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On the Least (Ordered) Semilattice Congruence in Ordered Γ-Semigroups

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Abstract : In this paper, we firstly characterize the relationship between the (ordered) filters, (ordered) s-prime ideals and (ordered) semilattice congruences in ordered Γ-semigroups. Finally, we give some characterizations of semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups and prove that

1. n is the least semilattice congruence,
2. $\mathcal{N}$ is the least ordered semilattice congruence,
3. $\mathcal{N}$ is not the least semilattice congruence in general.

Keywords : Ordered Γ-semigroup; (ordered) filter; (ordered) s-prime ideal; Least (ordered) semilattice congruence.
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1 Preliminaries

In 1998, Gao [8] gives some characterizations of semilattice congruences and ordered semilattice congruences on ordered semigroups. Now we also characterize the semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups and give some characterizations of semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups analogous to the characterizations of semilattice congruences and ordered semilattice congruences on ordered semigroups.

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [3, 4] if there exists a mapping $M \times \Gamma \times M \longrightarrow M$, written as $(a, \gamma, b) \longrightarrow a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup M is called a *commutative* Γ-*semigroup* if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset K of a Γ-semigroup M is called a *sub*-Γ-*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

For examples of Γ-semigroups, see [1, 3, 4].

A partially ordered Γ-semigroup M is called an *ordered* Γ-*semigroup* (*po*-Γ-semigroup) if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Example 1. For $a, b \in [0, 1]$, let $M = [0, a]$ and $\Gamma = [0, b]$. Then M is an ordered Γ-semigroup under usual multiplication and usual partial order relation.

Example 2. Fix $m \in \mathbb{Z}$, let M be the set of all integers of the form $mn + 1$ and Γ denote the set of all integers of the form $mn + m - 1$ where n is an integer. Then M is an ordered Γ-semigroup under usual addition and usual partial order relation.

Throughout this paper, M stands for an ordered Γ-semigroup. For nonempty subsets A and B of M and a nonempty subset Γ' of Γ, let $A\Gamma'B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma'B$ as $a\Gamma'B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. A nonempty subset A of M is called a *left* (*right*) *ideal* of M [7] if $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$). A is called an *ideal* of M if it is both a left ideal and a right ideal of M. A left ideal (right ideal, ideal) A of M is called an *ordered left ideal* (*right ideal, ideal*) of M if for any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

The following definitions in this paper are introduced analogous some definitions in [5, 7, 8].

A left ideal (right ideal, ideal) A of M is called an *s-prime left ideal* (*right ideal, ideal*) of M if for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in A$ implies $a \in A$ or $b \in A$. Equivalently, for any subsets B and C of M and $\gamma \in \Gamma, B\gamma C \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$. An s-prime left ideal (right ideal, ideal) A of M is called an *ordered s-prime left ideal* (*right ideal, ideal*) of M if A is an ordered left ideal (right ideal, ideal) of M. Let

$$SP(M) := \{A : A \text{ is an } s\text{-prime ideal of } M\},$$
$$OSP(M) := \{A : A \text{ is an ordered } s\text{-prime ideal of } M\}.$$

Then $\emptyset \neq OSP(M) \subseteq SP(M)$.

For a subset H of M and $a \in M$, denote $(H] := \{t \in M : t \leq h \text{ for some } h \in H\}$, $[H) := \{t \in M : h \leq t \text{ for some } h \in H\}$ and $a \cup H := \{a\} \cup H$. For $H = \{a\}$, we also write $(\{a\}]$ as $(a]$. Clearly, $H \subseteq (H], ((H]] = (H]$ and for any subsets A and B of M with $A \subseteq B$, we have $(A] \subseteq (B]$. A sub-Γ-semigroup F of M is called a *left* (*right*) *filter* of M if for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $b \in F$ ($a \in F$). F is called a *filter* of M if it is both a left filter and a right filter of M. A left filter (right filter, filter) F of M is called an *ordered left filter* (*right filter, filter*) of M if for any $b \in M$ and $a \in F, a \leq b$ implies $b \in F$. The intersection of all filters (ordered filters) of M containing a nonempty subset A of M is the filter (ordered filter) of M generated by A. For $A = \{x\}$, let

$n(x)$ denote the filter of M generated by $\{x\}$,
$N(x)$ denote the ordered filter of M generated by $\{x\}$.

