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Seat Arrangement Problems

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Abstract : Let there be n students, and n row seats. For n days, a seat is arranged for each student on each day, and each student is required to sit on different seat on each of the n days. Also for these n days, it is required that each student shall has one chance to sit next to every other $(n-1)$ students on one of his side, and shall has one chance to sit next to every other $(n-1)$ students on the other one of his side. In this paper, we provide sufficient conditions and an algorithm for the arrangements.

Keywords : combinatorics; derangement, seat arrangement.
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1 Introduction

Let $1, 2, 3, \ldots, n$ be n different objects. From the set of all $n!$ possible arrangements, numbers of questions can be asked about some subsets of the arrangements. Some known results are about the number of arrangements which require that some particular objects are not allowed to be placed at certain positions. A particular case of this kind of arrangement, known as derangement, can be found from the formula $D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$, where D_n represent the number of arrangements of n objects which require that each object is not allowed to be placed on the original position. See [1–4], for more details. The more generalized formula is

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$D(n,r,k) = \dfrac{\binom{r}{k}}{(n-r)!} \sum_{i=0}^{r-k} (-1)^i \binom{r-k}{i} (n-k-i)!$, where $D(n,r,k)$ is the number of arrangements of r objects from n objects that allow each object to be placed on original position for k times. These widely known arrangements are discussed in some elementary texts on combinatorics, see [2], for examples.

In this paper, we propose questions and answers related to some arrangements. For motivation, we shall name the problems as Seat Arrangement Problems (SAP). Suppose a teacher wants to arrange n row seats for n students for n days. It is natural enough that each student wants to know who shall sit next to him on each day, and each may want to have different seat every day. The teacher, for social reason, may want to arrange seats such that all pairs of students shall have at least one day to sit next to each other. Let $1, 2, 3, \ldots, n$ be n students, and $s_1, s_2, s_3, \ldots, s_n$ be n row seats. For example, when $n = 4$, let $1, 2, 3, 4$ be four students.

The teacher may arrange row seats for the four students for four days as follow:

s_1	s_2	s_3	s_4
1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

For another example, when $n = 6$, the teacher may try to arrange seats as follow:

s_1	s_2	s_3	s_4	s_5	s_6
1	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

For simple obvious case, when $n = 2$, the arrangements for two days are

s_1	s_2
1	2
2	1

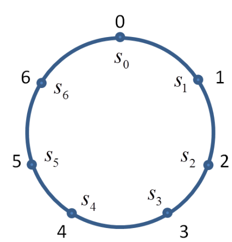
We can see that the arrangements of the cases $n = 2, n = 4$, and $n = 6$ above satisfy the following conditions of which we call Seat Arrangement Conditions (SAC).

1. Every student has different seat every day, so each student has a chance to sit on all n seats in n days.
2. Every student has two chances to sit next to every other students, once on one of his side and once on the other one of his side.

One may try to construct the above arrangements, and would soon find that, without proper algorithm, it is not an obvious work to do. If $n = 3$, there are 6 permutations 123, 132, 213, 231, 312, 321 of which no three permutations can be used for the three day seat arrangements that satisfies SAC. For the cases $n = 5$, and $n = 7$, since we have not provided any proof here, we can only say that we hypothesize that it is not possible to do the required arrangements for these cases. We have seen that the arrangements are possible when $n = 2, 4$, and 6, so one may expect that the case when $n = 8$ would also be possible for the required arrangement. However, when $n = 8$, we have reasons for hypothesizing that the required arrangements are not possible. From these, we can see that for some values of n it is not possible to construct the required arrangements. Now, the first question is about the values of n that are certainly possible for the arrangements. The next question is, when possible, how to construct the required arrangements. In section 2, we find sufficient conditions for the values of n that are possible for the arrangements and find an algorithm that can be used for the arrangements.

2 Seat Arrangement Algorithm (SAA)

First, for examples, we provide an algorithm that we have used for the case $n = 6$, and after that we shall prove that the algorithm can be used for more general cases. Let $1, 2, 3, 4, 5, 6$ represent the six students, and 0 represent the teacher. Let s_0 be the seat for the teacher, and let $s_1, s_2, s_3, s_4, s_5, s_6$ be six seats for six students. To serve our proof, we arrange the seats $s_0, s_1, s_2, s_3, s_4, s_5$, and s_6 in circular form as follow:



Arrangement for the first day
Figure 2.1

All seats shall be fixed as shown in Figure 2.1, teacher shall sit on s_0 every day, but in each day teacher shall assign students for their seats. For the first day, students $1, 2, 3, 4, 5, 6$ are assigned to sit on $s_1, s_2, s_3, s_4, s_5, s_6$ respectively, see Figure 2.1. Though the main goal is about arrranging 6 students on 6 row seats $s_1, s_2, s_3, s_4, s_5, s_6$, it is helpful to arrange 7 circular seats for 7 persons, i.e. to arrange seats $s_0, s_1, s_2, s_3, s_4, s_5, s_6$ for teacher 0, and students $1, 2, 3, 4, 5, 6$ respectively.

