Thai Journal of Mathematics
Volume 14 (2016) Number 2 : 341–351
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On the Strong and Δ-Convergence of NSP-Iteration on CAT(0) Spaces

Piyanan Pasom[†] **and Asawathep Cuntavepanit**[‡,1]

[†]Department of Mathematics, Faculty of Science, Mahidol University, Rama IV Road, Ratchathewi District, Bangkok 10400, Thailand
e-mail : piyanan.pas@mahidol.ac.th

[‡]Division of Sciences and Liberal Arts, Mahidol University Kanchanaburi Campus, Sai Yok District, Kanchanaburi 71150, Thailand
e-mail : asawathep.cun@mahidol.ac.th

Abstract : Let C be a nonempty closed convex subset of a complete CAT(0) space X. We define a modified NSP-iteration for a nonexpansive selfmapping T on C as follows: $x_1 \in C$, and for each $n \in \mathbb{N}$,

$$\begin{cases} z_n & = (1-c_n)x_n \oplus c_n T x_n \\ y_n & = (1-b_n)x_n \oplus b_n((1-\beta_n)z_n \oplus \beta_n T z_n) \\ x_{n+1} & = (1-a_n)x_n \oplus a_n((1-\alpha_n)y_n \oplus \alpha_n T y_n). \end{cases}$$

The strong and Δ-convergence of the above iteration scheme under some certain conditions of the sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ are shown. Our results extend and generalize many results in the literature.

Keywords : nonexpansive mappings; NSP-iteration; Δ-convergence; strong convergence ; CAT(0) spaces.

2010 Mathematics Subject Classification : 47H09; 47H10.

---

[1]Corresponding author.

1 Introduction

A mapping T on a subset C of a metric space X is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \tag{1.1}$$

for all $x, y \in C$. An element $x \in C$ is said to be a *fixed point* of T if $Tx = x$. The set of all fixed points of T will be denoted by $F(T)$. The fixed point theorem for nonexpansive mappings in a special metric space, so called CAT(0) space, was shown by Kirk (see [1, 2]). He proved that any nonexpansive selfmapping $T : C \to C$, where C is a nonempty bounded closed convex subset of a complete CAT(0) space, has a fixed point. The iteration for finding the fixed point of this mapping have developed in parallel. It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}-$trees) can be applied to graph theory, biology and computer science (see e.g., [3–7]).

Dhompongsa and Panyanak [8] defined the Mann iteration process in a CAT(0) space by $x_1 \in C$,

$$x_{n+1} = (1 - t_n)x_n \oplus t_nTx_n, \quad n \in \mathbb{N} \tag{1.2}$$

where $\{t_n\}$ is a sequence in $[0, 1]$, and also defined the Ishikawa iteration process as follows: $x_1 \in C$,

$$\begin{cases} y_n &= (1 - s_n)x_n \oplus s_nTx_n, \\ x_{n+1} &= (1 - t_n)x_n \oplus t_nTy_n, \end{cases} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0, 1]$. They proved the convergence theorems for Mann and Ishikawa iteration processes on a nonempty bounded closed convex subset C of a complete CAT(0) space X.

Xu and Noor [9] defined the three-step iteration scheme for an asymptotically nonexpansive mapping, a generalization of nonexpansive mapping, in Banach space setting as follows: $x_1 \in C$, and

$$\begin{cases} z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n, \\ y_n &= (1 - \beta_n)x_n + \beta_nTz_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in [0,1]. We can see that (1.4) reduces to the Ishikawa iteration if we set $\gamma_n = 0$ for all n and it becomes the Mann iteration if $\beta_n = \gamma_n = 0$ for all n.

Phuengrattana and Suantai [10] defined the SP-iteration which are independent of Mann and Ishikawa iterations for continuous and nondecreasing function T on an arbitrary interval in $\mathbb{R}$ by $x_1 \in C$,

$$\begin{cases} z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n, \\ y_n &= (1 - \beta_n)z_n + \beta_nTz_n, \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n, \end{cases} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in [0,1].They showed that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges faster than the others.

