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New Approximation Schemes for Two Asymptotically Perturbed Nonexpansive Nonself Mappings[1]

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Abstract : In this paper, we introduce and study a new type of two-step iterative scheme for two asymptotically perturbed nonexpansive nonself-mappings in a uniformly convex Banach space. Weak and strong convergence theorems are established for the new two-step iterative scheme in a uniformly convex Banach space. The results obtained in this paper generalize and refine some known results is the current literature.

Keywords : asymptotically perturbed nonexpansive nonself-mapping; Kadec-Klee property; completely continuous; Opial's condition; common fixed points.
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1 Introduction

Let C be a nonempty closed convex subset of a real normed linear space X. A self-mapping $T : C \to C$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. A self-mapping $T : C \to C$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\| \tag{1.1}$$

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for all $x, y \in C$ and $n \geq 1$. A mapping $T : C \to C$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly $L-$Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Fixed-point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes have been studied extensively by various authors [1–6] For nonexpansive nonself-mappings, some authors [7–12] have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space. In 1972, Goebel and Kirk [13] introduced the class of asymptotically nonexpansive self-mappings, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

In 1991, Schu [14] introduced the following modified Mann iteration process

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.3}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space. Since then, Schu's iteration process (1.3) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces [4, 14–17].

In all the above results, the operator T, remains a self-mapping of a nonempty closed convex subset C of X. If, however, the domain of T, $D(T)$, is a proper subset of X (and this is the case in several applications), and T maps $D(T)$ into X, then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

The concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume, Ofoedu and Zegeye [18] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows:

Definition 1.1 ([18]). Let C be a nonempty subset of a real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$.

By studying the following iteration process: $x_1 \in C$,

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \tag{1.6}$$

Chidume, Ofoedu and Zegeye [18] gave some strong and weak convergence theorems for asymptotically nonexpansive nonself-mapping in a uniformly convex Banach space.

If T is a self-mapping, then P becomes the identity mapping so that (1.4) and (1.5) reduce to (1.1) and (1.2), respectively. (1.6) reduces to (1.3).

In 2006, Wang [19] generalized the iteration process (1.6) as follows: $x_1 \in C$,

$$\begin{cases} x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.7}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$. He studied the strong and weak convergence of the iterative scheme (1.7) under proper conditions. Meanwhile, the results of [19] generalized the results of [18].

The asymptotically perturbed nonexpansive and uniformly perturbed $L-$Lipschitzian (nonself) mappings are defined as follows:

Definition 1.2 ([20])**.** Let C be a nonempty subset of a real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called *asymptotically perturbed $P-$nonexpansive* if there exists a sequence $\{k_n\} \subset [1-\epsilon, \infty)$, $k_n \to 1-\epsilon$ as $n \to \infty$ for some $\epsilon > 0$ such that

$$\|T(PT)^{n-1}x - Tx\| \quad \leq \quad k_n\|x - Tx\|$$

for all $x \in C$ and $n \geq 1$. T is said to be *uniformly perturbed $L-$Lipschitzian with respect to retraction* P if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - Tx\| \quad \leq \quad L\|x - Tx\|$$

for all $x \in C$ and $n \geq 1$. A nonself asymptotically perturbed $P-$nonexpansive mapping $T : C \to X$ is said to satisfy *ball condition* if there exists $r > 0$ such that

$$\|x - Tx\| + \|y - Ty\| \quad \leq \quad (\sup_{i\geq 1} k_i)^{-1}\|x - y\|$$

for all $x, y \in C \cap \bar{B}_r(0)$ with $x \neq y$, where $\bar{B}_r(0)$ is the closed ball in X centre 0 and radius r.

Now, since $T : C \to X$ is an asymptotically perturbed $P-$nonexpansive satisfying the ball condition, so is $T : C \cap \bar{B}_r(0) \to X$. Using Definition 1.2, we have

$$\begin{aligned}\|T(PT)^{n-1}x-T(PT)^{n-1}y\| &\le \|T(PT)^{n-1}x-Tx\|+\|Tx-Ty\| \\ &\quad +\|Ty-T(PT)^{n-1}y\| \\ &\le k_n\|x-Tx\|+\|Tx-Ty\|+k_n\|y-Ty\| \\ &= k_n\|x-Tx\|+k_n\|y-Ty\| \\ &\quad +\|Tx-x+x-y+y-Ty\| \\ &\le k_n\|x-Tx\|+k_n\|y-Ty\|+\|x-Tx\| \\ &\quad +\|y-Ty\|+\|x-y\| \\ &= (1+k_n)\|x-Tx\|+(1+k_n)\|y-Ty\|+\|x-y\| \\ &= (1+k_n)(\|x-Tx\|+\|y-Ty\|)+\|x-y\| \\ &\le (1+k_n)(\sup_{i\ge 1} k_i)^{-1}\|x-y\|+\|x-y\| \\ &= K_n\|x-y\|\end{aligned}$$

for all $x, y \in C \cap \bar{B}_r(0)$ with $x \neq y$, where $K_n = (1+k_n)(\sup_{i\ge1} k_i)^{-1} + 1$.