We use teacher seat as reference point, and use students and their seats on day 1 arrangement in referring for other day arrangements. We define that any two consecutive seats have distance one. Teacher can count the distances of seats from s_0, in clockwise direction, around all seven seats. So, on day 1, if teacher counts from s_0 with distance 4, he shall arrive s_4 where student 4 is sitting. If he counts with distance 7 or 14, he shall arrive back at his seat s_0. If he counts with distances 3, or 10, or 17 he shall arrive at seat s_3 where student 3 is sitting. So, for any day k arrangement, we shall use the day 1 arrangement for our reference, that is when we want to assign students to have seat $s_1, s_2, s_3, s_4, s_5, s_6$ we shall use his distance, on day 1, from s_0 in referring the students. We note that teacher can also count the distances of seats from s_0, in anticlockwise direction, around all 7 seats but the distances would have minus signs. For examples, if teacher counts from s_0 with distance -4, he shall arrive s_3, and if he counts from s_0 with distance -9, he shall arrive s_5. If we plus distance i to the distance of seat s_j, we mean that we move further, in clockwise direction, to the distance $j+i$, and if we minus distance i to the distance of seat s_j, we mean that we move backward, in anticlockwise direction, to the distance $j-i$.

We have the first day arrangement as described above where students on $s_1, s_2, s_3, s_4, s_5, s_6$ are $1, 2, 3, 4, 5, 6$ respectively. Every day the teacher sits at s_0 and he shall assign 6 students to sit at seats $s_1, s_2, s_3, s_4, s_5, s_6$. For the first day, we write down the teacher and students on seats $s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_0$ and we have the arrangement for the first day as

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 0$$

We note that, since we use circular arrangement, it is helpful to write down teacher 0 at s_0 at the left end and at the right end of arrangement so that we can identify two students who sit next to the teacher on that day.

For second day arrangement, using the first day arrangement as reference, teacher shall assign students whose distances from s_0 are $2, 4, 6, 8, 10, 12$, i.e. students $2, 4, 6, 1, 3, 5$, to sit on seats $s_1, s_2, s_3, s_4, s_5, s_6$ respectively. So, the second day arrangement is

$$0 \quad 2 \quad 4 \quad 6 \quad 1 \quad 3 \quad 5 \quad 0$$

For the third day arrangement, using the first day arrangement as reference, teacher shall assign students whose distances from s_0 are $3, 6, 9, 12, 15, 18$, i.e. students $3, 6, 2, 5, 1, 4$, to sit on seats $s_1, s_2, s_3, s_4, s_5, s_6$ respectively. So, the third day arrangement is

$$0 \quad 3 \quad 6 \quad 2 \quad 5 \quad 1 \quad 4 \quad 0$$

For the fourth day arrangement, using the first day arrangement as reference, teacher shall assign students whose distances from s_0 are $4, 8, 12, 16, 20, 24$, i.e.

students $4, 1, 5, 2, 6, 3$, to sit on seats $s_1, s_2, s_3, s_4, s_5, s_6$ respectively. So the fourth day arrangement is

$$0 \quad 4 \quad 1 \quad 5 \quad 2 \quad 6 \quad 3 \quad 0$$

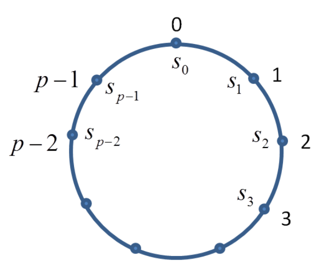
For the fifth day arrangement, using the first day arrangement as reference, teacher shall assign students whose distances from s_0 are $5, 10, 15, 20, 25, 30$, i.e. students $5, 3, 1, 6, 4, 2$ to sit on seats $s_1, s_2, s_3, s_4, s_5, s_6$ respectively. So the fifth day arrangement is

