} \\ \left(\frac{k}{1-k}\right)^n d(x_0, y_1) + \left(\frac{1-2k}{1-k}\right) \left[\sum_{i=0}^{n-1} \left(\frac{k}{1-k}\right)^i\right] d(A, B) & \text{if } n \text{ is even.} \end{cases}$$

In the above inequality, as $\lim_{n\to\infty} d(x_n, y_{n+1}) = d(A, B)$ and Similarly, as $\lim_{n\to\infty} d(y_n, x_{n+1}) = d(A, B)$.
Claim: $\{x_n\}$ is a Cauchy sequence.
Let $m > n$.

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, F(x_{m-1}, y_{m-1})) - d(A, B) \\ &\le d(F(y_{n-1}, x_{n-1}, F(x_{m-1}, y_{m-1})) - d(A, B) \\ &\le k[d(y_{n-1}, x_n) + d(x_{m-1}, y_m)] + (1-2k)d(A, B) - d(A, B) \\ &\le k[d(y_{n-1}, x_n) + d(x_{m-1}, y_m)] + (-2k)d(A, B). \end{aligned}$$

Hence, as $\lim_{n\to\infty} d(x_n, x_m) = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence and hence converges to some point $x \in A$. Similarly, $\{y_n\}$ is a Cauchy sequence and hence converges to some point $y \in B$. Therefore, $d(x, y) = d(A, B)$.
Now,

$$\begin{aligned} d(x, F(x, y)) &\le d(x, F(y_n, x_n)) + d(F(y_n, x_n), F(x, y)) \\ &\le d(x, x_{n+1}) + k[d(y_n, F(y_n, x_n) + d(x, F(x, y))] + (1-2k)d(A, B) \end{aligned}$$

and hence,

$$(1-k)d(x, F(x, y)) \le d(x_n, x_{n+1}) + k\, d(y_n, x_{n+1}) + (1-2k)d(A, B).$$

Now letting $n \to \infty$, the above inequality reduces to

$$\begin{aligned} d(x, F(x, y)) &\le \left(\frac{k}{1-k}\right) d(A, B) + \left(\frac{1-2k}{1-k}\right) d(A, B) \\ &\le d(A, B). \end{aligned}$$

Therefore, $d(x, F(x, y)) = d(A, B)$ and similarly, $d(y, F(y, x)) = d(A, B)$ with $d(x, y) = d(A, B)$ concludes that (x, y) is the strong coupled proximal point of F. □

Example 2.7. Consider $A = \{(0, a) \mid a \in [-1, 0]\}$ and $B = \{(1, b) \mid b \in [-1, 0]\}$ on $\mathbb{R}^2$ under $1-$norm with $d(A, B) = 1$.
Define

$$F(a', b') = \begin{cases} (1, \frac{ab}{4}) \text{ if } (a', b') \in A \times B, \text{where } a' = (0, a) \text{and } b' = (1, b); \\ (0, \frac{ab}{4}) \text{ if } (a', b') \in B \times A, \text{where } a' = (1, a) \text{and } b' = (0, b). \end{cases}$$

It is easy to verify that F satisfies all conditions of Theorem 2.6. The strong coupled proximal points of F on A and B are $(0, 0)$ and $(1, 0)$.

3 Strong Coupled Fixed Point for Cyclic Coupled Multi-Valued Mapping

In this part we introduce the concept of multi-valuedness on cyclic coupled mapping and established strong coupled fixed point for cyclic coupled multi-valued contraction maps. Let (X,d) be a metric space and $CB(M)$ denote collection of all non-empty closed and bounded subsets of X with metric defined for $A, B \in CB(M)$ as

$$H(A,B) = \max\{\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\},$$

where $d(x,A) = \inf_{y\in A} d(x,y)$. Such a map H is called *Hausdorff metric induced by* d.

Definition 3.1. Let A and B be two nonempty subsets of complete metric space (X,d). A mapping $F : X \times X \to CB(X)$ is said to be *cyclic coupled ϕ-multivalued contraction* mapping with respect to A and B if

$$H(F(x,y), F(u,v)) \le \phi[d(x,u)]d(x,u), \text{ for all } x,v \in A \text{ and } y,u \in B,$$

where $\phi : [0,\infty) \to [0,1)$ is a increasing function with $0 = \phi(0) < \phi(r) \le r$ for all $r \in (0,\infty)$.