Şahin and Başarır [11] modified the SP-iteration in a CAT(0) space by $x_1 \in C$, for all $n \in \mathbb{N}$,

$$\begin{cases} z_n &= (1-\gamma_n)x_n \oplus \gamma_n T x_n, \\ y_n &= (1-\beta_n)z_n \oplus \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)y_n \oplus \alpha_n T y_n, \end{cases} \tag{1.6}$$

where C is a nonempty convex subset of a CAT(0) space, $T : C \to C$ is nonexpansive $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1].

By using the idea of the SP-iteration and Noor iteration, Cholamjiak and Pholasa [12] defined the new iteration scheme, called the NSP-iteration process, as follows: $x_1 \in C$, for $n \in \mathbb{N}$,

$$\begin{cases} z_n &= (1-\mu_n)x_n + \mu_n T x_n, \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n z_n + \beta_n T z_n, \\ x_{n+1} &= (1-\gamma_n-\alpha_n)x_n + \gamma_n y_n + \alpha_n T y_n, \end{cases} \tag{1.7}$$

where T is a continuous real-valued function on an interval C in $\mathbb{R}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\tau_n\}$ are sequences in [0,1].

Motivated by the NSP-iteration defined by Cholamjiak and Pholasa, we introduce an iteration scheme for a nonexpansive mapping T in a CAT(0) space by $x_1 \in C$, and for $n \in \mathbb{N}$,

$$\begin{cases} z_n &= (1-c_n)x_n \oplus c_n T x_n, \\ y_n &= (1-b_n)x_n \oplus b_n((1-\beta_n)z_n \oplus \beta_n T z_n), \\ x_{n+1} &= (1-a_n)x_n \oplus a_n((1-\alpha_n)y_n \oplus \alpha_n T y_n), \end{cases} \tag{1.8}$$

where $\{a_n\}, \{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Our iteration generalizes the SP-iteration in CAT(0) space defined by Şahin and Başarır, indeed by letting $a_n = b_n = 1$ for all n, our iteration reduces to the iteration (1.6).

In this paper, we give the sufficient conditions to ensure strong convergence and Δ-convergence of our iteration scheme.

2 Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. The precise definition is stated below. Any complete, simply connected Riemannian manifold having nonpositive sectional curvature and $\mathbb{R}-$trees are examples of CAT(0) space. A thorough discussion in these spaces and the important role they play in many branches in mathematics hemicompact, see in [13].

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0)=x, c(l)=y$, and $d(c(t),c(t'))=|t-t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subset X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points. A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points x_1,x_2,x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i,\bar{x}_j)=d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT(0) inequality* if for all $x,y \in \triangle$, and all comparison points $\bar{x},\bar{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1,y_2]$, then the CAT(0) inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2.1}$$

This is the (CN) inequality of Bruhat and Tits [14]. In fact (see [13], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

Let $x,y \in X$, by Lemma 2.1(iv) of [8], for each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y). \tag{2.2}$$

From now on we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.2). Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see [15],Proposition 7). Also, every CAT(0) space has the *Opial* property, *i.e.*, if $\{x_n\}$ is a sequence in C and $\Delta\text{-}\lim_{n\to\infty} x_n = x$, then for each $y(\neq x) \in C$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

We now give the definition and collect some basic properties of the Δ-convergence.

Definition 2.1. A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-*convergent* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is called the Δ-*limit* of $\{x_n\}$.

Lemma 2.2 ([8], Lemma 2.7)**. (i)** *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

(ii) *Let C be a nonempty closed convex subset of a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in C. Then the asymptotic center of $\{x_n\}$ is in C.*

(iii) *Let K be a nonempty closed convex subset of a complete CAT(0) space X, $\{x_n\}$ be a bounded sequence in C and let $f : C \to X$ be a nonexpansive mapping. Then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in C$ and $f(x) = x$.*

The following lemmas can be found in [8].

Lemma 2.3. *Let X be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z),$$

for all $t \in [0, 1]$, and $x, y, z \in X$.

Lemma 2.4. *Let X be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2,$$

for all $t \in [0, 1]$, and $x, y, z \in X$.

3 Main Results

Before proving our main results, we need the following lemmas.