An equivalence relation σ on M is called a *congruence* [2] if for any $a, b, c \in M$ and $\gamma \in \Gamma, (a, b) \in \sigma$ implies $(a\gamma c, b\gamma c) \in \sigma$ and $(c\gamma a, c\gamma b) \in \sigma$. A congruence σ

on M is called a *semilattice congruence* [6] if for all $a, b \in M$ and $\gamma \in \Gamma, (a\gamma a, a) \in \sigma$ and $(a\gamma b, b\gamma a) \in \sigma$. A semilattice congruence σ on M is called an *ordered semilattice congruence* if for any $a, b \in M$ and $\gamma \in \Gamma, a \leq b$ implies $(a, a\gamma b) \in \sigma$. Now, let

$$SC(M) := \{\sigma : \sigma \text{ is a semilattice congruence on } M\},$$
$$OSC(M) := \{\sigma : \sigma \text{ is an ordered semilattice congruence on } M\}.$$

Then $\emptyset \neq OSC(M) \subseteq SC(M)$.

For a nonempty subset A of M, define equivalence relations on M as follows:

$$\sigma_A := \{(x, y) \in M \times M : x, y \in A \text{ or } x, y \notin A\},$$
$$n := \{(x, y) \in M \times M : n(x) = n(y)\},$$
$$\mathcal{N} := \{(x, y) \in M \times M : N(x) = N(y)\}.$$

We note here that $\sigma_A = \sigma_{M \setminus A}$.

For any congruence σ on M and $x \in M$, let

$f(x)_\sigma$ denote the filter of M generated by σ-class $(x)_\sigma$,

t denote the filter of M generated by $\bigcup_{y \in (x)_\sigma} n(y)$,

$F(x)_\sigma$ denote the ordered filter of M generated by σ-class $(x)_\sigma$,

T denote the ordered filter of M generated by $\bigcup_{y \in (x)_\sigma} N(y)$.

The following results are also necessary for our considerations.

Theorem 1.1. *Let F be a nonempty subset of M. Then F is a left filter of M if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an s-prime left ideal of M.*

Proof. Assume that F is a left filter of M and $M \setminus F \neq \emptyset$. First to show that $M \setminus F$ is a left ideal of M, let $x \in M, y \in M \setminus F$ and $\gamma \in \Gamma$. Since F is a left filter of M and $y \notin F, x\gamma y \in M \setminus F$. Thus $M \setminus F$ is a left ideal of M. Next, let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in M \setminus F$. Since F is a sub-Γ-semigroup of $M, x \in M \setminus F$ or $y \in M \setminus F$. Thus $M \setminus F$ is an s-prime left ideal of M.

Conversely, if $M \setminus F = \emptyset$, then $F = M$. Hence F is a left filter of M. Assume that $M \setminus F$ is an s-prime left ideal of M. First to show that F is a sub-Γ-semigroup of M, let $x, y \in F$ and $\gamma \in \Gamma$. Then $x\gamma y \in F$ because $M \setminus F$ is an s-prime left ideal of M. Thus F is a sub-Γ-semigroup of M. Next, let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in F$. Then $y \in F$ because $M \setminus F$ is a left ideal of M, so F is a left filter of M. □

A similar result holds if we replace the word "left" by "right". Then we get the following.

Corollary 1.2. *Let F be a nonempty subset of M. Then F is a filter of M if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an s-prime ideal of M.*

Theorem 1.3. *Let F be a nonempty subset of M. Then F is an ordered left filter of M if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an ordered s-prime left ideal of M.*

Proof. Assume that F is an ordered left filter of M and $M \setminus F \neq \emptyset$. By Theorem 1.1, $M \setminus F$ is an s-prime left ideal of M. Now, let $x \in M$ and $y \in M \setminus F$ be such that $x \leq y$. Then $x \in M \setminus F$ because F is an ordered left filter of M, so $M \setminus F$ is an ordered s-prime left ideal of M.