Therefore, for all $n \geq 1$, $T(PT)^{n-1}$ is Lipschitzian mapping with the Lipschitzian constant $K_n \geq 1$.

Pathak, Cho and Kang [20] generalized the iteration process (1.7) as follows: $x_1 \in C_1 \cap C_2$,

$$\begin{cases} x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_2(QT_2)^{n-1}y_n) \in C_1, \\ y_n = (1-\beta_n)x_n + \beta_n T_1(PT_1)^{n-1}x_n \in C_2, \quad n \geq 1, \end{cases} \tag{1.8}$$

where $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ are two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. In [20], they got the following strong and weak convergence theorems.

Theorem 1.3 ([20])**.** *Let C_1 and C_2 be two nonempty closed convex subsets of a uniformly convex Banach space X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two nonself asymptotically perturbed P-nonexpansive and Q-nonexpansive mappings satisfying ball condition and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$ with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty} k_n' < \infty$, $\sum_{n=1}^{\infty} l_n' < \infty$, where $k_n' = (1+k_n)(\sup_{i\ge1} k_i)^{-1}$ and $l_n' = (1+l_n)(\sup_{i\ge1} l_i)^{-1}$. Let $\{x_n\} \subset C_1$ and $\{y_n\} \subset C_2$ be defined by (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[\epsilon, 1-\epsilon)$. If one of T_1 and T_2 is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ and $\{y_n\}$ both converge strongly to a common fixed point of T_1 and T_2.*

Theorem 1.4 ([20])**.** *Let C_1 and C_2 be two nonempty closed convex subsets of a uniformly convex Banach space X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two nonself asymptotically perturbed P-nonexpansive and Q-nonexpansive mappings satisfying ball condition and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$ with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ for some $\epsilon > 0$ and $\sum_{n=1}^{\infty} k_n' <$*

∞, $\sum_{n=1}^{\infty} l_n^{'} < \infty$, *where* $k_n^{'} = (1+k_n)(\sup_{i\geq 1} k_i)^{-1}$ *and* $l_n^{'} = (1+l_n)(\sup_{i\geq 1} l_i)^{-1}$. *Let* $\{x_n\} \subset C_1$ *and* $\{y_n\} \subset C_2$ *be defined by (1.8), where* $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequences in* $[\epsilon, 1-\epsilon)$. *If one of* T_1 *and* T_2 *is demicompact and* $F(T_1) \cap F(T_2) \neq \emptyset$, *then* $\{x_n\}$ *and* $\{y_n\}$ *both converge strongly to a common fixed point of* T_1 *and* T_2.

Theorem 1.5 ([20]). *Let* C_1 *and* C_2 *be two nonempty closed convex subsets of a uniformly convex Banach space* X *satisfying Opial's condition. Let* $T_1 : C_1 \to X$ *and* $T_2 : C_2 \to X$ *be two nonself asymptotically perturbed* P*-nonexpansive and* Q*-nonexpansive mappings satisfying ball condition and* $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ *for each* $\lambda \in [\epsilon, 1-\epsilon)$ *for some* $\epsilon > 0$ *with sequences* $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ *for some* $\epsilon > 0$ *and* $\sum_{n=1}^{\infty} k_n^{'} < \infty$, $\sum_{n=1}^{\infty} l_n^{'} < \infty$, *where* $k_n^{'} = (1+k_n)(\sup_{i\geq 1} k_i)^{-1}$ *and* $l_n^{'} = (1+l_n)(\sup_{i\geq 1} l_i)^{-1}$. *Let* $\{x_n\} \subset C_1$ *and* $\{y_n\} \subset C_2$ *be defined by (1.8), where* $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequences in* $[\epsilon, 1-\epsilon)$. *If* $F(T_1) \cap F(T_2) \neq \emptyset$, *then* $\{x_n\}$ *and* $\{y_n\}$ *both converge weakly to a common fixed point of* T_1 *and* T_2.

If $T_1 = T_2$, $P = Q$ and $\beta_n = 0$ for all $n \geq 1$, then the iteration scheme (1.8) reduces to (1.6).

Recently, an iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings was defined and constructed by Thianwan [21]. It is given as follows:

$$\begin{cases} x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.9}$$

where $T_1, T_2 : C \to X$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$. He studied the strong and weak concergence of the iterative scheme (1.9) under proper conditions in a uniformly convex Banach space.

The iterative schemes (1.9) and (1.7) are independent: neither reduces to the other. If $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then (1.9) reduces to (1.6). It also can be reduces to Schu process (1.3).

Inspired and motivated by these facts, a new type of two-step iterative scheme is introduced and studied in this paper. The scheme is defined as follows.