Lemma 3.1. *Let X be a complete CAT(0) space, C be a nonempty closed convex subset of X, and T be a nonexpansive selfmapping on C with $F(T) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence defined by (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq a_n, b_n \leq 1$ for some $0 < a < 1$, and $c_n \in [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. Then by Lemma 2.3,

$$\begin{aligned} d(z_n, p) &= d((1-c_n)x_n \oplus c_n T x_n, p) \\ &\le (1-c_n)d(x_n, p) + c_n d(Tx_n, p) \\ &\le (1-c_n)d(x_n, p) + c_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.1}$$

And also,

$$\begin{aligned} d(y_n, p) &= d((1-b_n)x_n \oplus b_n((1-\beta_n)z_n \oplus \beta_n T z_n), p) \\ &\le (1-b_n)d(x_n, p) + b_n d((1-\beta_n)z_n \oplus \beta_n T z_n, p) \\ &\le (1-b_n)d(x_n, p) + b_n\big[(1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p)\big] \\ &\le (1-b_n)d(x_n, p) + b_n\big[(1-\beta_n)d(z_n, p) + \beta_n d(z_n, p)\big] \\ &= (1-b_n)d(x_n, p) + b_n d(z_n, p). \end{aligned} \tag{3.2}$$

This, together with (3.1), we obtain that

$$d(y_n, p) \le d(x_n, p). \tag{3.3}$$

Therefore

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)x_n \oplus a_n((1-\alpha_n)y_n \oplus \alpha_n T y_n), p) \\ &\le (1-a_n)d(x_n, p) + a_n d((1-\alpha_n)y_n \oplus \alpha_n T y_n, p) \\ &\le (1-a_n)d(x_n, p) + a_n\big[(1-\alpha_n)d(y_n, p) + \alpha_n d(Ty_n, p)\big] \\ &\le (1-a_n)d(x_n, p) + a_n\big[(1-\alpha_n)d(y_n, p) + \alpha_n d(y_n, p)\big] \\ &= (1-a_n)d(x_n, p) + a_n d(y_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{3.4}$$

This implies that the sequence $\{d(x_n, p)\}$ is bounded below and nonincreasing. Thus it is convergent which completes the proof. ☐

Lemma 3.2. *Let X, C, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the hypotheses of Lemma 3.1. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Let $p \in F(T)$. By Lemma 3.1, there exists a number c with

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.5}$$

Firstly, we will prove that $\lim_{n\to\infty} d(y_n, p) = c$. We have from (3.4) that

$$d(x_{n+1}, p) \le (1-a_n)d(x_n, p) + a_n d(y_n, p).$$

Therefore

$$\begin{aligned} d(x_n, p) &\le d(y_n, p) + \frac{1}{a_n}\big[d(x_n, p) - d(x_{n+1}, p)\big] \\ &\le d(y_n, p) + \frac{1}{a}\big[d(x_n, p) - d(x_{n+1}, p)\big]. \end{aligned}$$

By taking $\liminf$ as $n \to \infty$ both sides, we obtain that

$$c = \liminf_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(y_n, p).$$

As a consequence of (3.3) and (3.5), $\limsup_{n\to\infty} d(y_n, p) \leq c$. Therefore

$$c \leq \liminf_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq c.$$

Thus, $\lim_{n\to\infty} d(y_n, p) = c$.

Next, we will show $\lim_{n\to\infty} d(z_n, p) = c$. From (3.2), we have

$$d(y_n, p) \leq (1 - b_n)d(x_n, p) + b_n d(z_n, p).$$

This implies that

$$d(x_n, p) \leq d(z_n, p) + \frac{1}{b_n}\big[d(x_n, p) - d(y_n, p)\big] \leq d(z_n, p) + \frac{1}{a}\big[d(x_n, p) - d(y_n, p)\big].$$

This yields

$$c = \liminf_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(z_n, p). \tag{3.6}$$