Conversely, if $M \setminus F = \emptyset$, then $F = M$. Hence F is an ordered left filter of M. Assume that $M \setminus F$ is an ordered s-prime left ideal of M. By Theorem 1.1, F is a left filter of M. Now, let $x \in M$ and $y \in F$ be such that $y \leq x$. Then $x \in F$ because $M \setminus F$ is an ordered left ideal of M, so F is an ordered left filter of M. □

Corollary 1.4. *Let F be a nonempty subset of M. Then F is an ordered filter of M if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an ordered s-prime ideal of M.*

2 Semilattice Congruences and Ordered Semilattice Congruences

In this section, we characterize the relationship between the semilattice congruences, filters and s-prime ideals in ordered Γ-semigroups. Likewise, the relationship between the ordered semilattice congruences, ordered filters and ordered s-prime ideals in ordered Γ-semigroups are characterized.

The following lemmas are necessary for the main results and the first two lemmas are easy to verify.

Lemma 2.1. *An equivalence relation σ on M is a congruence if and only if for any $a, b, c, d \in M$ and $\gamma \in \Gamma, (a, b) \in \sigma$ and $(c, d) \in \sigma$ imply $(a\gamma c, b\gamma d) \in \sigma$.*

Lemma 2.2. *If $\sigma \in SC(M)$, then the following statements hold.*

(*a*) *For each $x \in M$, the σ-class $(x)_\sigma$ is a sub-Γ-semigroup of M.*

(*b*) *The set $M/\sigma := \{(x)_\sigma : x \in M\}$ is a commutative Γ-semigroup under the multiplication defined by $(x)_\sigma\gamma(y)_\sigma = (x\gamma y)_\sigma$ for all $(x)_\sigma, (y)_\sigma \in M/\sigma$ and $\gamma \in \Gamma$.*

Lemma 2.3. *Let A be a subset of M and $\sigma_A \in SC(M)$. If $x \in M \setminus A$ and $a \in A$ with $x\mu a \notin A$ (resp. $a\mu x \notin A$) for some $\mu \in \Gamma$, then $x\gamma a \notin A$ (resp. $a\gamma x \notin A$) for all $\gamma \in \Gamma$.*

Proof. Assume that $x \in M \setminus A$, $a \in A$ and $x\mu a \notin A$ for some $\mu \in \Gamma$. Then $(x, x\mu a) \in \sigma_A$, so $(x)_{\sigma_A} = (x\mu a)_{\sigma_A}$. Suppose that there exists $\gamma \in \Gamma$ such that

$x\gamma a \in A$. Then $(a, x\gamma a) \in \sigma_A$. Thus $(a)_{\sigma_A} = (x\gamma a)_{\sigma_A}$. By Lemma 2.2 (b), $(x)_{\sigma_A} = (x\mu a)_{\sigma_A} = (x\mu x\gamma a)_{\sigma_A} = (x\gamma a)_{\sigma_A} = (a)_{\sigma_A}$. Thus $(x, a) \in \sigma_A$, so $a \notin A$. This is a contradiction. Therefore $x\gamma a \notin A$ for all $\gamma \in \Gamma$. □

As a consequence of this result, we obtain

Lemma 2.4. *Let A be a nonempty subset of M. Then $\sigma_A \in SC(M)$ if and only if one of A or $M \setminus A$ is an s-prime ideal of M.*

