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The Determinant of Graphs Joined by j Edges

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Abstract : In this paper, we refine and extend a result obtained in [1] to compute the determinant of a graph that is constructed by joining two graphs with two new non-coinciding undirected edges. Here, we join the two graphs with j edges and create a procedure to decompose the determinant of the combined graph. For that, we implement two graph operations: vertex deletion and directed graph handle. We demonstrate the obtained results with constructions that involve different basic (di)graphs: (di)paths, (di)cycles and generalized tournaments-special types of directed complete graphs.

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1 Introduction and Preliminaries

The study of the algebraic properties of graphs and in particular the computation of the determinant of a graph has proven to be a productive and useful exercise and gives us valuable information about the structure of the graph. As usual, by the *determinant of a graph* G with a *vertex set* $V(G) = \{1, 2, ..., m\}$, we understand the determinant of its *adjacency matrix* $A(G)$. Recall that the adjacency matrix of a graph is a square matrix of order the *order of the graph,* with the a_{ij} element of $A(G)$ equal to 1 when the ordered pair of vertices (i, j) is an element of the *edge set* $E(G)$ and $a_{ij} = 0$ otherwise. Clearly, if a graph G is undirected,

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$a_{ij} = a_{ji}$ and $a_{ij} = 1$ when the unordered pair of vertices $\{i, j\}$ is an element of $E(G)$. From this point on, we work with directed graphs (or digraphs) unless specifically stated otherwise. We denote the determinant of the adjacency matrix of a graph G with $\det(G)$ or $|G|$. When S is a set, $|S|$ denotes the cardinality of the set S. When not specified, all modular computations are modulo 4.

The following elementary graphs and their determinants are widely used in the demonstrations of the results.

A *directed path* $\vec{P}_m$, $m \geq 1$ is a digraph with m vertices, a vertex set $V(\vec{P}_m) = \{1, 2, ..., m\}$ and an edge set $E(\vec{P}_m) = \{(1, 2), (2, 3), ..., (m-1, m)\}$. Pictorially,

$$\overset{1}{\bullet} \to \overset{2}{\bullet} \to \cdots \to \overset{m}{\bullet}$$

A *directed cycle* $\vec{C}_m$, $m \geq 2$ is a digraph with m vertices, a vertex set $V(\vec{C}_m) = \{1, 2, ..., m\}$ and an edge set $E(\vec{C}_m) = \{(1, 2), (2, 3), ..., (m-1, m), (m, 1)\}$.

Clearly, $\det(\vec{P}_m) = 0$ and for the determinant of $\vec{C}_m$ we have

$$\det(\vec{C}_m) = (-1)^{m-1}. \tag{1.1}$$

In [2], Harary introduced a general technique for the computation of determinant of graphs.

Definition 1.1. A *spanning subgraph* S of a digraph (or graph) G is a subgraph such that $V(S) = V(G)$.

Definition 1.2. A *directed linear subgraph* D_i of a digraph G is a spanning subgraph of G such that D_i is a disjoint union of directed cycles.

Lemma 1.3 ([2]). *Let G be a digraph of order m and D_i denote the n directed linear subgraphs of G. Then,*

$$\det(G) = \sum_{i=1}^{n} \det(D_i).$$

From the above Lemma it follows directly, that the determinant of a disjoint union of graphs is the product of the determinants of its components.

A *path graph* P_m, $m \geq 1$ is a undirected graph with m vertices, vertex set $V(P_m) = \{1, 2, ..., m\}$ and an edge set $E(P_m) = \{\{1, 2\}, \{2, 3\}, ..., \{m-1, m\}\}$.

The length of a path is the number of edges of the path.

A *path in a graph* G is a sequence of distinct vertices, such that adjacent vertices in the sequence are adjacent in the graph. The *distance* between two vertices u and v from $V(G)$, denoted $d(u, v)$, is the length of the shortest path in G that starts on u and ends on v, also called the $u - v$ *geodesic.*

A *cycle graph* C_m, $m \geq 3$ is a undirected graph with m vertices, vertex set $V(C_m) = \{1, 2, ..., m\}$ and an edge set $E(C_m) = \{\{1, 2\}, \{2, 3\}, ..., \{m-1, m\}, \{m, 1\}\}$.

The determinants of these graphs are as follows, where all modulo computations are modulo 4: (see [1])

$$\det(P_m) = \begin{cases} 1, & m \equiv 0, \\ 0, & m \equiv 1,3, \\ -1, & m \equiv 2. \end{cases} \qquad \det(C_m) = \begin{cases} 0, & m \equiv 0, \\ 2, & m \equiv 1,3, \\ -4, & m \equiv 2. \end{cases} \tag{1.2}$$

The determinant of *the complete graph* K_m, $m \geq 1$, that is a undirected graph with m vertices any two of which are connected, is given by

$$\det(K_m) = (-1)^{m-1}(m-1). \tag{1.3}$$

Recall further that a graph where every pair of distinct vertices is connected by a single directed edge is called *a tournament* [3]. In the context of tournaments, the vertices represent players (actors) and a directed edge represents the outcome of the game with the directed edge pointing from a winner to a loser. Here, we define *a generalized tournament* to be a graph where every pair of distinct vertices is connected by a single directed or undirected edge. Since each undirected edge can be considered as a pair of directed edges pointing in opposite directions, we extend the context of tournaments by allowing an undirected edge (or double arrow edges) to represent a draw for the game between the connected players.