By taking $\limsup$ as $n \to \infty$ both sides of (3.1), we obtain that

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c.$$

This, together with (3.6), yields

$$\lim_{n\to\infty} d(z_n, p) = c. \tag{3.7}$$

It follows from Lemma 2.4 that

$$\begin{aligned} d(z_n, p)^2 &= d\big((1 - c_n)x_n \oplus c_n T x_n, p\big)^2 \\ &\leq (1 - c_n)d(x_n, p)^2 + c_n d(Tx_n, p)^2 - c_n(1 - c_n)d(x_n, Tx_n)^2 \\ &\leq (1 - c_n)d(x_n, p)^2 + c_n d(x_n, p)^2 - c_n(1 - c_n)d(x_n, Tx_n)^2 \\ &= d(x_n, p)^2 - c_n(1 - c_n)d(x_n, Tx_n)^2, \end{aligned}$$

so we get

$$d(x_n, Tx_n)^2 \leq \frac{1}{c_n(1 - c_n)}\big[d(x_n, p)^2 - d(z_n, p)^2\big] \leq \frac{1}{\epsilon^2}\big[d(x_n, p)^2 - d(z_n, p)^2\big].$$

By using (3.5) and (3.7), $\limsup_{n\to\infty} d(x_n, Tx_n) \leq 0$ and hence, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ as desired. □

Now, we are ready to prove the strong and Δ-convergence theorem of NSP-iteration on a CAT(0) space.

Theorem 3.3. *Let X be a complete CAT(0) space, C be a nonempty closed convex subset of X, and T be a nonexpansive selfmapping on C with $F(T) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence defined by (1.8), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences in $[0,1]$ such that $0 < a \leq a_n, b_n \leq 1$ for some $0 < a < 1$, and $c_n \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. By Lemma 3.1 and Lemma 3.2, we have $\lim\limits_{n\to\infty} d(x_n, p) = 0$ exists for all $p \in F(T)$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, respectively. Thus $\{x_n\}$ is bounded. Let $\omega_\Delta(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

To show that $\omega_\Delta(x_n) \subset F(T)$, let $u \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2(i) and (ii), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v \in C$. It follows from Lemma 2.2(iii) that $v \in F(T)$. And so $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.1. Now, we claim that $u = v$. If $u \neq v$, then we get from the Opial property of X that

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\
&\leq \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v), \qquad (3.8)
\end{aligned}$$

which is a contradiction. Therefore $u = v \in F(T)$ and thus $\omega_\Delta(x_n) \subset F(T)$.

Finally we will show that the sequence $\{x_n\}$ Δ-converges to a fixed point of T. It suffices to show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. If $x \neq v$, then the same argument as (3.8) gives a contradiction and hence $x = v \in F(T)$. Therefore, $\omega_\Delta(x_n) = \{x\}$. This implies that $\{x_n\}$ Δ-converges to a fixed point of T. □

Theorem 3.4. *Let X, C, T, $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the hypotheses of Theorem 3.3. Then the iteration process $\{x_n\}$ defined by (1.8) converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) := \inf\{d(x,p) : p \in F(T)\}$.

Proof. Necessity is obvious. Conversely, assume that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. We have from the proof of Lemma 3.1 that

$$d(x_{n+1}, p) \leq d(x_n, p),$$

for all $p \in F(T)$. It follows that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)),$$

which implies that the sequence $\{d(x_n, F(T))\}$ is nonincreasing and bounded below. Thus $\lim_{n\to\infty} d(x_n, F(T))$ exists. By the hypothesis, we can conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{2}, \quad \text{for all } n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{2}$. Thus there exists $p^* \in F(T)$ such that

$$d(x_{n_0}, p^*) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq 2d(x_{n_0}, p^*) \\ &< 2(\frac{\varepsilon}{2}) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed in a complete CAT(0) space X, the sequence $\{x_n\}$ must be convergent to a point in C. Let $\lim_{n\to\infty} x_n = q \in C$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, give that $d(q, F(T)) = 0$. Moreover $q \in F(T)$ because $F(T)$ is closed. Therefore, the sequence $\{x_n\}$ converges strongly to a fixed point q of T. □

Senter and Dotson [16] introduced *condition (I)* as follows.

A mapping $T : C \to C$ is said to satisfy the *condition (I)* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f\big(d(x, F(T))\big)$ for all $x \in K$. It's worth mentioning that in the case of nonexpansive mappings T, the *condition (I)* is weaker than the requirement that T be hemicompact.

