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On the Solutions of the Recursive Sequence $x_{n+1}=\frac{ax_{n-k}}{a-\prod_{i=0}^{k}x_{n-i}}$

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Abstract : We obtain in this paper the solutions of the difference equation

$$x_{n+1}=\frac{ax_{n-k}}{a-\prod_{i=0}^{k}x_{n-i}} \quad \text{for} \quad n=0,1,2,...$$

where k is a positive number and initial conditions are non zero real numbers with $\prod_{i=0}^{k}x_{-i}\neq a$.

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1 Introduction

Recently there has been a lot of interest in studying the solution of nonlinear difference equations. For some results in this area, see for example [1–14].

Cinar [2] investigated the positive solutions of the rational difference equation

$$x_{n+1}=\frac{x_{n-1}}{-1+x_nx_{n-1}}.$$

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Elsayed [3] investigated the qualitative behavior of the solution of the difference equation

$$x_{n+1} = \frac{x_n}{x_{n-1}\left(x_n \pm 1\right)}.$$

Aloqeili [1] studied the solutions, stability character, semi-cycle behavior of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_{n-1}x_n}$$

and gave the following formulation

$$x_n = \begin{cases} x_0 \prod\limits_{i=1}^{\frac{n}{2}} \frac{a^{2i-1}(1-a)-(1-a^{2i-1})x_{-1}x_0}{a^{2i}(1-a)-(1-a^{2i})x_{-1}x_0}, & n \ \ even, \\ x_{-1} \prod\limits_{i=0}^{\frac{n+1}{2}} \frac{a^{2i-1}(1-a)-(1-a^{2i})x_{-1}x_0}{a^{2i+1}(1-a)-(1-a^{2i+1})x_{-1}x_0}, & n \ \ odd. \end{cases}$$

Hamza et al. [4] studied the global stability, periodic nature, oscillation and the boundedness of solutions of the difference equation

$$x_{n+1} = \frac{A\prod_{i=l}^{k} x_{n-2i-1}}{B + C\prod_{i=l}^{k-1} x_{n-2i}}.$$

Elabbasy et al. [5] investigated some qualitative behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod\limits_{i=0}^{k} x_{n-i}}.$$

Karatas [6] studied the dynamics of the solution of the difference equation

$$x_{n+1} = \frac{A x_{n-m}}{B + C \prod\limits_{i=0}^{2k+1} x_{n-i}}.$$

Our goal in this paper is to obtain the solutions of the difference equation

$$x_{n+1} = \frac{a x_{n-k}}{a - \prod\limits_{i=0}^{k} x_{n-i}} \text{ for } n = 0, 1, 2, ... \tag{1.1}$$

where k is a positive number and initial conditions are non zero real numbers with $\prod\limits_{i=0}^{k} x_{-i} \neq a$.

Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}\right),\ n = 0, 1, ... \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [7].

2 Main Results

Theorem 2.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1.1) and assume that $\prod_{i=0}^{k} x_{-i} = p$ and $p \neq a$. Then for $n = 0, 1, ...$*

$$x_{(k+1)n+1} = \frac{ax_{-k} \prod_{i=1}^{n} [a - (k+1)\,ip]}{(a-p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 1]\,p\}},$$

$$x_{(k+1)n+2} = \frac{x_{-(k-1)}\,(a-p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 1]\,p\}}{(a-2p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 2]\,p\}},$$

$$x_{(k+1)n+3} = \frac{x_{-(k-2)}\,(a-2p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 2]\,p\}}{(a-3p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 3]\,p\}},$$

$$x_{(k+1)n+4} = \frac{x_{-(k-3)}\,(a-3p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 3]\,p\}}{(a-4p) \prod_{i=1}^{n} \{a - [(k+1)\,i + 4]\,p\}},$$

$$\vdots$$

$$x_{(k+1)n+k} = \frac{x_{-1}\,[a - (k-1)\,p] \prod_{i=1}^{n} \{a - [(k+1)\,i + k - 1]\,p\}}{(a-kp) \prod_{i=1}^{n} \{a - [(k+1)\,i + k]\,p\}},$$

$$x_{(k+1)n+k+1} = \frac{x_0\,(a-kp) \prod_{i=1}^{n} \{a - [(k+1)\,i + k]\,p\}}{[a - (k+1)\,p] \prod_{i=1}^{n} \{a - [(k+1)\,i + k + 1]\,p\}}.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$x_{(k+1)n-k} = \frac{ax_{-k} \prod_{i=1}^{n-1} [a - (k+1) ip]}{(a-p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 1] p\}},$$

$$x_{(k+1)n-(k-1)} = \frac{x_{-(k-1)} (a-p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 1] p\}}{(a-2p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 2] p\}},$$

$$x_{(k+1)n-(k-2)} = \frac{x_{-(k-2)} (a-2p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 2] p\}}{(a-3p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 3] p\}},$$

$$x_{(k+1)n-(k-3)} = \frac{x_{-(k-3)} (a-3p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 3] p\}}{(a-4p) \prod_{i=1}^{n-1} \{a - [(k+1) i + 4] p\}},$$

$$\vdots$$

$$x_{(k+1)n-1} = \frac{x_{-1} [a - (k-1) p] \prod_{i=1}^{n-1} \{a - [(k+1) i + k - 1] p\}}{(a-kp) \prod_{i=1}^{n-1} \{a - [(k+1) i + k] p\}},$$

$$x_{(k+1)n} = \frac{x_0 (a-kp) \prod_{i=1}^{n-1} \{a - [(k+1) i + k] p\}}{[a - (k+1) p] \prod_{i=1}^{n-1} \{a - [(k+1) i + k + 1] p\}}.$$