Let $\vec{K}_m^w$ denote a generalized tournament with *a sole winner,* that is a generalized tournament in which the player w wins against all other players, while all other players draw among themselves.

Similarly, let $\vec{K}_m^{(w_1,w_2,\ldots,w_k)}$ with $k \leq m$, denote a generalized tournament with *a chain of winners,* that is a tournament where w_1 beats all other players, w_2 beats all others but w_1 and so on, with all vertices not listed in the winners chain drawing among each other.

By analogy, let $\vec{K}_m^l$ denote a generalized tournament with *a sole loser,* and $\vec{K}_m^{(l_1,l_2,\ldots,l_k)}$ with $k \leq m$, denote a generalized tournament with *a chain of losers.*

Clearly, for the classes of generalized tournaments above if $k = m$ we obtain *transitive directed complete graphs*, that is - transitive tournaments. Furthermore, the determinants for all winners' (respectively losers') classes are 0, since the adjacency matrix contains a zero column (row).

2 Graph Operations

We solve the underlying problem of this work, the determinant of graphs joined by j edges, with the help of two "editing" operations on graphs: vertex deletion and appending directed graph handles.

2.1 Vertex Deletion

Definition 2.1. (Vertex Deletion) For a graph G and a vertex $v \in V(G)$ we denote by $G \backslash v$ the subgraph of G obtained by removing the vertex v from $V(G)$

and all edges that are incident with v from the $E(G)$.

Further, if W is a subset of vertices of G, we denote by $G\backslash W$ the subgraph of G obtained by deleting all vertices in W from G.

Clearly, for any graph G, the determinant of $G\backslash i$, is equal to the minor $A_{ii}(G)$, obtained by removing the ith row and column from $A(G)$.

Example 2.2. Let $W \subset V(G)$ be proper non-empty subset of vertices of a graph G. For a complete graph K_m, the determinant of $K_m\backslash W$ is equal to the determinant of the complete graph $K_{m-|W|}$. For a cycle graph with a single vertex removed we have $\det(C_m\backslash v) = \det(P_{m-1})$ and $\det(C_m\backslash W)$ equals the product of the determinants of the disjoint paths that remain after the deletion of the vertices in W from C_m. Furthermore, $\det(\vec{C}_m\backslash W) = 0$ and $\det(\vec{K}_m^w\backslash W)$ is $(-1)^{m-|W|-1}(m-|W|-1)$ if $w \in W$ and 0 otherwise.

2.2 Directed Graph Handles

Next we define a new graph editing operation, appending directed graph handles, which, in the context of the determinant of a graph, is a generalization of vertex deletion.

Definition 2.3. (Directed Graph Handle) For a graph G of order m and vertices $u, v \in V(G)$, we denote $G_{(u,v)}$ to be the graph where a new vertex w is added to $V(G)$ and a directed edge from vertex u to vertex w and a directed edge from vertex w to vertex v are added to $E(G)$. Vertex w is called *the directed graph handle vertex of the directed graph handle* (u, v).

Further, if B is a set of ordered pairs of elements of $V(G)$, we denote G_B to be the graph where for each $(u_i, v_i) \in B$ a new directed graph handle is appended to G.

The following results for directed graph handles on arbitrary graphs are straightforward to obtain, so we list them without a proof.

Lemma 2.4. *Let G be a graph and $i \in V(G)$ be any vertex of G, with $|V(G)| > 1$. Then*

$$\det(G_{(i,i)}) = -\det(G\backslash i).$$

Further,

Lemma 2.5. *Let G be a graph and $B = \{(u_i, v_i) \mid u_i, v_i \in V(G)\}$. If there exists $(u_i, v_i), (u_j, v_j) \in B$ such that $u_i = u_j$ or $v_i = v_j$ for some $i \neq j$ then*

$$\det(G_B) = 0.$$

By the symmetry of the adjacency matrix of an undirected graph we have the following lemma.

Lemma 2.6. *Let G be an undirected graph and let $i, j \in V(G)$. Then*

$$\det(G_{(i,j)}) = \det(G_{(j,i)}).$$

Example 2.7. Recall that the determinant of a graph G is the sum of the determinants of the directed linear subgraphs of G. (See Lemma 1.3).

The only way to obtain a directed linear subgraph in $(\vec{P}_m)_{(i,j)}$ is when $i = m$ and $j = 1$. The only directed linear subgraph in that case is isomorphic to $\vec{C}_{m+1}$, thus

$$\det((\vec{P}_m)_{(i,j)}) = \begin{cases} (-1)^m, & j = 1, i = m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

In the case of $(\vec{C}_m)_{(i,j)}$ clearly, if the distance $d(i,j)$ is not equal to 1, a directed linear subgraph does not exist, and in the case of $d(i,j) = 1$ the only existing directed linear subgraph is isomorphic to $\vec{C}_{m+1}$. Thus

$$\det((\vec{C}_m)_{(i,j)}) = \begin{cases} (-1)^m, & d(i,j) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Next, let us consider a directed graph handle on a generalized tournament with a sole winner.