Proof. Assume that $\sigma_A \in SC(M)$. If $A = M$, then $A \in SP(M)$. Suppose that $A \subset M$. Then $M \setminus A \neq \emptyset$. First to show that A and $M \setminus A$ are sub-Γ-semigroups of M, let $x, y \in A$ and $\gamma \in \Gamma$. Then $(x\gamma y, y\gamma y) \in \sigma_A$ and $(y\gamma y, y) \in \sigma_A$ because $(x, y) \in \sigma_A$, so $(x\gamma y, y) \in \sigma_A$. Hence $x\gamma y \in A$, so A is a sub-Γ-semigroup of M. The same argument applies to $M \setminus A$, we have $M \setminus A$ is a sub-Γ-semigroups of M. Next, consider the following two cases:
Case 1: $M\Gamma A \subseteq A$. Then $A\Gamma M \subseteq A$ because $(x\gamma a, a\gamma x) \in \sigma_A$ and $x\gamma a \in A$ for all $x \in M, a \in A$ and $\gamma \in \Gamma$. Hence A is an ideal of M.
Case 2: $M\Gamma A \nsubseteq A$. Then there exist $x \in M, a \in A, \mu \in \Gamma$ but $x\mu a \notin A$. Since A is a sub-Γ-semigroup of M, $x \notin A$. By Lemma 2.3, $x\gamma a \notin A$ for all $\gamma \in \Gamma$. Thus $(x, x\gamma a) \in \sigma_A$ for all $\gamma \in \Gamma$. By Lemma 2.2 $(b), (x)_{\sigma_A} = (x\gamma a)_{\sigma_A} = (x)_{\sigma_A}\gamma(a)_{\sigma_A}$ for all $\gamma \in \Gamma$. Obviously, $M \setminus A = (x)_{\sigma_A}$ and $A = (a)_{\sigma_A}$, so $M \setminus A = (M \setminus A)\gamma A$ for all $\gamma \in \Gamma$. This implies that

$$M \setminus A = \bigcup_{\gamma \in \Gamma} (M \setminus A)\gamma A = (M \setminus A)\Gamma A.$$

Therefore

$$(M \setminus A)\Gamma M = (M \setminus A)\Gamma(A \cup (M \setminus A)) \subseteq ((M \setminus A)\Gamma A) \cup (M \setminus A) = M \setminus A,$$

so $M \setminus A$ is a right ideal of M. Since $(x\mu a, a\mu x) \in \sigma_A$ and $x\mu a \notin A, a\mu x \notin A$. By symmetry, $M \setminus A$ is a left ideal of M. This proves that $M \setminus A$ is an ideal of M.

Assume that A is an ideal of M. Let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in A$. If $x, y \notin A$, then $(x, y) \in \sigma_A$. Thus $(x\gamma x, x) \in \sigma_A$ and $(x\gamma x, x\gamma y) \in \sigma_A$, so $(x, x\gamma y) \in \sigma_A$. Thus $x\gamma y \notin A$, which is impossible. Hence $A \in SP(M)$. Similarly, we can show that if $M \setminus A$ is an ideal of M, then $M \setminus A \in SP(M)$.

Conversely, assume that $A \in SP(M)$. Now, let $x, y \in M$ be such that $(x, y) \in \sigma_A, c \in M$ and $\gamma \in \Gamma$. Then we have the following two cases:
Case 1: $x, y \in A$. Then $c\gamma x, c\gamma y, x\gamma c, y\gamma c \in A$ because A is an ideal of M. Thus $(c\gamma x, c\gamma y) \in \sigma_A$ and $(x\gamma c, y\gamma c) \in \sigma_A$.
Case 2: $x, y \notin A$. Then $c\gamma x \in A$ if and only if $c\gamma y \in A$. Thus $(c\gamma x, c\gamma y) \in \sigma_A$. By symmetry, $(x\gamma c, y\gamma c) \in \sigma_A$.

Hence σ_A is a congruence on M. Next, let $a, b \in M$ and $\gamma \in \Gamma$. Then $a \in A$ if and only if $a\gamma a \in A$, so $(a, a\gamma a) \in \sigma_A$. Similarly, we have $a\gamma b \in A$ if and only if $b\gamma a \in A$, so $(a\gamma b, b\gamma a) \in \sigma_A$. This proves that $\sigma_A \in SC(M)$. Similarly, we can show that if $M \setminus A \in SP(M)$, then $\sigma_A \in SC(M)$.
Hence the proof is completed. □

Lemma 2.5. *If A is a nonempty subset of M, then the following statements are equivalent.*

(a) $\sigma_A \in OSC(M)$.

