Thai Journal of Mathematics
Volume 14 (2016) Number 2 : 369–381
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

The Regularization Method for Solving Variational Inclusion Problems

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Abstract : In this paper, we introduce a new regularization method for solving inclusion problems in Banach spaces. We then prove the strong convergence theorems under some wild conditions. Finally, we also provide some concrete examples including its numerical experiments.

Keywords : accretive operator; Banach space; splitting method; forward-backward splitting method.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

Let X be a real Banach space. We consider the inclusion problem:
Find $\hat{x} \in X$ such that

$$0 \in (A+B)\hat{x}, \tag{1.1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form.

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A classical method for solving the problem (1.1) is the forward-backward splitting method [1–4] which is defined by the following manner: for any fixed $x_1 \in X$,

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n)$$

for each $n \geq 1$, where $r > 0$. We see that each step of the iteration involves only with A as the forward step and B as the backward step, but not the sum of B. In fact, this method includes, in particular, the proximal point algorithm [5–9] and the gradient method [10, 11]. In 1979, Lions-Mercier [2] introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n$$

for each $n \geq 1$, where $J_r^T = (I + rT)^{-1}$. The first one is often called the *Peaceman-Rachford algorithm* [12] and the second one is called the *Douglas-Rachford algorithm* [13]. We note that both algorithms can be weakly convergent in general [3].

In 2006, Hong-Kun Xu [14] proved some strong convergence theorems for a maximal monotone operator in a Hilbert space H, which is defined by the following manner: for any $x_1 \in H$,

$$x_{n+1} = J_{r_n}^T((1 - \alpha_n)x_n + \alpha_n u + e_n), \tag{1.2}$$

for each $n \geq 1$, where $u \in H$, $\{r_n\} \subseteq (0, \infty), \{\alpha_n\} \subseteq (0, 1)$ and $\{e_n\}$ is a sequence of errors satisfy the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(d) there are constants $0 < \underline{r} \leq \overline{r}$ such that $\underline{r} \leq r_n \leq \overline{r}$ for all $n \geq 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(e) $\sum_{n=1}^{\infty} \|e_n\| < \infty$,

then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{T^{-1}(0)}u$.

Recently, López et al. [15] introduced the following Halpern-type forward-backward method: for any $x_1 \in X$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n) \tag{1.3}$$

for each $n \geq 1$, where $u \in X$, A is an α-inverse strongly accretive mapping on X and B is an m-accretive operator on X, $\{r_n\} \subseteq (0, \infty)$, $\{\alpha_n\} \subseteq (0, 1]$ and $\{a_n\}, \{b_n\}$ are the error sequences in X. They proved that the sequence $\{x_n\}$ generated by (1.3) strongly converges to a zero point of the sum of A and B under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces). For more details, see [4, 16–18].

In this paper, we study the modified regularization methods (1.2) for solving the problem (1.1) for accretive operators and inverse strongly accretive operators in Banach spaces and prove its strong convergence for the proposed methods under some mild conditions. We also give some applications and numerical examples to support our main results.

Remark 1.1. *We note that our obtained results can be viewed as the improvement of the results of Hong-Kun Xu [14]. In fact, we remove the conditions that* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ *and there are constants* $0<\underline{r}\leq\overline{r}$ *such that* $\underline{r}\leq r_n\leq\overline{r}$ *for all* $n\geq 0$*, and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$ *in our results. Moreover, we extend their results in Hilbert spaces to certain Banach spaces.*

2 Preliminaries

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel. The *modulus of convexity* of a Banach space X is the function $\delta_X:(0,2]\to[0,1]$ defined by

$$\delta_X(\epsilon) \quad = \quad \inf\Big\{1-\frac{\|x+y\|}{2}:\|x\|=\|y\|=1,\|x-y\|\geq\epsilon\Big\}.$$

Then X is said to be *uniformly convex* if $\delta_X(\epsilon)>0$ for any $\epsilon\in(0,2]$.

The *modulus of smoothness* of X is the function $\rho_X:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$\rho_X(t) \quad = \quad \sup\Big\{\frac{\|x+ty\|+\|x-ty\|}{2}-1:\|x\|=1,\|y\|=1\Big\}.$$

Then X is said to be *uniformly smooth* if $\rho'_X(0)=\lim\limits_{t\to 0}\dfrac{\rho_X(t)}{t}=0$. For any $q\in(1,2]$, a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c_q>0$ such that $\rho_X(t)>c_qt^q$ for any $t>0$.