Clearly, for $(\vec{K}_m^w)_{(i,j)}$ if $j \neq w$ then $|(\vec{K}_m^w)_{(i,j)}| = 0$, since no directed cycle subgraph of $(\vec{K}_m^w)_{(i,j)}$ can contain w and thus, a directed linear subgraph does not exist.

If $i = j = w$ then $|(\vec{K}_m^w)_{(w,w)}| = -|K_{m-1}|$, from Lemma 2.4. Finally, if $j = w$ and $i \neq w$, a simple adjacency matrix argument yields that $\det(\vec{K}_m^w)_{(i,j)} = (-1)^m$.

Thus, we obtain

$$\det((\vec{K}_m^w)_{(i,j)}) = \begin{cases} (-1)^m, & i \neq w,\ j = w, \\ (-1)^{m-1}(m-2), & i = j = w, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that some of the arguments made above are easier to obtain using the adjacency matrix and Lemma 2.4 instead of the directed linear subgraphs. Putting the two arguments together, however, yields a nontrivial combinatorial result. In particular, a directed cycle subgraph that contains w, must contain the directed graph handle vertex and the vertex i. Thus the determinant of $(\vec{K}_m^w)_{(i,j)}$ in this case is a sum of the form

$$\begin{aligned} |(\vec{K}_m^w)_{(i,j)}| = {} & P(m-2,0)|\vec{C}_3||K_{m-2}| + P(m-2,1)|\vec{C}_4||K_{m-3}| + \cdots + \\ + {} & P(m-2,m-4)|\vec{C}_{m-1}||K_2| + P(m-2,m-2)|\vec{C}_{m+1}|, \end{aligned} \tag{2.4}$$

where $P(n,r)$ denotes the number of permutations of r elements from a set of n elements.

Substituting (1.1), (1.3) and (2.3) in (2.4) we get

$$(-1)^m = \sum_{i=0}^{m-4} [P(m-2,i)(-1)^{i+2}(-1)^{m-i-3}(m-i-3)] + (-1)^m P(m-2,m-2)$$

and thus the nontrivial combinatorics identity:

$$n! - \sum_{i=0}^{n-1} \frac{n!}{(n-i)!}(n-i-1) = 1.$$

Now, let us consider directed handles appended to some undirected graphs.

Example 2.8. For the case of computing the determinant of $(K_m)_{(i,j)}$ observe that any directed cycle that contains the vertex j must also contain the directed handle vertex $m+1$ and thus will only have an outgoing edge from j towards the vertices of K_m. Thus, computing $\det((K_m)_{(i,j)})$ is equivalent to computing $\det((\vec{K}^w_m)_{(i,j)})$ with $w = j$, that is

$$\det((K_m)_{(i,j)}) = \begin{cases} (-1)^{m-1}(m-2), & i = j, \\ (-1)^m, & \text{otherwise.} \end{cases} \tag{2.5}$$

Next, consider a path graph with directed handle (i,j). Without loss of generality let us assume that $i < j$. (See Lemma 2.6). Note further, that the edges $\{i-1, i\}$ and $\{j, j+1\}$, for $i > 1$ and $j < m$, can never be part of a dicycle of a directed linear subgraph of $(P_m)_{(i,j)}$, thus they can be deleted, which yields a three component subgraph of $(P_m)_{(i,j)}$ whose determinant is equal to the determinant of $(P_m)_{(i,j)}$. That is, $|(P_m)_{(i,j)}| = |P_{i-1}||\overrightarrow{C}_{j-i+2}||P_{m-j}|$ that yields the following result, where $d = j - i = d(i,j)$.

$$\det((P_m)_{(i,j)}) = \begin{cases} 1, & i \text{ odd and } d+m \equiv 3, \\ -1, & i \text{ odd and } d+m \equiv 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

A direct check shows that the formula also holds when $i = 1$, $j = m$ and when $i = j$.

As a final example, for the determinant of a cycle graph C_m with a directed handle (i,j) observe that when $i \neq j$ any directed linear subgraph of $(C_m)_{(i,j)}$ must contain one of two possible dicycles through the directed graph handle vertex w : the dicycle through w that contains the $i-j$ geodesic or the dicycle through w that contains the complementary $i-j$ path. Clearly, those cycles are isomorphic to $\overrightarrow{C}_{d+2}$ and $\overrightarrow{C}_{m-(d-1)+1}$, where $d = d(i,j)$. Thus,

$$|(C_m)_{(i,j)}| = |\overrightarrow{C}_{d+2}||P_{m-(d+1)}| + |\overrightarrow{C}_{m-(d-1)+1}||P_{d-1}|.$$

Substituting (1.1) and (1.2) in the above equation yields the following result:

$$\det((C_m)_{(i,j)}) = \begin{cases} 0, & m \equiv 0 \text{ or } (m \equiv 2 \text{ and } d \equiv 0, 2), \\ 1, & (m \equiv 1 \text{ and } d \equiv 0, 1) \text{ or } (m \equiv 3 \text{ and } d \equiv 1, 2), \\ -1, & (m \equiv 1 \text{ and } d \equiv 2, 3) \text{ or } (m \equiv 3 \text{ and } d \equiv 0, 3), \\ 2, & m \equiv 2 \text{ and } d \equiv 1 \\ -2, & m \equiv 2 \text{ and } d \equiv 3. \end{cases} \tag{2.7}$$

Modifying the directed linear subgraphs argument or using Lemma 2.4 shows the the formula holds and in the case $i = j$.