Let X be a uniformly convex Banach space and C_1, C_2 two nonempty closed convex subsets of X which are also nonexpansive retracts of X with retractions P and Q, respectively. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed $P-$nonexpansive and $Q-$nonexpansive nonself-mappings satisfying the ball condition with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ for some $\epsilon > 0$ such that $\lim_{n\to\infty} k_n = 1-\epsilon$, $\lim_{n\to\infty} l_n = 1-\epsilon$, respectively and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. Then for a given $x_1 \in C_1 \cap C_2$, we now introduce the following iteration scheme:

$$\begin{cases} x_{n+1} = P((1-\alpha_n)y_n + \alpha_n T_2(QT_2)^{n-1}y_n) \in C_1, \\ y_n = (1-\beta_n)x_n + \beta_n T_1(PT_1)^{n-1}x_n \in C_2, \quad n \geq 1, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$.

If $T_1 = T_2$, $P = Q$ and $\beta_n = 0$ for all $n \geq 1$, then (1.10) reduces to (1.6). Pathak, Cho and Kang process (1.8) and our process (1.10) are independent: neither reduces to the other.

Note that each l^p $(1 \leq p < \infty)$ satisfies the Opial's condition, while all L^p do not have the property unless $p = 2$ and the dual of reflexive Banach spaces with a Fréchet differentiable norm have the Kadec–Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial property; however, their dual does have the Kadec–Klee property (see [22, 23]).

The purpose of this paper is to construct a more general iteration scheme (see (1.10) above) than iteration scheme (1.9) for approximating common fixed points of two asymptotically perturbed nonexpansive nonself-mappings and to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space. Furthermore, we prove weak convergence of the iteration process (1.10) in a uniformly convex Banach space whose dual has the Kadec–Klee property. The result applies not only to L^p spaces with $(1 \leq p < \infty)$ but also to other spaces which do not satisfy Opial's condition or have a Fréchet differentiable norm.

2 Preliminaries

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x - y\|\}.$$

A Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A subset C of X is said to be a *retract* if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a *retraction* if $P^2 = P$. It follows that if a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, the range of P. A set C is optimal if each point outside C can be moved to be closer to all points of C. It is well known (see [24]) that

(1) If X is a separable, strictly convex, smooth, reflexive Banach space, and if $C \subset$X is an optimal set with interior, then C is a nonexpansive retract of X.

(2) A subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. Moreover, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that a Banach space X is said to satisfy *Opial's condition* [25] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A mapping $T : C \to X$ is said to be *semi-compact (or demicompact)* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in C$. A Banach space X is said to have the *Kadec–Klee property* if for every sequence $\{x_n\}$ in X, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$ for more details on Kadec-Klee property, the reader is referred to [26, 27] and the references therein.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.1 ([17])**.** *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + t_n \text{ for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2 ([14])**.** *Let X be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3 ([20])**.** *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and let $T : C \to X$ be an asymptotically perturbed $P-$nonexpansive map satisfying the ball condition with a sequence $\{k_n\} \subset [1-\epsilon, \infty)$ and $k_n \to 1 - \epsilon$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 2.4 ([28])**.** *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.5 ([23])**.** *Let X be a real reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $x^*, y^* \in \omega_w(x_n)$; where $\omega_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1 - t)x^* - y^*\|$ exists for all $t \in [0, 1]$. Then $x^* = y^*$.*

We denote by Γ the set of strictly increasing, continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a convex subset of the Banach space X. A mapping $T : C \to C$ is said to be type (γ) [29] if $\gamma \in \Gamma$ and $0 \leqslant \alpha \leqslant 1$,

$$\gamma(\|\alpha Tx + (1 - \alpha)Ty - T(\alpha x + (1 - \alpha)y)\|) \leqslant \|x - y\| - \|Tx - Ty\|$$

for all x, y in C. Obviously, every type (γ) mapping is nonexpansive. For more information about mappings of type(γ), see [30–32].

Lemma 2.6 ([33,34])**.** *Let X be a uniformly convex Banach space and C a convex subset of X. Then there exists $\gamma \in \Gamma$ such that for each mapping $S : C \to C$ with* Lipschitz constant L,

$$\|\alpha Sx + (1 - \alpha)Sy - S(\alpha x + (1 - \alpha)y)\| \leqslant L\gamma^{-1}(\|x - y\| - \frac{1}{L}\|Sx - Sy\|)$$

for all $x, y \in C$ and $0 < \alpha < 1$.