The *subdifferential* of a proper convex function $f:X\to(-\infty,+\infty]$ is the set-valued operator $\partial f:X\to 2^X$ defined as

$$\partial f(x)=\{x^*\in X^*:\langle x^*,y-x\rangle+f(x)\leq f(y)\}.$$

If f is proper convex and lower semicontinuous, then the subdifferential $\partial f(x)\neq\emptyset$ for any $x\in int\mathcal{D}(f)$, the interior of the domain of f.

The *generalized duality mapping* $J_q:X\to 2^{X^*}$ is defined by

$$J_q(x)=\{j(x)\in X^*\langle j_q(x),x\rangle=\|x\|^q,\ \|j_q(x)\|=\|x\|^{q-1}\}.$$

If $q=2$, then the corresponding duality mapping is called the *normalized duality mapping* and denoted by J. We know that the following subdifferential inequality holds: for any $x, y \in X$,

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle,\ j_q(x+y) \in J_q(x+y). \tag{2.1}$$

In particular, it follows that, for all $x, y \in X$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,\ j(x+y) \in J(x+y).$$

Proposition 2.1 ([19]). *Let $1 < q < \infty$. Then we have the following:*

(1) *The Banach space X is smooth if and only if the duality mapping J_q is single valued.*

(2) *The Banach space X is uniformly smooth if and only if the duality mapping J_q is single valued and norm-to-norm uniformly continuous on bounded sets of X.*

A set-valued operator $A : X \to 2^X$ with the domain $\mathcal{D}(A)$ and the range $\mathcal{R}(A)$ is said to be *accretive* if, for all $t > 0$ and $x, y \in \mathcal{D}(A)$,

$$\|x-y\| \leq \|x-y+t(u-v)\|$$

for all $u \in Ax$ and $v \in Ay$.

Recall that A is accretive if and only if, for each $x, y \in \mathcal{D}(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u-v, j(x-y)\rangle \geq 0$$

for all $u \in Ax$ and $v \in Ay$. An accretive operator A is said to be *m-accretive* if the range

$$\mathcal{R}(I+\lambda A) = X$$

for some $\lambda > 0$. It can be shown that an accretive operator A is m-accretive if and only if

$$\mathcal{R}(I+\lambda A) = X$$

for all $\lambda > 0$.

For any $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is *α-inverse strongly accretive* (shortly, *α-isa*) of order q if, for each $x, y \in \mathcal{D}(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle u-v, j_q(x-y)\rangle \geq \alpha\|u-v\|^q$$

for all $u \in Ax$ and $v \in Ay$.

Let C be a nonempty closed and convex subset of a real Banach space X and K be a nonempty subset of C. A mapping $T : C \to K$ is called a *retraction* of C onto K if $Tx = x$ for all $x \in K$. We say that T is *sunny* if, for each $x \in C$ and $t \geq 0$,

$$T(tx+(1-t)Tx) = Tx,$$

whenever $tx+(1-t)Tx \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

Theorem 2.2 ([20]). *Let X be a uniformly smooth Banach space and $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define a mapping $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D.*

Lemma 2.3 ([21], Lemma 3.1). *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+, \{\alpha_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be the sequences such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n + c_n$$

for all $n \geq 1$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:
(1) *If $b_n \leq \alpha_n M$ where $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(2) *If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4 ([22]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n$$

for all $n \geq 1$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\tau_n\}$ and $\{\rho_n\}$ are real sequences such that
(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\lim_{n\to\infty} \rho_n = 0$;
(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([15], Lemm 3.1). *For any $r > 0$, if*

$$T_r := J_r^B(I - rA) = (I + rB)^{-1}(I - rAx),$$

then $Fix(T_r) = (A+B)^{-1}(0)$.