The following lemma shows that more than one directed graph handle on a complete graph yields a zero determinant graph as long as the end of one directed graph handle is not the beginning of another.

Lemma 2.9. *Let B be a collection of directed graph handles on K_m such that $|B| \geq 2$ and for any $(u_i, v_i), (u_j, v_j) \in B$, $v_i \neq u_j$. Then,*

$$\det((K_m)_B) = 0.$$

Proof. If $u_i = u_j$ or $v_i = v_j$ for any $(u_i, v_i), (u_j, v_j) \in B$, then $|(K_m)_B| = 0$ from Lemma 2.5. Otherwise, expanding twice along row $m + 1$ from the adjacency matrix of $(K_m)_B$, gives a minor with two equal rows, because $v_i \neq u_j$ for any $(u_i, v_i), (u_j, v_j) \in B$. $\square$

3 Main Results

We move to the main goal of the work - creating a procedure to calculate the determinant of a graph that is a joint of two graphs with j new non-coinciding undirected edges. We develop the procedure by modifying the Laplace expansion formula with the help of the previously defined graph operations: vertex deletion and appending of directed graph handle.

For the purpose of the discussion going forward let us introduce the following notations.

For a square matrix A of order $n \times n$ and k-tuples $\mathbf{r} = (r_1, r_2, ..., r_k)$, $\mathbf{c} = (c_1, c_2, ..., c_k)$ with $1 \leq k < n$, $1 \leq r_1 < r_2 \cdots < r_k \leq n$, $1 \leq c_1 < c_2 \cdots < c_k \leq n$ let $S(A; \mathbf{r}, \mathbf{c})$ denote the submatrix of A obtained by selecting the rows indicated in $\mathbf{r}$ and the columns indicated in $\mathbf{c}$. Let $S^*(A; \mathbf{r}, \mathbf{c})$ denote the submatrix of A obtained by deleting the rows indicated in $\mathbf{r}$ and the columns indicated in $\mathbf{c}$. Finally, for a k-tuple $\mathbf{a} = (a_1, a_2, ..., a_k)$, let $\varepsilon(\mathbf{a}) = \sum_{i=1}^{k} a_i$ and for an integer n let $\varepsilon(n) = \sum_{i=1}^{n} i$.

Recall the following result, attributed to Laplace.

Theorem 3.1 ([4]). (**Laplace Expansion Formula**)
Let A be an $n \times n$ matrix and let $\boldsymbol{r} = (r_1, r_2, ..., r_k)$ be k-tuples of row indices, where $1 \leq k \leq n$ and $1 \leq r_1 < r_2 \cdots < r_k \leq n$. Then

$$\det(A) = (-1)^{\varepsilon(\boldsymbol{r})} \sum_{c} (-1)^{\varepsilon(\boldsymbol{c})} |S(A; \boldsymbol{r}, \boldsymbol{c})||S^*(A; \boldsymbol{r}, \boldsymbol{c})|$$

where the summation is over all k-tuples $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ for which $1 \leq c_1 < c_2 \cdots < c_k \leq n$.

Next, observe that the determinant of a graph does not depend of the labeling of the vertices, since a relabeling yields a graph isomorphic to the original one. So, the following definition is not restrictive in the ways two graphs can be joined by j new vertices, but rather creates structure and removes the ambiguity from the notation $G \overset{j}{\asymp} H$.

Definition 3.2. Let G and H be two graphs with vertex sets $V(G) = \{1, ..., m\}$ and $V(H) = \{m+1, ..., m+n\}$ respectively and let $j \leq \min(m, n)$. We define $G \overset{j}{\asymp} H$ to be the graph formed by joining “the last” j distinct vertices of G (by natural order) with ”the first” j distinct vertices of H, so that the edge set of $G \overset{j}{\asymp} H$ is $E(G \overset{j}{\asymp} H) = E(G) \cup E(H) \cup \{\{m+1-i, m+i\} \mid 1 \leq i \leq j\}$ and the vertex set is $V(G \overset{j}{\asymp} H) = V(G) \cup V(H)$. We call the pair of connected vertices $\{m+1-i, m+i\}$ mirror of each other.

From the definition above, the adjacency matrix of $G \overset{j}{\asymp} H$ is a block matrix of the form:

$$A(G \overset{j}{\asymp} H) = \left(\begin{array}{ccc|ccc} & & & 0 & \iddots & \\ & G & & & 1 & 0 \\ & & & 1 & & \\ \hline 0 & & 1 & & & \\ & 1 & & & & \\ \iddots & & & & H & \\ & & 0 & & & \end{array}\right).$$

Consider the Laplace expansion on the block matrix $A(G \overset{j}{\asymp} H)$ above, with $\mathbf{r} = (1, ..., m)$.