Theorem 3.5. *Let X, C, $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the hypotheses of Lemma 3.1 and let $T : C \to C$ be a nonexpansive mapping satisfying condition (I). Then the iteration process $\{x_n\}$ defined by (1.8) converges strongly to a fixed point of T.*

Proof. Let $p \in F(T)$. By Lemma 3.1, there exists a real number c such that

$$\lim_{n\to\infty} d(x_n, p) = c.$$

If $c = 0$, then it's done. Suppose that $c > 0$. In the proof of Lemma 3.1, we have that

$$d(x_{n+1}, p) \leq d(x_n, p),$$

for all $p \in F(T)$, thus $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$. This implies that $\lim_{n\to\infty} d(x_n, F(T))$ exists. It follows by Lemma 3.2 and *Condition (I)* that

$$\lim_{n\to\infty} f\big(d(x_n, F(T))\big) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Thus,

$$\lim_{n\to\infty} f\big(d(x_n, F(T))\big) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

So the conclusion follows from Theorem 3.4, □

Since the NSP-iteration reduces to the SP-iteration when $a_n = b_n = 1$ for all $n \in \mathbb{N}$ and to the Noor iteration when $\alpha_n = \beta_n = 1$ for all $n \in \mathbb{N}$, we have the following corollaries.

Corollary 3.6. *Let X be a complete CAT(0) space, C be a nonempty closed convex subset of X, and T be a nonexpansive selfmapping on C with $F(T) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be the sequence of SP-iteration process defined by (1.6), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq a_n, b_n \leq 1$ for some $0 < a < 1$ and $c_n \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a fixed point of T. Furthurmore, if T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 3.7. *Let X be a complete CAT(0) space, C be a nonempty closed convex subset of X, and T be a nonexpansive selfmapping on C with $F(T) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be the sequence of Noor iteration process defined by (1.4) (replacing + with $\oplus$), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq a_n, b_n \leq 1$ for some $0 < a < 1$ and $c_n \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a fixed point of T. Furthurmore, if T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Acknowledgements : The authors are grateful to a referee for his/her careful reading and valuable comments and suggestions which led to improve the presentation in the paper. This research was supported by Mahidol University

References

[1] W.A. Kirk, Geodesic geometry and fixed point theory, in: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), in: Colecc. Abierta, vol. 64, Univ. Sevilla Secr. Publ. Seville (2003), 195-225.

[2] W.A. Kirk, Geodesic geometry and fixed point theory II, in: International Conference on Fixed Point Theory Appl., Yokohama Publ., Yokohama (2004), 113-142.

[3] W. A. Kirk, Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees, Fixed Point Theory Appl. 4 (2004) 309-316.

[4] W. A. Kirk, Some recent results in metric fixed point theory, Journal of Fixed Point Theory and Applications 2 (2) (2007) 195-207.

[5] M. Bestvina, R-trees in topology, geometry, and group theory, in Handbook of Geometric Topology, North-Holland, Amsterdam, The Netherlands (2002), 55-91.

[6] R. Espinola, W.A. Kirk, Fixed point theorems in R-trees with applications to graph theory, Topology and Its Appl. 153 (7) (2006) 1046-1055.

[7] C. Semple, M. Steel, Phylogenetics, of Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, Oxford, UK, 2003.

[8] S. Dhompongsa, B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 56 (2008) 2572-2579.

[9] B. Xu, MA. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 (2002) 444-453.

[10] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235 (2011) 3006-3014.

[11] A. Şahin, M. Başarır, On the strong and Δ-convergence of SP-iteration on CAT(0) space, Journal of Inequalities and Applications 2013, 2013:311.

[12] P. Cholamjiak, N. Pholasa, Convergence theorems for continuous functions on an arbitrary interval, Rend. Circ. Mat. Palermo (2013) 253-260.

[13] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

[14] F. Bruhat, J. Tits, Groupes réductifs sur un corps local, Inst. Hautes Etudes Sci. Publ. Math. 41 (1972) 5-251.

[15] S. Dhompongsa, W.A. Kirk, B. Sims, Fixed points of uniformly Lipschitzian mappings, Nonlinear Anal. 65 (2006) 762-772.

[16] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings. Proc. Am. Math. Soc. 44 (1974) 375-380.

(Received 12 April 2015)
(Accepted 15 March 2016)