3 Main Results

In this section, we prove strong and weak convergence theorems for the two-step iterative scheme given in (1.10) to a common fixed point for two asymptotically perturbed nonexpansive nonself-mappings satisfying the ball condition in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let X be a normed linear space and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$ with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ for some $\epsilon > 0$ such that $k_n \to 1-\epsilon$, $l_n \to 1-\epsilon$ as $n \to \infty$, respectively and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. From an arbitrary $x_1 \in C_1 \cap C_2$, define the sequences $\{x_n\}$ and $\{y_n\}$ using (1.10). If $q \in F(T_1) \cap F(T_2)$, then $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} \|y_n - q\|$ exist.*

Proof. Let $q \in F(T_1) \cap F(T_2)$. Since $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ are two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition, for all $n \geq 1$, $T_1(PT_1)^{n-1}$ and $T_2(QT_2)^{n-1}$ are Lipschitzian with Lipschitzian constants K_n, $L_n \geq 1$, respectively, where $K_n = (1 + k_n)(\sup_{i\geq 1} k_i)^{-1} + 1$ and $L_n = (1 + l_n)(\sup_{i\geq 1} l_i)^{-1} + 1$. Since $k_n \to 1-\epsilon$, $l_n \to 1-\epsilon$ as $n \to \infty$ for some $\epsilon > 0$, so $\sum_{n=1}^{\infty}(K_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(L_n - 1) < \infty$. Setting $k_n' = (1 + k_n)(\sup_{i\geq 1} k_i)^{-1}$ and $l_n' = (1 + l_n)(\sup_{i\geq 1} l_i)^{-1}$. We have $1 + k_n' = K_n$ and $1 + l_n' = L_n$. It follows that $\sum_{n=1}^{\infty} k_n' < \infty$ and $\sum_{n=1}^{\infty} l_n' < \infty$. Using (1.10), we have

$$\begin{aligned}
\|y_n - q\| &= \|(1-\beta_n)x_n + \beta_n T_1(PT_1)^{n-1}x_n - q\| \\
&\leq \|(1-\beta_n)(x_n - q) + \beta_n(T_1(PT_1)^{n-1}x_n - q)\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n\|T_1(PT_1)^{n-1}x_n - q\| \\
&\leq (1-\beta_n)\|x_n - q\| + \beta_n K_n\|x_n - q\| \\
&= (1-\beta_n)\|x_n - q\| + \beta_n(1 + k_n')\|x_n - q\| \\
&= (1-\beta_n)\|x_n - q\| + (\beta_n + \beta_n k_n')\|x_n - q\| \\
&\leq (1 + k_n')\|x_n - q\|, \qquad (3.1)
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P((1-\alpha_n)y_n + \alpha_n T_2(QT_2)^{n-1}y_n) - P(q)\| \\
&\leq \|(1-\alpha_n)(y_n - q) + \alpha_n(T_2(QT_2)^{n-1}y_n - q)\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n\|T_2(QT_2)^{n-1}y_n - q\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n L_n\|y_n - q\| \\
&= (1-\alpha_n)\|y_n - q\| + \alpha_n(1 + l_n')\|y_n - q\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1+l_n^{'})\|y_n-q\| \\
&\leq (1+l_n^{'})(1+k_n^{'})\|x_n-q\| \\
&= (1+k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})\|x_n-q\| \\
&< e^{\sum_{n=1}^{\infty}(k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})}\|x_1-q\|.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})<\infty$, then $\{x_n\}$ is bounded. It implies that there exists constant $M>0$ such that $\|x_n-q\|\leq M$ for all $n\geq 1$. So,

$$\|x_{n+1}-q\|\leq\|x_n-q\|+(k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})M.$$

It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists. By using (3.1) and Lemma 2.1, it follows easily that $\lim_{n\to\infty}\|y_n-q\|$ exists. This completes the proof. □

Lemma 3.2. *Let X be a uniformly convex Banach space and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1: C_1\to X$ and $T_2: C_2\to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2\supseteq(1-\lambda)C_1+\lambda T_1(C_1)$ for each $\lambda\in[\epsilon,1-\epsilon)$ for some $\epsilon>0$ with sequences $\{k_n\}$, $\{l_n\}\subset[1-\epsilon,\infty)$ for some $\epsilon>0$ such that $k_n\to 1-\epsilon$, $l_n\to 1-\epsilon$ as $n\to\infty$, respectively and $F(T_1)\cap F(T_2)\neq\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon,1-\epsilon)$ for some $\epsilon>0$. From an arbitrary $x_1\in C_1\cap C_2$, define the sequences $\{x_n\}$ and $\{y_n\}$ using (1.10). Then $\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|y_n-T_2y_n\|=0$.*

Proof. Let $q\in F(T_1)\cap F(T_2)$. Since $T_1(PT_1)^{n-1}$ and $T_2(QT_2)^{n-1}$ are Lipschitzian with Lipschitzian constants K_n, $L_n\geq 1$, respectively,
where $K_n=(1+k_n)(\sup_{i\geq 1}k_i)^{-1}+1$ and $L_n=(1+l_n)(\sup_{i\geq 1}l_i)^{-1}+1$. Set $1+k_n^{'}=K_n$ and $1+l_n^{'}=L_n$. By Lemma 3.1, we see that $\lim_{n\to\infty}\|x_n-q\|$ exists. Assume that $\lim_{n\to\infty}\|x_n-q\|=c$. Using (3.1), we have

$$\|y_n-q\| \leq (1+k_n^{'})\|x_n-q\|. \tag{3.2}$$

Taking the lim sup on both sides in the inequality (3.2), we have

$$\limsup_{n\to\infty}\|y_n-q\| \leq c. \tag{3.3}$$

In addition, $\|T_2(QT_2)^{n-1}y_n-q\|\leq(1+l_n^{'})\|y_n-q\|$, taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|T_2(QT_2)^{n-1}y_n-q\| \leq c. \tag{3.4}$$