Lemma 2.6 ([15], Lemma 3.2). *For any $s \in (0, r]$ and $x \in X$, we have*

$$\|x - T_s x\| \leq 2\|x - T_r x\|.$$

Lemma 2.7 ([15], Lemma 3.3). *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1,2]$. Assume that A is a single-valued α-isa of order q in X. Then, for any $s > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in \mathcal{B}_r$,*

$$\begin{aligned} \|T_r x - T_r y\|^q &\leq \|x-y\|^q - r(\alpha q - r^{q-1}\kappa_q)\|Ax - Ay\|^q \\ &- \phi_q(\|(I-J_r)(I-rA)x - (I-J_r)(I-rA)y\|), \end{aligned}$$

where κ_q is the q-uniform smoothness coefficient of X.

3 Main Results

We next prove the main result in this paper.

Theorem 3.1. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ be an m-accretive operator. Assume that $S = (A+B)^{-1}(0) \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme: $u, x_1 \in X$,*

$$\begin{aligned} z_n &= \alpha_n u + (1-\alpha_n)x_n \\ x_{n+1} &= J_{r_n}^B(z_n - r_n A z_n), \end{aligned} \tag{3.1}$$

for each $n \geq 1$, $J_{r_n}^B = (I + r_n B)^{-1}$, $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ converges strongly to a point $z = Q(u)$, where Q is the sunny nonexpansive retraction of X onto S.

Proof. Let $z = Q(u)$. Put $T_n = J_{r_n}^B(I - r_n A)$ for each $n \geq 1$. Then we have,

$$\begin{aligned} \|z_n - z\| &= \|\alpha_n(u-z) + (1-\alpha_n)(x_n - z)\| \\ &\leq \alpha_n\|u-z\| + (1-\alpha_n)\|x_n - z\|. \end{aligned}$$

So we obtain, by Lemma 2.5 and condition (b),

$$\begin{aligned} \|x_{n+1} - z\| &= \|J_{r_n}^B(z_n - r_n A z_n) - z\| \\ &= \|T_n z_n - z\| \\ &\leq \|z_n - z\| \\ &\leq \alpha_n\|u-z\| + (1-\alpha_n)\|x_n - z\|. \end{aligned}$$

By Lemma 2.3 gives that $\{x_n\}$ is bounded. Using (2.1), we have

$$\begin{aligned} \|z_n - z\|^q &= \|\alpha_n(u-z) + (1-\alpha_n)(x_n - z)\|^p \\ &\leq (1-\alpha_n)^q\|x_n - z\|^q + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\ &\leq (1-\alpha_n)\|x_n - z\|^q + q\alpha_n\langle u - z, J_q(z_n - z)\rangle. \end{aligned} \tag{3.2}$$

So, by Lemma 2.7 and (3.2), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^q &= \|J_{r_n}^B(z_n - r_n A z_n) - z\|^q \\ &= \|T_n z_n - z\|^q \\ &\leq \|z_n - z\|^q - r_n(\alpha q - r_n^{q-1}\kappa_q)\|Az_n - Az\|^q \\ &\quad -\phi_q(\|(I - J_{r_n}^B)(I - r_n A)z_n - (I - J_{r_n}^B)(I - r_n A)z\|) \end{aligned}$$

$$
\begin{aligned}
&= \|z_n - z\|^q - r_n(\alpha q - r_n^{q-1}\kappa_q)\|Az_n - Az\|^q \\
&\quad -\phi_q(\|z_n - T_n z_n - r_n Az_n + r_n Az\|) \\
&\leq (1-\alpha_n)\|x_n - z\|^q + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&\quad -r_n(\alpha q - r_n^{q-1}\kappa_q)\|Az_n - Az\|^q \\
&\quad -\phi_q(\|z_n - T_n z_n - r_n Az_n + r_n Az\|). \qquad (3.3)
\end{aligned}
$$

By condition (b), we see that $r_n(\alpha q - r_n^{q-1}\kappa_q)$ is positive. Then, by (3.3), it follows that

$$
\|x_{n+1} - z\|^q \leq (1-\alpha_n)\|x_n - z\|^q + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \qquad (3.4)
$$

and also

$$
\begin{aligned}
\|x_{n+1} - z\|^q &\leq \|x_n - z\|^q + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&\quad -r_n(\alpha q - r_n^{q-1}\kappa_q)\|Az_n - Az\|^q \\
&\quad -\phi_q(\|z_n - T_n z_n - r_n Az_n + r_n Az\|). \qquad (3.5)
\end{aligned}
$$