By construction, the (possible) non-zero summands in the expansion are those with $\mathbf{c} = (1, ..., m-j, p_1, ..., p_j)$, where $(p_1, ..., p_j)$ is an ordered j-tuple with $p_1 < p_2 < \cdots < p_j$, chosen from the set $J = \{m-j+1, ..., m, m+1, ..., m+j\}$. Observe, that the set J lists all vertices from G and H that are connected, that is $|J| = 2j$ and thus, there are at most $\binom{2j}{j}$ non-zero minors. Observe further, that the (possible) non-zero minors in the expansion fall in one of the following four categories:

Case I. In the case when $(p_1, ..., p_j) = (m-j+1, ..., m)$ (that is all vertices of G are selected) the submatrix $S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ is the adjacency matrix of G and $S^*(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ is the adjacency matrix of H. Thus, the summand in the Laplace expansion is $(-1)^{\varepsilon(\mathbf{r})+\varepsilon(\mathbf{c})} \det(G)\det(H) = \det(G)\det(H)$ and so knowledge of the determinants of the individual graphs G and H is required.

Case II. In the case when $(p_1, ..., p_j)$ is in the form where for every missing vertex from G the mirror vertex from H is present, (for example $(m-j+1, ..., m-1, m+1)$ or $(m-j+1, ..., m-2, m, m+2)$ and so on) then the determinant of $S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ is equal (up to a sign) to the determinant of $G \setminus R$ where R is the set of the missing vertices of G in $(p_1, ..., p_j)$ and the determinant of $S^*(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ is equal (up to a sign) to $|H \setminus R^*|$, where R^* is the set of the added vertices of H in $(p_1, ..., p_j)$. Thus, in these cases the computations of the minors can be performed using the vertex deletion operation. Clearly, there are $2^j - 1$ such minors in the Laplace expansion and we can calculate the sets R and R^* explicitly from $(p_1, ..., p_j)$.

Case III. In the case when $(p_1, ..., p_j)$ is in the form where for every missing vertex from G the mirror vertex from H is also not present (for example $(m-j+1, ..., m-1, m+2)$ or $(m-j+1, ..., m-2, m+3, m+4)$ and so on), then the determinant of $S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ is equal (up to a sign) to the determinant of G with directed graph handles appended in the following order: the smallest missing vertex from G is the end of a handle that starts at the mirror of the largest vertex that is added from H, and so on following the natural order.

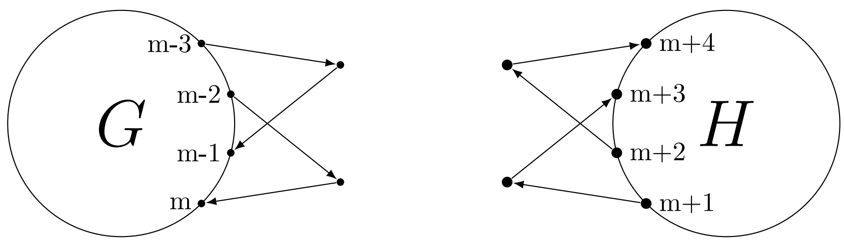
To demonstrate this finding, consider $\mathbf{c} = (1, ..., m-j, m-j+1, ..., m-1, m+2)$, that is, a choice in which the only vertex from G that is not present in $\mathbf{c}$ is m and the substitute from H is $m+2$, which is not a mirror of m. In this case the determinant of $S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ is equal (up to sign) to the determinant of $G_{(m-1,m)}$. The handle starts at $m-1$ – the mirror of $m+2$ and ends at m.

Similarly, in the case of $\mathbf{c} = (1, ..., m-j, m-j+1, ..., m-2, m+3, m+4)$, the computation of the determinant of $S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ can be obtained (up to a sign), through the computation of the determinant of G_B where B is a two handles set, $B = \{(m-2, m), (m-3, m-1)\}$. That is, the handles are: a handle from $m-3$ (the mirror of $m+4$) to $m-1$ and a handle from $m-2$ to m.

Following the process described above, the submatrix $S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})$ can be naturally build up to the adjacency matrix of G_B by adding to $A(G)$ two rows with only non-zero entries corresponding to the missing elements of G in $(p_1, ..., p_j)$ and two columns with only non-zero entries corresponding to the mirrors of the added vertices of H in $(p_1, ..., p_j)$. Clearly, the determinants of the two matrices are equal (up to a sign). In particular,

$$\begin{aligned}|S(G \overset{j}{\asymp} H;\mathbf{r},\mathbf{c})| &= -|G_{\{(m-2,m),(m-3,m-1)\}}| \\ &= -\left|\begin{array}{ccccccc} & & & & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & G & & & & \\ & & & & & 0 & 0 \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ 0 & \cdots & & 0 & 1 & 0 & 0 \\ 0 & \cdots & & 1 & 0 & 0 & 0 \end{array}\right|\end{aligned}$$

The appended graphs G_B and H_{B^*} in this case are:



Similar build up can be done for the symmetric submatrix $S^*(G \overset{j}{\asymp} H;\mathbf{r},\mathbf{c})$ to the adjacency matrix of H_{B^*}. Thus, in these cases the computations of the minors can be performed using the directed graph handle operation.