$$0 \quad 5 \quad 3 \quad 1 \quad 6 \quad 4 \quad 2 \quad 0$$

Finally, for the sixth day arrangement, using the first day arrangement as reference, teacher shall assign students whose distances from s_0 are $6, 12, 18, 24, 30$, i.e. students $6, 5, 4, 3, 2, 1$ to sit on seats $s_1, s_2, s_3, s_4, s_5, s_6$ respectively. So the sixth day day arrangement is

$$0 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0$$

If we remove teacher 0 from the above 6 day arrangements, we then obtain the required arrangements for the 6 students. Next, we describe SAA for general cases. Theorem 2.1 provides the values of n that can be used with the algorithm. Let 0 represent the teacher, and $1, 2, 3, \ldots, n$ represent n students. According to Theorem 2.1, the values of $n = p - 1$, where p is any prime number which is equal or greater than 3. Let s_0 be the seat for the teacher, and $s_1, s_2, s_3, \ldots, s_{p-1}$ be n seats provided for students. Let the seats be arranged in circle as follow.



Arrangement for the first day
Figure 2.2

For convenience, we may represent the above circular arrangement of seats in one line as

$$s_0 \quad s_1 \quad s_2 \quad s_3 \quad \ldots \quad s_{p-2} \quad s_{p-1} \quad s_0$$

For k day arrangements, we can have day k arrangement by assigning proper students to sit on seat $s_1, s_2, s_3, \ldots, s_{p-1}$ respectively. The SAA for $p-1$ days is as follow:

1. Every day the teacher sits at s_0.

2. For first day arrangement, teacher shall assign students $1, 2, 3, \dots, p-1$ to sit on $s_1, s_2, s_3, \dots, s_{p-1}$ respectively. So, the distances of students $1, 2, 3, \dots, p-1$ from s_0 are $1, 2, 3, 4, \dots, p-1$ respectively. Therefore, corresponding to positions of seats $s_0, s_1, s_2, s_3, \dots, s_{p-1}, s_0$, the first day arrangement is

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad p-1 \quad 0.$$

3. For day k arrangements ($k = 2, 3, 4, \dots, p-1$), using the first day arrangement as reference, teacher shall assign students whose distances from s_0 are $k, 2k, 3k, \dots, (p-1)k$ to sit on seats $s_1, s_2, s_3, \dots, s_{p-1}$ respectively. So, for day k arrangement , using distances of students in assigning them to sit on $s_1, s_2, s_3, \dots, s_{p-1}$, we have the day k arrangement as

$$0 \quad k \quad 2k \quad 3k \quad 4k \quad \dots \quad (p-1)k \quad 0.$$

Next, in Theorem 2.1, we explain why the arrangements are possible when using SAA.

Theorem 2.1. *The SAA are possible when the number of students $n = p - 1$ for any given prime number $p \geq 3$.*

Proof. Let 0 represents the teacher, and $1, 2, 3, \dots, p-1$ represent $p-1$ students. Let seats of the teacher be s_0 and student seats be $s_1, s_2, s_3, \dots, s_{p-1}$ and the seats be arranged in circle as described in Figure 2.2.

In order to prove that the SAC for the arrangement are satisfied, we need to show that the following (1), (2), (3) are satisfied for the $n = p-1$ day arrangements

1. Every day all students shall have their own seats for sitting.

2. For all n students, each has a chance to sit on all n seats

3. Each student has a chance to sit next to all other $n-1$ students once on one of his side, and once on the other of his side.

Remind that, using the distances of the first day arrangement as references, the arrangements, for example on day k, are distances (of the first day arrangement) of $(p-1)$ students who shall be assigned on day k to sit on $s_1, s_2, s_3, \dots, s_{p-1}$ respectively. That is, students whose distances are $1k, 2k, 3k, \dots, (p-1)k$ shall be assigned for seat $s_1, s_2, s_3, \dots, s_{p-1}$ respectively on day k. So, on any day k, referring to the first day arrangement, the distance of any two consecutive people is equal to k.