For each $n \geq 1$, set

$$
\begin{aligned}
s_n &= \|x_n - z\|^q, \\
\gamma_n &= \alpha_n, \\
\tau_n &= q\langle u - z, J_q(z_n - z)\rangle, \\
\eta_n &= r_n(\alpha q - r_n^{q-1}\kappa_q)\|Az_n - Az\|^q \\
&\quad +\phi_q(\|z_n - T_n z_n - r_n Az_n + r_n Az\|), \\
\rho_n &= q\alpha_n\langle u - z, J_q(z_n - z)\rangle.
\end{aligned}
$$

From (3.4) and (3.5), it follows that

$$
s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n
$$

and

$$
s_{n+1} \leq s_n - \eta_n + \rho_n
$$

for each $n \geq 1$. We see that $\sum_{n=1}^{\infty}\gamma_n = \infty$. By the boundedness of $\{z_n\}$ and $\lim_{n\to\infty}\alpha_n = 0$, we see that $\lim_{n\to\infty}\rho_n = 0$.

In order to complete the proof, using Lemma 2.4, it remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$
\lim_{k\to\infty}\|Az_{n_k} - Az\| = \lim_{k\to\infty}\|z_{n_k} - r_{n_k}Az_{n_k} - T_{n_k}z_{n_k} + r_{n_k}Az\| = 0,
$$

which gives, by the triangle inequality, that

$$
\lim_{k\to\infty}\|T_{n_k}z_{n_k} - z_{n_k}\| = 0. \qquad (3.6)
$$

By the condition (b), there exists $\epsilon > 0$ such that $r_n \geq \epsilon$ for all $n > 0$. Lemma 2.6, we yields tht

$$\|T_\epsilon z_{n_k} - z_{n_k}\| \quad \leq \quad 2\|T_{n_k} z_{n_k} - z_{n_k}\|.$$

It follows from (3.6) and (3.7) that

$$\limsup_{k\to\infty} \|T_\epsilon z_{n_k} - z_{n_k}\| \leq 2 \limsup_{k\to\infty} \|T_{n_k} z_{n_k} - z_{n_k}\| = 0.$$

We conclude that

$$\lim_{k\to\infty} \|T_\epsilon z_{n_k} - z_{n_k}\| = 0. \tag{3.7}$$

Since $\|x_{n_k} - z_{n_k}\| \to 0$, we have $\|T_\epsilon x_{n_k} - x_{n_k}\| \to 0$. Let $z_t = tu + (1-t)T_\epsilon z_t$ for any $t \in (0,1)$. Employing Theorem 2.2, we have $z_t \to Qu = z$ as $t \to 0$. So we obtain

$$\begin{aligned}
\|z_t - z_{n_k}\|^q &= \|t(u - z_{n_k}) + (1-t)(T_\epsilon z_t - z_{n_k})\|^q \\
&\leq (1-t)^q \|T_\epsilon z_t - z_{n_k}\|^q + qt\langle u - z_{n_k}, J_q(z_t - z_{n_k})\rangle \\
&= (1-t)^q \|T_\epsilon z_t - z_{n_k}\|^q + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle \\
&\quad + qt\langle z_t - z_{n_k}, J_q(z_t - z_{n_k})\rangle \\
&= (1-t)^q \|T_\epsilon z_t - T_\epsilon z_{n_k} + T_\epsilon z_{n_k} - z_{n_k}\|^q \\
&\quad + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q \\
&\leq (1-t)^q \Big[\|T_\epsilon z_t - T_\epsilon z_{n_k}\| + \|T_\epsilon z_{n_k} - z_{n_k}\|\Big]^q \\
&\quad + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q \\
&\leq (1-t)^q \Big[\|z_t - z_{n_k}\| + \|T_\epsilon z_{n_k} - z_{n_k}\|\Big]^q \\
&\quad + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\langle z_t - u, J_q(z_t - z_{n_k})\rangle \\
\leq\ & \frac{(1-t)^q}{qt}\Big[\|z_t - z_{n_k}\| + \|T_\epsilon z_{n_k} - z_{n_k}\|\Big]^q + \frac{(qt-1)}{qt}\|z_t - z_{n_k}\|^q.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), it follows that