Finally, note that there are $\displaystyle\sum_{i=1}^{\lfloor j/2 \rfloor} \binom{j}{i} \cdot \binom{j-i}{i}$ minors in the Laplace expand of this type.

Remark 3.1. Clearly, the build up of $S(G \overset{j}{\asymp} H;\mathbf{r},\mathbf{c})$ to that of an adjacency matrix of G with appended handles can be done in more than one way, with any permutation of the added rows and columns leading to equal up to sign determinants. Thus, for consistency we impose the build up of $S(G \overset{j}{\asymp} H;\mathbf{r},\mathbf{c})$ to the matrix of G with directed graph handles appended in the following order: the smallest missing vertex from G is the end of a handle that starts at the mirror of the largest vertex that is added from H, and so on following the natural order.

Case IV. The remaining minors in the Laplace expansion, there are $\displaystyle\binom{2j}{j} - [2^j + \sum_{i=1}^{\lfloor j/2 \rfloor} \binom{j}{i} \cdot \binom{j-i}{i}]$ of those, are of type that is a combination of the cases II and III, thus they can be computed through the computation of the determinant of an adjacency matrix of a graph obtained from G (respectively H) with both operations - vertex deletion and directed handles simultaneously implemented.

Thus the computation of the determinant of $A(G \overset{j}{\asymp} H)$ boils down to the computation of the determinants of G, H and the different sub-determinants of G and

H obtained by vertex deletion, directed graph handle appending or combination of the two.

To formally compute the sets of the removed vertices R and the set of the appended handles B for G and respectively R^* and B^* for H in each of the Laplace expansion terms let us introduce the following notations.

Recall that for $A(G \overset{j}{\asymp} H)$ the set $J = \{m-j+1, ..., m, m+1, ..., m+j\}$ lists all vertices from G and H that are connected. Let $J_G = \{m-j+1, ..., m\}$ be the set of vertices from G and $J_H = \{m+1, ..., m+j\}$ the set of vertices from H.

Further, let X^* be *the set of vertices of* H *that are in* $(p_1, ..., p_j)$, that is, $X^* = \{p_1, ..., p_j\} \cap J_H$ and let X be *the set of the mirror vertices of the vertices in* X^**,* that is, $X = \{2m+1-y_i \mid y_i \in X^*\}$. Finally, let Y be *the set of vertices of* G *that are not in* $(p_1, ..., p_j)$, that is $Y = J_G \backslash \{p_1, ..., p_j\}$. Then, **the set of removed vertices** of G is $R = X \cap Y$.

Note that $|X| = |Y|$ so **the set of directed graph handles appended to** G is $B = \{(x_i, y_i) \mid x_i > x_{i+1}, y_i > y_{i+1}, x_i \in X\backslash R, y_i \in Y\backslash R\}$.

A corresponding construction holds for the set of removed vertices R^* and the set of appended directed directed graph handles B^* on H. Thus, the set of removed vertices is $R^* = \{v \in V(H) \mid r \in R \text{ and } \{r, v\} \in E(G \overset{j}{\asymp} H)\}$, that is R^* is the set of the mirror vertices of those in R. And the set of appended directed graph handles is $B^* = \{(r^*, c^*) \mid r^*, c^* \in V(H),\ (c, r) \in B \text{ and } \{c, c^*\}, \{r, r^*\} \in E(G \overset{j}{\asymp} H)\}$.

Thus, we established the following lemma:

Lemma 3.3. *Let* G *and* H *be two graphs with vertex sets* $V(G) = \{1, ..., m\}$ *and* $V(H) = \{m+1, ..., m+n\}$ *respectively and let* $G \overset{j}{\asymp} H$ *be the joining of* G *and* H*. For a fixed* $\boldsymbol{r} = (1, ..., m)$ *and* $\boldsymbol{c} = (1, ..., m-j, p_1, ..., p_j)$ *the minor* $|S(G \overset{j}{\asymp} H; \boldsymbol{r}, \boldsymbol{c})|$ *is equal up to a sign to the determinant of the graph* $(G\backslash R)_B$*, where the sets* B *and* R *are as defined above.*

And similarly, the minor $|S^*(G \overset{j}{\asymp} H; \boldsymbol{r}, \boldsymbol{c})|$ *is equal up to a sign to the determinant of the graph* $(G\backslash R^*)_{B^*}$*.*

To address the sign issue, let us calculate the sign for each of the Laplace terms in the process of calculating the determinant of $A(G \overset{j}{\asymp} H)$.

Lemma 3.4. *Let* G *and* H *be two graphs with vertex sets* $V(G) = \{1, ..., m\}$ *and* $V(H) = \{m+1, ..., m+n\}$ *respectively. Let* $G \overset{j}{\asymp} H$ *be the joining of* G *and* H *via* j *new edges and let* $\boldsymbol{r} = (1, ..., m)$ *and* $\boldsymbol{c} = (1, ..., m-j, p_1, ..., p_j)$ *where* $p_1, ..., p_j$ *is an ordered j-tuple chosen from the set* $J = \{m-j+1, ..., m, m+1, ..., m+j\}$*. Then*

$$(-1)^{\varepsilon(\boldsymbol{r})}(-1)^{\varepsilon(\boldsymbol{c})}|S(G \overset{j}{\asymp} H; \boldsymbol{r}, \boldsymbol{c})||S^*(G \overset{j}{\asymp} H; \boldsymbol{r}, \boldsymbol{c})| = (-1)^{|R|+|B|}|(G\backslash R)_B||(H\backslash R^*)_{B^*}|.$$

Proof. Consider the graph $\bar{G}_B$, where $\bar{G}_B$ is the graph G_B with appended directed graph handles from r_i to itself for every r_i in R.