From (1.10), we have

$$\begin{aligned}
\|x_{n+1}-q\| &\leq \|(1-\alpha_n)(y_n-q)+\alpha_n(T_2(QT_2)^{n-1}y_n-q)\| \\
&\leq (1+k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})\|x_n-q\|. \qquad (3.5)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})<\infty$ and $\lim_{n\to\infty}\|x_{n+1}-q\|=c$, letting $n\to\infty$ in the inequality (3.5), we have

$$\lim_{n\to\infty}\|(1-\alpha_n)(y_n-q)+\alpha_n(T_2(QT_2)^{n-1}y_n-q)\| \;=\; c. \tag{3.6}$$

By using (3.3), (3.4), (3.6) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|T_2(QT_2)^{n-1}y_n-y_n\| \;=\; 0. \tag{3.7}$$

In addition, $\|T_1(PT_1)^{n-1}x_n-q\|\le(1+k_n^{'})\|x_n-q\|$, taking lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-q\| \;\le\; c. \tag{3.8}$$

Using (1.10), we have

$$\begin{aligned}\|x_{n+1}-q\| &\le (1-\alpha_n)\|y_n-q\|+\alpha_n\|T_2(QT_2)^{n-1}y_n-q\|\\ &= (1-\alpha_n)\|y_n-q\|+\alpha_n\|T_2(QT_2)^{n-1}y_n-y_n+y_n-q\|\\ &\le (1-\alpha_n)\|y_n-q\|+\alpha_n\|T_2(QT_2)^{n-1}y_n-y_n\|+\alpha_n\|y_n-q\|\\ &\le \|y_n-q\|+\|T_2(QT_2)^{n-1}y_n-y_n\|. \end{aligned}\tag{3.9}$$

Taking the lim inf on both sides in the inequality (3.9), by (3.7) and $\lim_{n\to\infty}\|x_{n+1}-q\|=c$, we have

$$\liminf_{n\to\infty}\|y_n-q\| \;\ge\; c. \tag{3.10}$$

It follows from (3.3) and (3.10) that $\lim_{n\to\infty}\|y_n-q\|=c$. This implies that

$$\begin{aligned}c=\lim_{n\to\infty}\|y_n-q\| &\le \lim_{n\to\infty}\|(1-\beta_n)(x_n-q)+\beta_n(T_1(PT_1)^{n-1}x_n-q)\|\\ &\le \lim_{n\to\infty}\|x_n-q\|=c,\end{aligned}$$

and so

$$\lim_{n\to\infty}\|(1-\beta_n)(x_n-q)+\beta_n(T_1(PT_1)^{n-1}x_n-q)\|=c.$$

Using (3.8) and Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-x_n\| \;=\; 0. \tag{3.11}$$

From $y_n=(1-\beta_n)x_n+\beta_nT_1(PT_1)^{n-1}x_n$, we have

$$\begin{aligned}\|y_n-x_n\| &= \|(1-\beta_n)x_n+\beta_nT_1(PT_1)^{n-1}x_n-x_n\|\\ &\le \|(1-\beta_n)(x_n-x_n)+\beta_n(T_1(PT_1)^{n-1}x_n-x_n)\|\\ &\le (1-\beta_n)\|x_n-x_n\|+\beta_n\|T_1(PT_1)^{n-1}x_n-x_n\|\\ &\le \|T_1(PT_1)^{n-1}x_n-x_n\|.\end{aligned}\tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

We now prove that $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$. Suppose, on the contrary, that $\lim_{n\to\infty} \|x_n - T_1x_n\| > 0$. In addition,

$$\begin{aligned} \|x_n - T_1x_n\| &= \|x_n - T_1(PT_1)^{n-1}x_n + T_1(PT_1)^{n-1}x_n - T_1x_n\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1x_n\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + k_n\|x_n - T_1x_n\|. \end{aligned} \tag{3.13}$$

It follows from (3.11) and (3.13) that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - T_1x_n\| &\leq (1-\varepsilon) \lim_{n\to\infty} \|x_n - T_1x_n\| \\ &< \lim_{n\to\infty} \|x_n - T_1x_n\|, \end{aligned}$$

a contradiction. So, we must have

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = 0.$$

Next, we prove that $\lim_{n\to\infty} \|y_n - T_2y_n\| = 0$. Indeed, we have

$$\begin{aligned} \|y_n - T_2y_n\| &= \|y_n - T_2(QT_2)^{n-1}y_n + T_2(QT_2)^{n-1}y_n - T_2y_n\| \\ &\leq \|y_n - T_2(QT_2)^{n-1}y_n\| + \|T_2(QT_2)^{n-1}y_n - T_2y_n\| \\ &\leq \|y_n - T_2(QT_2)^{n-1}y_n\| + l_n\|y_n - T_2y_n\|. \end{aligned} \tag{3.14}$$