Each row of the following is the distances of teacher and all $p-1$ students from s_0. The 1st, 2nd, ..., $(p-1)$-th rows are for day 1, day 2, day 3,..., day $p-1$ respectively.

	s_0	s_1	s_2	s_3	$\dots$	s_{p-1}	s_0
Day 1	0	$1\cdot 1$	$2\cdot 1$	$3\cdot 1$	$\dots$	$(p-1)\cdot 1$	0
Day 2	0	$1\cdot 2$	$2\cdot 2$	$3\cdot 2$	$\dots$	$(p-1)\cdot 2$	0
Day 3	0	$1\cdot 3$	$2\cdot 3$	$3\cdot 3$	$\dots$	$(p-1)\cdot 3$	0
$\vdots$							
Day k	0	$1\cdot k$	$2\cdot k$	$3\cdot k$	$\dots$	$(p-1)\cdot k$	0
$\vdots$							
Day $p-2$	0	$1\cdot(p-2)$	$2\cdot(p-2)$	$3\cdot(p-2)$	$\dots$	$(p-1)\cdot(p-2)$	0
Day $p-1$	0	$1\cdot(p-1)$	$2\cdot(p-1)$	$3\cdot(p-1)$	$\dots$	$(p-1)\cdot(p-1)$	0

Since on each day for the arrangement of seats, no difference of any two distances are multiple of p, so each of the assigned students will be on different seats, i.e. each student on any day shall have his own seat for sitting. Similarly, on each column of day 1, day 2, day 3,..., day $p-1$ arrangements the difference of any two values in the column is not the multiple value of p, so for any particular student seat, for $p-1$ days, there would have $p-1$ different students to sit on. That is, no student occupies any given seat twice or more. From the above arguments, we can conclude that every student has his own different seat every day.

Next, we want to prove (3).

Consider any three peoples (teacher, or students) whose distances $j-i, j, j+i$ who sit on three seats s_{j-i}, s_j, s_{j+i} of day 1 ($j = 0, 1, 2, 3, \dots, p-1; i = 1, 2, 3, \dots, p-1$). According to the algorithm, these three peoples shall be assigned to sit on some three consecutive seats s_{r-1}, s_r and s_{r+1} of day i. That is, peoples (teacher or students) with distances $j-1, j$, and $j+1$ would sit next to each other on day 1, peoples with distances $j-2, j$, and $j+2$ would sit next to each other on day 2, ..., peoples with distances $j-(p-1), j$, and $j+(p-1)$ would sit next to each other on day $p-1$. Consider Figure 2.2, with the above arguments for the people with distance j, we can see that j shall has chances to sit next to every other people once on one of his side, and once on the other of his side. Therefore, we have proved the theorem. □

To clarify the proof, consider the known case $n = p-1 = 7-1 = 6$. For example, consider Figure 2.1, when $j = 5$, then peoples with distances $j-1 = 4, j = 5$, and $j+1 = 6$ would sit next to each other on day 1, peoples with distances $j-2 = 3, j = 5$ and $j+2 = 7(=0)$ would sit next to each other on day 2, peoples with distances $j-3 = 2, j = 5$, and $j+3 = 8(=1)$ would sit next to each other on day 3, peoples with distances $j-4 = 1, j = 5$, and $j+4 = 9(=2)$ would sit next to each other on day 4, peoples with distances $j-5 = 0, j = 5$, and $j+5 = 10(=3)$ would sit next to each other on day 5, peoples with distances $j-6 = 6, j = 5$,and $j+6 = 11(=4)$ would sit next to each other on day 6. So, we see that student 5 sit next on one of his side to all other persons (5 students and the teacher) in 6 days, and that the student 5 sit next, on his other side, to every other persons in 6 day arrangements.

From the proof of Theorem 2.1, we can readily have the following corollary.

Corollary 2.2. *If the number of student is $n = p - 1$, where p is any prime number ≥ 3, then teacher can arrange round table seat $s_0, s_1, s_2, s_3, \ldots, s_{p-1}$ for $p-1$ days such that in each day teacher sit on seat s_0, students have different seats every day, and every people has a chance to sit next to every other people on one of his side, and has a chance to sit next to every other people on the other one of his side.*

Also, from the results of Theorem 2.1, with less conditions, we can readily have Corollary 2.3.

Corollary 2.3. *The first $\frac{(p-1)}{2}$ day arrangements of the SAA, for the row seat arrangements, and round table seat arrangements, are the arrangements which satisfy the conditions that each student has new seat every day, and that every person has one chance to sit next to every other person.*

See the first three day arrangements of the 6 day arrangements in the case $n = 6$ above for verification of Corollary 2.3.

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References

[1] V.K. Balakrishnan, Combinatorics, Schaum's Outline Series, Mc Graw-Hill, New York, (1995).

[2] C. Chuan-Chong, K. Khee-Meng, Principles and Techniques in Combinatorics, World Scientific Publishing Co. Pte. Ltd., Singapore, (2004).

[3] F.S. Robert, Applied Combinatorics, Prentice-Hall, New Jersy, (1984).

[4] J.H. Van Lint, R.M. Wilson, A Course in Combinatorics, Cambridge University Press, (2001).

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