(b) *One of A or $M \setminus A$ is an ordered s-prime ideal of M.*

Proof. Assume that $\sigma_A \in OSC(M)$. By Lemma 2.4, $A \in SP(M)$ or $M \setminus A \in SP(M)$. Assume that $A \in SP(M)$. Now, let $x \in M$ and $a \in A$ be such that $x \leq a$ and $\gamma \in \Gamma$. Then $(x, x\gamma a) \in \sigma_A$, so $x \in A$ because $x\gamma a \in A$. Hence $A \in OSP(M)$. Similarly, we can show that if $M \setminus A \in SP(M)$, then $M \setminus A \in OSP(M)$.

Conversely, assume that $A \in OSP(M)$. Then $\sigma_A \in SC(M)$ by Lemma 2.4. Now, let $a, b \in M$ be such that $a \leq b$ and $\gamma \in \Gamma$. If $a \in A$, then $a\gamma b \in A$. If $a \notin A$, then $b \notin A$ and so $a\gamma b \notin A$. Hence $(a, a\gamma b) \in \sigma_A$, so $\sigma_A \in OSC(M)$. Similarly, we can show that if $M \setminus A \in OSP(M)$, then $\sigma_A \in OSC(M)$.
Hence the proof is completed. □

Lemma 2.6. *If $x \in M$ and $\sigma \in SC(M)$, then the following statements hold.*

(a) $f(x)_\sigma = \{a \in M : a \in (x)_\sigma$ *or* $u\gamma a \in (x)_\sigma$ *for some* $u \in f(x)_\sigma$ *and* $\gamma \in \Gamma\}$.

(b) $f(x)_\sigma = t$.

(c) *If* $b \in f(x)_\sigma$, *then* $f(b)_\sigma \subseteq f(x)_\sigma$.

(d) $\sigma = \{(x, y) \in M \times M : f(x)_\sigma = f(y)_\sigma\}$.

Proof. (a) Let

$$N := \{a \in M : a \in (x)_\sigma \text{ or } u\gamma a \in (x)_\sigma \text{ for some } u \in f(x)_\sigma \text{ and } \gamma \in \Gamma\}.$$

It is clear that $(x)_\sigma \subseteq N \subseteq f(x)_\sigma$. Conversely, to show that N is a filter of M, let $a, b \in N$ and $\gamma \in \Gamma$. If $u_1\gamma_1 a, u_2\gamma_2 b \in (x)_\sigma$ for some $u_1, u_2 \in f(x)_\sigma$ and $\gamma_1, \gamma_2 \in \Gamma$, then $u_1\gamma_1 a\gamma u_2\gamma_2 b \in (x)_\sigma$ by Lemma 2.2 (a). It follows from Lemma 2.2 (b) that

$$(x)_\sigma = (u_1\gamma_1 a\gamma u_2\gamma_2 b)_\sigma = (u_1\gamma_1 a\gamma b\gamma_2 u_2)_\sigma = (u_1\gamma_1 u_2\gamma_2 a\gamma b)_\sigma.$$

Thus $a\gamma b \in N$ because $u_1\gamma_1 u_2 \in f(x)_\sigma$. Similarly, it is easy to verify in the remain cases that $a\gamma b \in N$. Hence N is a sub-Γ-semigroup of M. We note here that for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in N$ implies $b\gamma a \in N$. Next, let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a\gamma b \in N$. Since $N \subseteq f(x)_\sigma$, we have $a, b \in f(x)_\sigma$. Since $a\gamma b \in N$, $a\gamma b \in (x)_\sigma$ or $u\alpha a\gamma b \in (x)_\sigma$ for some $u \in f(x)_\sigma$ and $\alpha \in \Gamma$. Thus $b \in N$. Since $b\gamma a \in N, a \in N$. Hence N is a filter of M, so $f(x)_\sigma \subseteq N$. Therefore $N = f(x)_\sigma$.