Suppose, on the contrary, that $\lim_{n\to\infty} \|y_n - T_2y_n\| > 0$. It follows from (3.7) and (3.14) that

$$\begin{aligned} \lim_{n\to\infty} \|y_n - T_2y_n\| &\leq (1-\varepsilon) \lim_{n\to\infty} \|y_n - T_2y_n\| \\ &< \lim_{n\to\infty} \|y_n - T_2y_n\|, \end{aligned}$$

a contradiction. Therefore, we must have $\lim_{n\to\infty} \|y_n - T_2y_n\| = 0$. This completes the proof. □

Theorem 3.3. *Let X be a uniformly convex Banach space and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$ with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ for some $\epsilon > 0$ such that $k_n \to 1-\epsilon$, $l_n \to 1-\epsilon$ as $n \to \infty$, respectively and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. From an arbitrary $x_1 \in C_1 \cap C_2$, define the sequences $\{x_n\}$ and $\{y_n\}$ using (1.10). If one of T_1 and T_2 is completely continuous, then $\{x_n\}$ and $\{y_n\}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 3.1, $\{x_n\}$ and $\{y_n\}$ both are bounded. In addition, by Lemma 3.2, $\lim_{n\to\infty}\|x_n-y_n\|=0$, $\lim_{n\to\infty}\|x_n-T_1x_n\|=0$ and $\lim_{n\to\infty}\|y_n-T_2y_n\|=0$, then $\{T_1x_n\}$ and $\{T_2x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j}\to q$ as $j\to\infty$. It follows from Lemma 3.2, that $\lim_{j\to\infty}\|x_{n_j}-T_1x_{n_j}\|=\lim_{j\to\infty}\|y_{n_j}-T_2y_{n_j}\|=0$. So by the continuity of T_1 and Lemma 2.3, we have $\lim_{j\to\infty}\|x_{n_j}-q\|=0$ and $q\in F(T_1)\cap F(T_2)$. Furthermore, by Lemma 3.1, we get that $\lim_{n\to\infty}\|x_n-q\|$ exists. Thus $\lim_{n\to\infty}\|x_n-q\|=0$. Similarly, if T_2 is completely continuous, then using the same argument we can show that $\{y_n\}$ converges strongly to a common fixed point of T_1 and T_2. The proof is completed. □

Theorem 3.4. *Let X be a uniformly convex Banach space and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2 \supseteq (1-\lambda)C_1+\lambda T_1(C_1)$ for each $\lambda\in[\epsilon,1-\epsilon)$ for some $\epsilon>0$ with sequences $\{k_n\}$, $\{l_n\}\subset[1-\epsilon,\infty)$ for some $\epsilon>0$ such that $k_n\to 1-\epsilon$, $l_n\to 1-\epsilon$ as $n\to\infty$, respectively and $F(T_1)\cap F(T_2)\neq\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon,1-\epsilon)$ for some $\epsilon>0$. From an arbitrary $x_1\in C_1\cap C_2$, define the sequences $\{x_n\}$ and $\{y_n\}$ using (1.10). If one of T_1 and T_2 is semi-compact, then $\{x_n\}$ and $\{y_n\}$ converge strongly to a common fixed point of T_1 and T_2.*

Proof. Since one of T_1 and T_2 is semi-compact, $\{x_n\}$ and $\{y_n\}$ both are bounded and $\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_n-T_1x_n\|=\lim_{n\to\infty}\|y_n-T_2y_n\|=0$, then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges strongly to q. It follows from Lemma 2.3 that $q\in F(T_1)\cap F(T_2)$. Thus $\lim_{n\to\infty}\|x_n-q\|$ exists by Lemma 3.1. Since the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to q, then $\{x_n\}$ converges strongly to the common fixed point $q\in F(T_1)\cap F(T_2)$. Similarly, if T_2 is semi-compact, then using the same argument we can show that $\{y_n\}$ converges strongly to a common fixed point of T_1 and T_2. The proof is completed. □

Next, we prove the weak convergence of the iterative scheme (1.10) for two asymptotically perturbed nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.5. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2\supseteq(1-\lambda)C_1+\lambda T_1(C_1)$ for each $\lambda\in[\epsilon,1-\epsilon)$ for some $\epsilon>0$ with sequences $\{k_n\}$, $\{l_n\}\subset[1-\epsilon,\infty)$ for some $\epsilon>0$ such that $k_n\to1-\epsilon$, $l_n\to1-\epsilon$ as $n\to\infty$, respectively and $F(T_1)\cap F(T_2)\neq\emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon,1-\epsilon)$ for some $\epsilon>0$. From an arbitrary $x_1\in C_1\cap C_2$, define the sequences $\{x_n\}$ and $\{y_n\}$ using (1.10). Then $\{x_n\}$ and $\{y_n\}$ converge weakly to a common fixed point of T_1 and T_2.*