Now, it follows from Eq.(1.1) that

$$x_{(k+1)n+1} = \frac{ax_{(k+1)n-k}}{a - \prod_{i=0}^{k} x_{(k+1)n-i}}.$$

Hence, we have

$$x_{(k+1)n+1} = \frac{a\dfrac{ax_{-k}\prod_{i=1}^{n-1}[a-(k+1)ip]}{(a-p)\prod_{i=1}^{n-1}\{a-[(k+1)i+1]p\}}}{a-\dfrac{ap\prod_{i=1}^{n-1}[a-(k+1)ip]}{[a-(k+1)p]\prod_{i=1}^{n-1}\{a-[(k+1)i+k+1]p\}}}$$

$$= \frac{a\dfrac{ax_{-k}\prod_{i=1}^{n-1}[a-(k+1)ip]}{(a-p)\prod_{i=1}^{n-1}\{a-[(k+1)i+1]p\}}}{a-\dfrac{ap\prod_{i=2}^{n-1}[a-(k+1)ip]}{\prod_{i=2}^{n-1}[a-(k+1)ip]}}$$

$$= \frac{ax_{-k}\prod_{i=1}^{n-1}[a-(k+1)\,ip]}{(a-p)\prod_{i=1}^{n-1}\{a-[(k+1)\,i+1]\,p\}}.\frac{a-(k+1)\,np}{a-[(k+1)\,n+1]\,p}.$$

Hence, we have

$$x_{(k+1)n+1} = \frac{ax_{-k}\prod_{i=1}^{n}[a-(k+1)\,ip]}{(a-p)\prod_{i=1}^{n}\{a-[(k+1)\,i+1]\,p\}}.$$

Similarly, we get from Eq(1.1) that

$$x_{(k+1)n+2} = \frac{ax_{(k+1)n-(k-1)}}{a-\prod_{i=-1}^{k-1}x_{(k+1)n-i}}.$$

Then

$$x_{(k+1)n+2}=\frac{a\dfrac{ax_{-(k-1)}(a-p)\prod_{i=1}^{n-1}\{a-[(k+1)i+1]p\}}{(a-2p)\prod_{i=1}^{n-1}\{a-[(k+1)i+2]p\}}}{a-\dfrac{ap\prod_{i=1}^{n}[a-(k+1)ip]}{\{a-[(k+1)n+1]p\}[a-(k+1)p]\prod_{i=1}^{n-1}\{a-[(k+1)i+k+1]p\}}}$$

$$=a\frac{ax_{-(k-1)}\left(a-p\right)\prod_{i=1}^{n-1}\left\{a-\left[\left(k+1\right)i+1\right]p\right\}}{(a-2p)\prod_{i=1}^{n-1}\left\{a-\left[\left(k+1\right)i+2\right]p\right\}}\cdot\frac{a-\left[\left(k+1\right)n+1\right]p}{a\left\{a-\left[\left(k+1\right)n+2\right]p\right\}}.$$

Hence, we have

$$x_{(k+1)n+2}=\frac{ax_{-(k-1)}\left(a-p\right)\prod_{i=1}^{n}\left\{a-\left[\left(k+1\right)i+1\right]p\right\}}{(a-2p)\prod_{i=1}^{n}\left\{a-\left[\left(k+1\right)i+2\right]p\right\}}.$$

Similarly, one can obtain the other cases. Thus, the proof is completed. □

Theorem 2.2. *Eq.*(1.1) *has periodic solutions of period* $(k+1)$ *iff one of the initial condition is zero and will be take the form*

$$\left\{x_{-k},x_{-(k-1)},...,x_{-1},x_0,x_1,x_2,...,x_{k+2},...\right\}.$$

Proof. Firstly, assume that there exists a prime period $(k+1)$ solution x_{-k}, $x_{-(k-1)}$, ..., $x_{-1},x_0,x_1,x_2,...,x_{k+2},...$ of *Eq.*(1.1). We have from the form of solution of *Eq.*(1.1) that

$$x_{-k}=\frac{ax_{-k}}{a-p},x_{-(k-1)}=\frac{x_{-(k-1)}\left(a-p\right)}{a-2p},...,x_0=\frac{x_0(a-kp)}{a-(k+1)p}.$$

Then $p=0$. That is, one of the initial condition is zero.

Now suppose that one of the initial condition is zero. Then we have

$$x_{(k+1)n+1}=x_{-k},x_{(k+1)n+2}=x_{-(k-1)},...,x_{(k+1)n+k+1}=x_0,$$
$$x_{(k+1)n+k+2}=x_{-k},x_{(k+1)n+k+3}=x_{-(k-1)},...,x_{(k+1)n+2(k+1)}=x_0.$$

Thus, we obtain a period $(k+1)$ solution. The proof is complete. □

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