Theorem 3.2. *Let (X,d) be a complete metric space and A and B be two non-empty closed subsets of X such that $A \cap B \neq \emptyset$. Suppose $F : X \times X \to CB(X)$ be cyclic coupled ϕ-multivalued contraction mapping with respect to A and B such that $F(x,y)$ is closed and bounded on $A \times B$ and $B \times A$, then F has strong coupled fixed point in $A \cap B$.*

Proof. Let $x_0 \in A, y_0 \in B$ be any two arbitrary elements of X. Choose $x_1 \in F(y_0,x_0), y_1 \in F(x_0,y_0)$ and $s > 1$ such that $s\phi[d(y_0,x_0)] < 1$. Then

$$\begin{aligned} d(x_1,y_1) &\le sH(F(y_0,x_0), F(x_0,y_0)) \\ &\le s\phi[d(y_0,x_0)]\, d(y_0,x_0) \\ &< d(y_0,x_0), \text{since } s\phi[d(y_0,x_0)] < 1. \end{aligned}$$

Hence, $\phi[d(x_1,y_1)] < \phi[d(y_0,x_0)]$.

Similarly, Choose $t > 1$ such that $t\phi[d(y_0,x_1)] < 1$ and $t\phi[d(x_0,y_1)] < 1$. Then, from definition, we can find $x_2 \in F(y_1,x_1)$ and $y_2 \in F(x_1,y_1)$ such that

$$\begin{aligned} d(x_2,y_2) &\le sH(F(y_1,x_1), F(x_1,y_1)) \\ &\le s\phi[d(y_1,x_1)]\, d(y_1,x_1) \\ &< s\phi[d(y_0,x_0)]\, d(y_1,x_1) \\ &< d(y_1,x_1), \text{since } s\phi[d(y_0,x_0)] < 1. \end{aligned}$$

Hence, $\phi[d(x_2,y_2)] < \phi[d(y_1,x_1)]$, with

$$\begin{aligned} d(x_1,y_2) &\le tH(F(y_0,x_0), F(x_1,y_1)) \\ &\le t\phi[d(y_0,x_1)]\, d(y_0,x_1) \\ &< d(y_0,x_1), \text{since } t\phi[d(y_0,x_1)] < 1. \end{aligned}$$

Therefore, $\phi[d(x_1, y_2)] < \phi[d(y_0, x_1)]$. Also, we get

$$\begin{aligned} d(y_1, x_2) &\leq tH(F(x_0, y_0), F(y_1, x_1)) \\ &\leq t\phi[d(x_0, y_1)]\, d(x_0, y_1) \\ &< d(x_0, y_1), \text{since } t\phi[d(y_0, x_1)] < 1. \end{aligned}$$

Therefore, $\phi[d(y_1, x_2)] < \phi[d(x_0, y_1)]$.
Now, choose $x_3 \in F(y_2, x_2)$ and $y_3 \in F(x_2, y_2)$ which gives

$$\begin{aligned} d(x_3, y_3) &\leq sH(F(y_2, x_2), F(x_2, y_2)) \\ &\leq s\phi[d(y_2, x_2)]\, d(y_2, x_2) \\ &< s\phi[d(x_1, y_1)]\, d(y_2, x_2) \\ &< s\phi[d(x_0, y_0)]\, d(y_2, x_2) \\ &< d(y_2, x_2), \text{since } s\phi[d(x_0, y_0)] < 1. \end{aligned}$$

Therefore, $\phi[d(x_3, y_3)] < \phi[d(y_3, x_2)]$ with

$$\begin{aligned} d(x_2, y_3) &\leq tH(F(y_1, x_1), F(x_2, y_2)) \\ &\leq t\phi[d(y_1, x_2)]\, d(y_1, x_2) \\ &< t\phi[d(x_0, y_1)]\, d(y_1, x_2) \\ &< d(y_1, x_2), \text{since } t\phi[d(x_0, y_1)] < 1. \end{aligned}$$

Therefore, $\phi[d(x_2, y_3)] < \phi[d(y_1, x_2)]$ and also from

$$\begin{aligned} d(x_2, y_3) &< t\phi[d(x_0, y_1)]\, d(y_1, x_2) \\ &< [t\phi[d(x_0, y_1)]]^2\, d(x_0, y_1). \end{aligned}$$

Also, we get

$$\begin{aligned} d(y_2, x_3) &\leq tH(F(x_1, y_1), F(y_2, x_2)) \\ &\leq t\phi[d(x_1, y_2)]\, d(x_1, y_2) \\ &< t\phi[d(y_0, x_1)]\, d(x_1, y_2) \\ &< d(x_1, y_2). \end{aligned}$$

Therefore, $\phi[d(y_2, x_3)] < \phi[d(x_1, y_2)]$ and also from

$$\begin{aligned} d(y_2, x_3) &< t\phi[d(y_0, x_1)]\, d(x_1, y_2) \\ &< [t\phi[d(y_0, x_1)]]^2\, d(y_0, x_1). \end{aligned}$$