Proof. It follows from Lemma 3.2 that $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|y_n - T_2y_n\| = 0$. By Lemma 3.1, $\{x_n\}$ and $\{y_n\}$ both are bounded. Since X is uniformly convex, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.3, we have $u \in F(T_1) \cap F(T_2)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.3, $u, v \in F(T_1) \cap F(T_2)$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 2.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. By using the same argument it also follows that $\{y_n\}$ converges weakly to a common fixed point of T_1 and T_2. This completes the proof of the theorem. □

In the remainder of this section, we deal with the weak convergence of the sequence $\{x_n\}$ defined by (1.10) in a uniformly convex Banach space X whose dual X^* has the Kadec-Klee property.

Theorem 3.6. *Let X be a uniformly convex Banach space and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$ with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon, \infty)$ for some $\epsilon > 0$ such that $k_n \to 1-\epsilon$, $l_n \to 1-\epsilon$ as $n \to \infty$, respectively and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. From an arbitrary $x_1 \in C_1 \cap C_2$, define the sequences $\{x_n\}$ and $\{y_n\}$ using (1.10). Then for all $u, v \in F(T_1) \cap F(T_2)$, the limit $\lim_{n\to\infty}\|tx_n - (1-t)u - v\|$ exists for all $t \in [0,1]$.*

Proof. Since $T_1(PT_1)^{n-1}$ and $T_2(QT_2)^{n-1}$ are Lipschitzian with Lipschitzian constants $K_n = (1+k_n)(\sup_{i\geq 1} k_i)^{-1} + 1$, $L_n = (1+l_n)(\sup_{i\geq 1} l_i)^{-1} + 1 \in [1,\infty)$, respectively. Set $1 + k_n^{'} = K_n$ and $1 + l_n^{'} = L_n$. It follows from Lemma 3.1 that the sequence $\{x_n\}$ is bounded. Then there exists $R > 0$ such that $\{x_n\} \subset B_R(0) \cap C_1$. Let $a_n(t) := \|tx_n + (1-t)u - v\|$ where $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|u - v\|$ and by Lemma 3.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\|x_n - v\|$ exists. Without loss of the generality, we may assume that $\lim_{n\to\infty}\|x_n - v\| = r$ for some positive number r. Let $x \in C_1$ and $t \in (0,1)$. For each $n \geq 1$, define $A_n : C_1 \to C_1$ by

$$A_n x = P((1-\alpha_n)y_n(x) + \alpha_n T_2(QT_2)^{n-1}y_n(x)),$$

where

$$y_n(x) = (1-\beta_n)x + \beta_n T_1(PT_1)^{n-1}x.$$

For $x, z \in C_1$, we have

$$\begin{aligned}
\|A_n x - A_n z\| &= \|P((1-\alpha_n)y_n(x) + \alpha_n T_2(QT_2^{n-1})y_n(x)) \\
&\quad - P((1-\alpha_n)y_n(z) + \alpha_n T_2(QT_2)^{n-1})y_n(z))\| \\
&\leq \|(1-\alpha_n)(y_n(x) - y_n(z)) + \alpha_n(T_2(QT_2)^{n-1}y_n(x) \\
&\quad - T_2(QT_2)^{n-1}y_n(z))\| \\
&\leq (1-\alpha_n)\|y_n(x) - y_n(z)\| + \alpha_n L_n\|y_n(x) - y_n(z)\| \\
&= (1-\alpha_n)\|y_n(x) - y_n(z)\| + \alpha_n(1 + l_n^{'})\|y_n(x) - y_n(z)\|
\end{aligned}$$

$$\leq (1+l_n^{'})\|y_n(x)-y_n(z)\| \tag{3.15}$$

and

$$\begin{aligned}\|y_n(x)-y_n(z)\| &= \|((1-\beta_n)x+\beta_n T_1(PT_1)^{n-1}x) \\ &\quad -((1-\beta_n)z+\beta_n T_1(PT_1)^{n-1}z)\| \\ &= \|(1-\beta_n)(x-z)+\beta_n(T_1(PT_1)^{n-1}x-T_1(PT_1)^{n-1}z)\| \\ &\leq (1-\beta_n)\|x-z\|+\beta_n\|T_1(PT_1)^{n-1}x-T_1(PT_1)^{n-1}z\| \\ &\leq (1-\beta_n)\|x-z\|+\beta_n K_n\|x-z\| \\ &= (1-\beta_n)\|x-z\|+\beta_n(1+k_n^{'})\|x-z\| \\ &\leq (1+k_n^{'})\|x-z\|. \end{aligned} \tag{3.16}$$