$$\begin{aligned}
&\limsup_{k\to\infty} \langle z_t - u, J_q(z_t - z_{n_k})\rangle \\
\leq\ & \limsup_{k\to\infty} \frac{(1-t)^q}{qt}\Big[\|z_t - z_{n_k}\| + \|T_\epsilon z_{n_k} - z_{n_k}\|\Big]^q + \limsup_{k\to\infty} \frac{(qt-1)}{qt}\|z_t - z_{n_k}\|^q \\
=\ & \frac{(1-t)^q}{qt} M^q + \frac{(qt-1)}{qt} M^q \\
=\ & \Big(\frac{(1-t)^q + qt - 1}{qt}\Big) M^q,
\end{aligned} \tag{3.9}$$

where $M = \limsup_{k\to\infty} \|z_t - z_{n_k}\|$, $t \in (0,1)$. We see that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Proposition 2.1 (2), we know that J_q is norm-to-norm uniformly continuous on bounded subset of X. Since $z_t \to z$ as $t \to 0$, we have

$$\|J_q(z_t - z_{n_k}) - J_q(z - z_{n_k})\| \to 0 \; as \; t \to 0.$$

On the other hand, we see that

$$\begin{aligned} & \Big|\langle z_t - u, J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big| \\ = & \Big|\langle (z_t - z) + (z - u), J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big| \\ \leq & \Big|\langle z_t - z, J_q(z_t - z_{n_k})\rangle\Big| + \Big|\langle z - u, J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big| \\ \leq & \|z_t - z\|\|z_t - z_{n_k}\|^{q-1} + \|z - u\|\|J_q(z_t - z_{n_k}) - J_q(z - z_{n_k})\|. \end{aligned}$$

So, as $t \to 0$, we get

$$\langle z_t - u, J_q(z_t - z_{n_k})\rangle \to \langle z - u, J_q(z - z_{n_k})\rangle. \tag{3.10}$$

From (3.9) and (3.10), as $t \to 0$, we see that

$$\limsup_{k\to\infty} \langle z - u, J_q(z - z_{n_k})\rangle \leq 0.$$

This shows that $\limsup_{k\to\infty} \tau_{n_k} \leq 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.4(c). Therefore $x_n \to z$ as $n \to \infty$. We thus completes the proof. □

We note that the following theorem can be proved in a similar fashion.

Theorem 3.2. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ be an m-accretive operator. Assume that $S = (A + B)^{-1}(0) \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme: $u, x_1 \in X$,*

$$\begin{aligned} z_n &= \alpha_n u + (1 - \alpha_n)x_n + e_n \\ x_{n+1} &= J^B_{r_n}(z_n - r_n A z_n + e_n), \end{aligned} \tag{3.11}$$

for each $n \geq 1$, $J^B_{r_n} = (I + r_n B)^{-1}$, $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$;

(c) $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z = Q(u)$, where Q is the sunny nonexpansive retraction of X onto S.

4 Applications and Numerical Examples

In this section, we apply Theorem 3.1 to the convex minimization problem. Let H be a real Hilbert space. Let $F : H \to \mathbb{R}$ be a convex smooth function and $G : H \to \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. We consider the problem of finding $\hat{x} \in H$ such that

$$F(\hat{x}) + G(\hat{x}) \leq F(x) + G(x) \tag{4.1}$$

for all $x \in H$. This problem (4.1) is equivalent, by Fermatfs rule, to the problem of finding $\hat{x} \in H$ such that

$$0 \in \nabla F(\hat{x}) + \partial G(\hat{x}), \tag{4.2}$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. In this point of view, we can set $A = \nabla F$ and $B = \partial G$ in Theorem 3.1. This is because if ∇F is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone and ∂G is maximal monotone. So we obtain the following result.

Theorem 4.1. *Let H be real Hilbert space. Let $F : H \to \mathbb{R}$ be a bounded linear operator with K-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function which $F + G$ attains a minimizer. Let $J_{r_n}^{\partial G} = (I + r_n \partial G)^{-1}$ and $\{x_n\}$ be a sequence generated by $u, x_1 \in H$ and*

$$\begin{aligned} z_n &= \alpha_n u + (1 - \alpha_n) x_n \\ x_{n+1} &= J_{r_n}^{\partial G}(z_n - r_n \nabla F(z_n)), \end{aligned} \tag{4.3}$$

for each $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, +\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2}{K}$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer of $F + G$.