Let us assume that $x_n \in F(y_{n-1}, x_{n-1})$ and $y_n \in F(x_{n-1}, y_{n-1})$, with

$$\begin{aligned} d(x_n, y_n) &< s\phi[d(x_0, y_0)]\, d(y_{n-1}, x_{n-1}) \\ &< d(y_{n-1}, x_{n-1}) \end{aligned}$$

which gives $\phi[d(x_n, y_n)] < \phi[d(y_{n-1}, x_{n-1})]$ and also assume

$$d(x_{n-1}, y_n) < \begin{cases} [t\phi[d(x_0, y_1)]]^{n-1}\, d(x_0, y_1), \text{if } n \text{ is odd;} \\ [t\phi[d(y_0, x_1)]]^{n-1}\, d(y_0, x_1), \text{if } n \text{ is even.} \end{cases}$$

with $\phi[d(x_{n-1}, y_n)] < \phi[d(y_{n-2}, x_{n-1})]$ and

$$d(y_{n-1}, x_n) < \begin{cases} [t\phi[d(y_0, x_1)]]^{n-1}\, d(y_0, x_1), \text{if } n \text{ is odd;} \\ [t\phi[d(x_0, y_1)]]^{n-1}\, d(x_0, y_1), \text{if } n \text{ is even.} \end{cases}$$

with $\phi[d(y_{n-1}, x_n)] < \phi[d(x_{n-1}, y_n)]$.
Now, for $x_{n+1} \in F(y_n, x_n)$ and $y_{n+1} \in F(x_n, y_n)$, we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq sH(F(y_n, x_n), F(x_n, y_n)) \\ &\leq s\phi[d(y_n, x_n)]\, d(y_n, x_n) \\ &< s\phi[d(x_0, y_0)]\, d(x_n, y_n) \\ &< [s\phi[d(x_0, y_0)]]^2\, d(x_{n-1}, y_{n-1}) \\ &\vdots \\ &< [s\phi[d(x_0, y_0)]]^n d(x_0, y_0). \end{aligned}$$

$$\begin{aligned} d(x_n, y_{n+1}) &\leq tH(F(y_{n-1}, x_{n-1}), F(x_n, y_n)) \\ &\leq t\phi[d(y_{n-1}, x_n)]\, d(y_{n-1}, x_n) \\ &< \begin{cases} [t\phi[d(y_0, x_1)]]^n d(y_0, x_1), \text{if } n \text{ is odd;} \\ [t\phi[d(x_0, y_1)]]^n d(x_0, y_1), \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Similarly, we obtain that

$$d(y_n, x_{n+1}) < \begin{cases} [t\phi[d(x_0, y_1)]]^n d(x_0, y_1), \text{if } n \text{ is odd;} \\ [t\phi[d(y_0, x_1)]]^n d(y_0, x_1), \text{if } n \text{ is even.} \end{cases}$$

$$\begin{aligned} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\leq d(x_n, y_n) + d(y_n, x_{n+1}) + d(y_n, x_n) + d(x_n, y_{n+1}) \\ &\leq d(x_n, y_n) + d(y_n, x_{n+1}) + d(x_n, y_n) + d(x_n, y_{n+1}) \\ &\leq 2d(x_n, y_n) + d(y_n, x_{n+1}) + d(x_n, y_{n+1}) \\ &< 2[s\phi[d(x_0, y_0)]]^n d(x_0, y_0) + [t\phi[d(y_0, x_1)]]^n [d(y_0, x_1) \\ &\quad + d(x_0, y_1)]. \end{aligned}$$

Since, $s\phi[d(x_0, y_0)] < 1$ and $t\phi[d(y_0, x_1)] < 1$ we get $\sum_{i=0}^{n} d(x_i, x_{i+1}) + d(y_i, y_{i+1}) < \infty$. Hence, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $A \cup B$. Let, $\{x_n\} \to x$ and $\{y_n\} \to y$. Also, $d(\{x_n\}, \{y_n\}) = 0$ as $n \to \infty$, which implies that $d(x, y) = 0$ i.e., $x = y$.

$$\begin{aligned} d(x, F(x, x)) &= d(x, F(x, y)) \\ &\leq d(x, x_n) + d(x_n, F(x, y)) \\ &\leq d(x, x_n) + d(F(y_{n-1}, x_{n-1}), F(x, y)) \\ &\leq d(x, x_n) + kd(y_{n-1}, x), \end{aligned}$$

for all $n \in \mathbb{N}$. Hence (x,x) is the strong coupled fixed point of F, since as $n \to \infty$, we get $d(x, F(x,x)) = 0$. Since, $x = y$, the equality becomes $d(x, F(x,x)) = 0$ and hence (x,x) is the strong coupled fixed point of F. □

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