Using (3.15) and (3.16), we have

$$\|A_n x-A_n z\| \leq (1+k_n^{'}+l_n^{'}+k_n^{'}l_n^{'})\|x-z\|.$$

Set $S_{n,m} := A_{n+m-1}A_{n+m-2}\ ...A_n, n,m\geq 1$, and $b_{n,m} = \|S_{n,m}(tx_n+(1-t)u)-(tS_{n,m}x_n+(1-t)u)\|$, where $0\leq t\leq 1$. Also

$$\begin{aligned}\|S_{n,m}x-S_{n,m}y\| &\leq \|A_{n+m-1}(A_{n+m-2}...A_n x)-A_{n+m-1}(A_{n+m-2}...A_n y)\| \\ &\leq (1+k_{n+m-1}^{'}+l_{n+m-1}^{'}+k_{n+m-1}^{'}l_{n+m-1}^{'}) \\ &\quad \|A_{n+m-2}(A_{n+m-3}...A_n x)-A_{n+m-2}(A_{n+m-3}...A_n y)\| \\ &\quad \vdots \\ &\leq \prod_{j=n}^{n+m-1}(1+k_j^{'}+l_j^{'}+k_j^{'}l_j^{'})\|x-y\|.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(K_n-1)<\infty$, $\sum_{n=1}^{\infty}(L_n-1)<\infty$, so $\sum_{n=1}^{\infty}k_n^{'}<\infty$, $\sum_{n=1}^{\infty}l_n^{'}<\infty$. It follows that $k_n^{'}\to 0$ and $l_n^{'}\to 0$ as $n\to\infty$. Applying Lemma 2.6 with $x=x_n$, $y=u$, $S=S_{n,m}$ and $\lim_{n\to\infty}\|x_n-x^*\|$ exists for all $x^*\in F(T_1)\cap F(T_2)$. We obtain $\lim_{n\to\infty}b_{n,m}=0$. Observe that

$$\begin{aligned}a_{n+m}(t) &= \|tx_{n+m}+(1-t)u-v\| \\ &= \|tS_{n,m}x_n+(1-t)u-S_{n,m}v\| \\ &= \|S_{n,m}v-(tS_{n,m}x_n+(1-t)u)\| \\ &= \|S_{n,m}v-S_{n,m}(tx_n+(1-t)u)+S_{n,m}(tx_n+(1-t)u) \\ &\quad -(tS_{n,m}x_n+(1-t)u)\| \\ &\leq \|S_{n,m}v-S_{n,m}(tx_n+(1-t)u)\|+b_{n,m} \\ &= \|S_{n,m}(tx_n+(1-t)u)-S_{n,m}v\|+b_{n,m} \\ &\leq \|tx_n+(1-t)u-v\|+b_{n,m} \\ &= a_n(t)+b_{n,m}.\end{aligned}$$

Consequently,

$$\begin{aligned}\limsup_{m\to\infty} a_m(t) &= \limsup_{m\to\infty} a_{n+m}(t)\\ &\le \limsup_{m\to\infty}(b_{n,m}+a_n(t))\\ &\le \gamma^{-1}(\|x_n-u\| - \lim_{m\to\infty}\|x_m-u\|)+a_n(t)\end{aligned}$$

and

$$\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t).$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t\in[0,1]$. This completes the proof. □

Theorem 3.7. *Let X be a uniformly convex Banach space and C_1, C_2 two nonempty closed convex subsets of X. Let $T_1 : C_1 \to X$ and $T_2 : C_2 \to X$ be two asymptotically perturbed P-nonexpansive and Q-nonexpansive nonself-mappings satisfying the ball condition and $C_2 \supseteq (1-\lambda)C_1 + \lambda T_1(C_1)$ for each $\lambda \in [\epsilon, 1-\epsilon)$ for some $\epsilon > 0$ with sequences $\{k_n\}$, $\{l_n\} \subset [1-\epsilon,\infty)$ for some $\epsilon>0$ such that $k_n \to 1-\epsilon$, $l_n \to 1-\epsilon$ as $n\to\infty$, respectively and $F(T_1)\cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. Then the sequence $\{x_n\}$ defined by the iterative scheme (1.10) converges weakly to a fixed point of T_1 and T_2.*

Proof. It follows from Lemma 3.1 that the sequence $\{x_n\}$ and $\{y_n\}$ both are bounded. By Lemma 3.2, we have $\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\|x_n-T_1x_n\| = \lim_{n\to\infty}\|y_n-T_2y_n\| = 0$. Since X is uniformly convex, applying Lemma 2.3, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to a point $x^*\in F(T_1)\cap F(T_2)$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $x^*, y^* \in \omega_w(x_n)\cap F(T_1)\cap F(T_2)$. By Theorem 3.6,

$$\lim_{n\to\infty}\|tx_n-(1-t)x^*-y^*\|$$

exists for all $t\in[0,1]$. It follows from Lemma 2.5 that $x^*=y^*$. As a result, $\omega_w(x_n)$ is a singleton, and so $\{x_n\}$ converges weakly to a fixed point of T_1 and T_2. □

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