Example 4.2. Solve the following minimization:

$$\min_{x\in\mathbb{R}^3} \|Ax + c\|_2 + \frac{1}{2}x^T x + d^T x + 9 \tag{4.4}$$

where

$$A = \begin{bmatrix} -3 & -5 & 3 \\ 1 & -1 & 7 \\ -3 & -2 & 4 \end{bmatrix}, \quad x = (y_1, y_2, y_3)^T, \quad c = (4, 2, 7)^T, \quad d = (1, 3, 9)^T.$$

For each $x \in \mathbb{R}^3$, we set $F(x) = \frac{1}{2}x^T x + d^T x + 9$ and $G(x) = \|Ax + c\|_2$. Then $\nabla F(x) = x + (1, 3, 9)^T$. We can check that F is convex and differentiable on $\mathbb{R}^3$ with 1-Lipschitz continuous gradient ∇F and G is convex and lower semi-continuous.

We choose $\alpha_n = \frac{1}{55n+1}$, $r_n = 0.2, x_1 = (1,-1,4)^T$ and $u = (-3,-1,0)^T$. We have that, for $r > 0$,

$$(I + r\partial G)^{-1}(x) = \begin{cases} \left(\frac{1-r}{\|x\|_2}\right)x, & \text{if } \|x\|_2 \geq r, \\ 0, & \text{otherwise.} \end{cases}$$

Using algorithm (4.3) in Theorem 4.1, we obtain the following numerical results:

n	x_n	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(1.000000, -1.000000, 4.000000)	69.536764	4.926816E+00
50	(-0.084228, -0.251688 , -0.754691)	19.180642	7.707892E-06
100	(-0.084044,-0.251638, -0.754729)	19.180992	1.908388E-06
150	(-0.083983, -0.251621, -0.754741)	19.181108	8.454559E-07
200	(-0.083953, -0.251613, -0.754747)	19.181167	4.748087E-07
250	(-0.083935, -0.251608, -0.754751)	19.181202	3.035865E-07
300	(-0.083923, -0.251605, -0.754753)	19.181225	2.106894E-07
⋮	⋮	⋮	⋮
800	(-0.083885, -0.251594, -0.754761)	19.181298	2.956922E-08
850	(-0.083884, -0.251594, -0.754761)	19.181301	2.619097E-08
900	(-0.083883, -0.251594, -0.754761)	19.181303	2.336024E-08
950	(-0.083882, -0.251594, -0.754761)	19.181305	2.096480E-08
1000	(-0.083881, -0.251593, -0.754761)	19.181307	1.891978E-08

Table 1. Numerical results of Example 4.4

From Table 1, we see that $x_{1000} = (-0.083881, -0.251593, -0.754761)$ is an approximation of the minimizer of $F + G$ with an error $1.891978E - 08$ and its minimum value is approximately 19.181307.

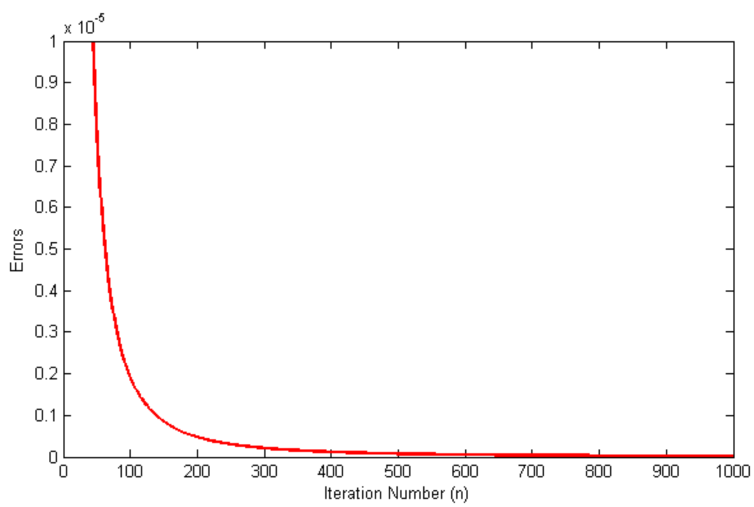

Figure 1. Errors of Example 4.4

Acknowledgement : The authors would like to thank University of Phayao for financial supports.

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(Received 2 May 2016)
(Accepted 20 May 2016)