By Lemma 2.4, removing the vertices r_i of $\bar{G}_B$ does not change its determinant up to sign. So we have $|\bar{G}_B| = (-1)^{|R|}|\bar{G}_B\backslash R| = (-1)^{|R|}|(G\backslash R)_B|$.

Expanding the adjacency matrix $A(\bar{G}_B)$ along the $m+1$ row $|Y|$ times gives us

$$|\bar{G}_B| = (-1)^{|Y|(m+1)}(-1)^{\varepsilon(Y)}|S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})|,$$

where $\varepsilon(Y)$ denotes the sum of the elements of the set Y.

Thus,

$$|S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})| = (-1)^{\varepsilon(Y)+|Y|(m+1)+|R|}|(G\backslash R)_B|. \tag{3.1}$$

Following a similar approach as above we get,

$$|S^*(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})| = (-1)^{\varepsilon(X^*)+|Y|(1+|Y|-m)+|R^*|}|(H\backslash R^*)_{B^*}|. \tag{3.2}$$

Recall that the m-tuple $\mathbf{c}$ from the Laplace expansion lists the columns selected from $A(G \overset{j}{\asymp} H)$, where the columns omitted from G are listed in Y and the columns added from H are listed in X^*. Thus, $\varepsilon(\mathbf{c}) = \varepsilon(m) - \varepsilon(Y) + \varepsilon(X^*)$ and therefore the sign in the Laplace expansion corresponding to that $\mathbf{c}$ is

$$(-1)^{\varepsilon(\mathbf{r})}(-1)^{\varepsilon(\mathbf{c})} = (-1)^{\varepsilon(m)}(-1)^{\varepsilon(m)-\varepsilon(Y)+\varepsilon(X^*)} \tag{3.3}$$

Combining the sign changes (3.1),(3.2),(3.3) and using the fact that $|Y| = |R| + |B|$ we have

$$(-1)^{|Y|^2} = (-1)^{(|R|+|B|)^2} = (-1)^{|R|+|B|},$$

which proves that for any term in the Laplace transform we have
$(-1)^{\varepsilon(\mathbf{r})}(-1)^{\varepsilon(\mathbf{c})}|S(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})||S^*(G \overset{j}{\asymp} H; \mathbf{r},\mathbf{c})| = (-1)^{|R|+|B|}|(G\backslash R)_B||(H\backslash R^*)_{B^*}|$.
□

We can state now the main theorem of our paper, which follows directly from Lemma 3.4 above.

Theorem 3.5. *Let G and H be graphs of order m and n respectively and let J_G be the set of vertices of G that are joined to H with j undirected edges. Then,*

$$\det(G \overset{j}{\asymp} H) = \sum_{R\subseteq J_G} \sum_{B} (-1)^{|R|+|B|} \det((G\backslash R)_B) \det((H\backslash R^*)_{B^*}),$$

where the inner summation is over all possible sets of directed graph handles B that can be made from $J\backslash R$ such that if $(x_i, y_i), (x_j, y_j) \in B$ and $y_i > y_j$ then $x_i > x_j$.

Next, we demonstrate the theorem by computing the determinant of several graph joinings.

Corollary 3.6. *Let G be any digraph and let K_m be a complete graph with $m \geq 0$. Let $J \subset V(G)$ be the set of vertices that are joined to K_m. Then,*

$$\det(G \overset{j}{\asymp} K_m) =$$

$$(-1)^{m-1} \left(\sum_{R \subseteq J} (m - |R| - 1)|G \backslash R| + \sum_{\substack{R \subseteq J \\ i,j \in J \backslash R}} |(G \backslash R)_{(i,j)}| \right).$$

Proof. From Lemma 2.9, the sum from Theorem 3.5 reduces to the sum of terms where either only vertices are removed from the graphs, in which case $B = \emptyset$, or at most one directed graph handle is appended and possibly other vertices are removed. Recall that removing a vertex from K_m produces K_{m-1} and thus using (1.3) and (2.5)we have

$$\det(G \overset{j}{\asymp} K_m) = \sum_{R \subseteq J} \sum_{B} (-1)^{|R|+|B|} |(G \backslash R)_B||(K_m \backslash R^*)_{B^*}|$$

$$= \sum_{R \subseteq J} (-1)^{|R|}(m - |R| - 1)(-1)^{m-|R|-1}|(G \backslash R)| +$$

$$\sum_{\substack{R \subseteq J \\ i,j \in J \backslash R}} (-1)^{|R|+1}(-1)^{m-|R|}|(G \backslash R)_{(i,j)}|$$

which simplifies to the conclusion. □

In particular, when we join K_m with K_n with j new edges we get:

Corollary 3.7.

$$\det(K_m \overset{j}{\asymp} K_n) = 0 \textit{ for } j \geq 2.$$

Proof. From Lemma 2.9 it follows we only need to consider one directed graph handle. For the case $j = 2$ by Theorem 3.5 the determinant of the joined graph simplifies to $|K_m||K_n| - |(K_m)_{(m,m-1)}||(K_n)_{(m+2,m+1)}| - |(K_m)_{(m-1,m)}||(K_n)_{(m+1,m+2)}| - |K_m \backslash \{m\}||K_n \backslash \{m+1\}| - |K_m \backslash \{m-1\}||K_n \backslash \{m+2\}| + |K_m \backslash \{m, m-1\}||K_n \backslash \{m+1, m+2\}|$=0 using (1.3) and (2.5).

For $j > 2$, using Theorem 3.5 we have

$$\det(K_m \overset{j}{\asymp} K_n) = \sum_{R \subseteq J} (-1)^{|R|}|K_m \backslash R||K_n \backslash R^*| +$$

$$+ \sum_{|R| \leq j-2, x \neq y} (-1)^{|R|+1}|(K_m \backslash R)_{(x,y)}||(K_n \backslash R^*)_{(x^*,y^*)}|$$

$$= \sum_{R} (-1)^{|R|}(m-|R|-1)(n-|R|-1)(-1)^{m-|R|-1}(-1)^{n-|R|-1}$$

$$+ \sum_{|R| \le j-2} (-1)^{|R|+1}(-1)^{m-|R|}(-1)^{n-|R|}.$$

Our second sum is only for $|R| \le j-2$, because otherwise a directed graph handle cannot be made. If $|R| = i$, then there are $\binom{j}{i}$ ways to do this. Also, given $j - |R|$ vertices, there are $2\binom{j-|R|}{2}$ ways to make one directed graph handle. Then

$$\sum_{i=0}^{j} \binom{j}{i} (-1)^i (m-i-1)(n-i-1)(-1)^{m+n}$$

$$- \sum_{i=0}^{j} \binom{j}{i} 2 \binom{j-i}{2} (-1)^i (-1)^{m+n} = 0,$$

where we use the combinatorial identity

$$\sum_{i=0}^{j} \binom{j}{i} (-1)^i (i)^p = 0$$

that is true for any non-negative integer $p < j$. We see that the highest power of i in our sum is 2 thus the formula applies. □

The following corollaries demonstrate how quick calculations for the determinant can be done by looking at the geometry of the joined graphs in terms of vertex deletions and directed handle appended. Further, series of results can be easily obtained imposing different restriction on the sets of vertices J_G and J_H that are connected from G and H.

Corollary 3.8. *Let J_G be the set of vertices of G that are connected to H and let $d(v_1, v_2)$ be even for any v_1, v_2 in J_G. Then*

$$\det(C_{2m} \overset{j}{\asymp} K_n) = \begin{cases} 0, & m \equiv 0, 2, \\ 4(n-1)(-1)^n, & m \equiv 1, 3. \end{cases}$$

Proof. From 2.7, $|(C_{2m})_{(x,y)}| = 0$ since d is even and $2m$ is even. Removing any number of points gives a disjoint union of odd paths. We only need to consider appending one directed graph handle, and any directed graph handle appended to the disjoint union of paths gives a zero determinant. So the only remaining non-zero term is $|C_{2m}||K_n|$ which gives the result. □

Let $\mathbf{w} = (w_1, w_2, ..., w_p)$ be a non-empty chain of winners and $\mathbf{l} = (l_1, l_2, ..., l_k)$ be a non-empty chain of losers of K_m, denoted $(\overrightarrow{K}_m^{\mathbf{w},\mathbf{l}})_{(i,j)}$. Because any directed linear subgraph must contain a dicycle that includes the dipath through the chain of losers from l_k to l_1, then the handle vertex and then the chain of winners from w_1 to w_p. So, this is just a $(|\mathbf{w}| + |\mathbf{l}| - 1)$-handle on $K_{m-(|\mathbf{w}|+|\mathbf{l}|-2)}$.

$$\det((\overrightarrow{K}_m^{\mathbf{w},\mathbf{l}})_{(i,j)}) = \begin{cases} (-1)^m, & \text{if } i = l_1, j = w_1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Finally, we consider joining a generalized tournament with a winner or a loser with another generalized tournament with a winner or loser. In the corollary below, we denote this as $\overrightarrow{K}_m^{p_1} \overset{j}{\asymp} \overrightarrow{K}_n^{p_2}$, where p_i is either a winner or a loser and $p_1 = p_2$ means that both are winners or both are losers. For the determinant of such construct we get:

Corollary 3.9.

$$\det(\overrightarrow{K}_m^{p_1} \overset{j}{\asymp} \overrightarrow{K}_n^{p_2}) = \begin{cases} (-1)^{m+n+1}(m-2)(n-2), & \text{if } j = 1, \\ (-1)^{m+n+1}(m+n-5), & \text{if } j = 2, p_1 = p_2, \\ (-1)^{m+n+1}(m+n-4), & \text{if } j = 2, p_1 \neq p_2, \\ 0, & \text{otherwise.} \end{cases}$$

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