(b) From the fact that $(x)_\sigma \subseteq \bigcup\limits_{y \in (x)_\sigma} n(y)$, we get $f(x)_\sigma \subseteq t$. On the other hand, we have $n(y) \subseteq f(x)_\sigma$ for all $y \in (x)_\sigma$. Thus $\bigcup\limits_{y \in (x)_\sigma} n(y) \subseteq f(x)_\sigma$, so $t \subseteq f(x)_\sigma$. Therefore $f(x)_\sigma = t$.

(c) Let $b \in f(x)_\sigma$. By (a), we have $b \in (x)_\sigma$ or $u\alpha b \in (x)_\sigma$ for some $u \in f(x)_\sigma$ and $\alpha \in \Gamma$. Thus $(x)_\sigma = (b)_\sigma$ or $(x)_\sigma = (u\alpha b)_\sigma$ which implies that $(b)_\sigma \subseteq f(x)_\sigma$. Therefore $f(b)_\sigma \subseteq f(x)_\sigma$.

(d) Let

$$\tau := \{(x, y) \in M \times M : f(x)_\sigma = f(y)_\sigma\}.$$

It is clear that $\sigma \subseteq \tau$. Conversely, let $x, y \in M$ be such that $(x, y) \in \tau$. Then $f(x)_\sigma = f(y)_\sigma$, so $x \in f(y)_\sigma$ and $y \in f(x)_\sigma$. By (a), if $x \in (y)_\sigma$ or $y \in (x)_\sigma$, then $(x)_\sigma = (y)_\sigma$. Let $u_1\gamma_1 x \in (y)_\sigma$ and $u_2\gamma_2 y \in (x)_\sigma$ for some $u_1, u_2 \in f(x)_\sigma = f(y)_\sigma$ and $\gamma_1, \gamma_2 \in \Gamma$. It follows from Lemma 2.2 (b) that $(x)_\sigma = (u_2\gamma_2 y)_\sigma = (u_2\gamma_2 y\gamma_2 y)_\sigma = (x\gamma_2 y)_\sigma = (x\gamma_2 u_1\gamma_1 x)_\sigma = (x\gamma_2 x\gamma_1 u_1)_\sigma = (x\gamma_1 u_1)_\sigma = (u_1\gamma_1 x)_\sigma = (y)_\sigma$. Hence $(x, y) \in \sigma$, so $\sigma = \tau$. □

Immediately from Lemma 2.6, we have

Corollary 2.7. *If $x \in M$ and $\sigma \in SC(M)$, then $f(x)_\sigma = \{a \in M : a \in (x)_\sigma$ or $u\gamma a \in (x)_\sigma$ or $a\mu v \in (x)_\sigma$ or $u\gamma a\mu v \in (x)_\sigma$ for some $u, v \in f(x)_\sigma$ and $\gamma, \mu \in \Gamma\}$.*

Corollary 2.8. *If $x \in M$, then the following statements hold.*

(a) $n \in SC(M)$.

(b) $f(x)_n = n(x)$.

(c) $n(x) = \{a \in M : a \in (x)_n$ *or* $u\gamma a \in (x)_n$ *for some* $u \in n(x)$ *and* $\gamma \in \Gamma\}$.

Proof. (a) Let $a, b \in M$ be such that $(a, b) \in n, c \in M$ and $\gamma \in \Gamma$. Then $n(a) = n(b)$. Since $b\gamma c \in n(b\gamma c)$, we have $b, c \in n(b\gamma c)$. Thus $n(a) = n(b) \subseteq n(b\gamma c)$, so $a, c \in n(b\gamma c)$. Hence $a\gamma c \in n(b\gamma c)$, so $n(a\gamma c) \subseteq n(b\gamma c)$. Similarly, $n(b\gamma c) \subseteq n(a\gamma c)$. Therefore $n(a\gamma c) = n(b\gamma c)$, so $(a\gamma c, b\gamma c) \in n$. Similarly, $(c\gamma a, c\gamma b) \in n$. This proves that n is a congruence on M. Next, let $a, b, c \in M$ and $\gamma \in \Gamma$. Then $a \in n(a\gamma a)$ because $a\gamma a \in n(a\gamma a)$, so $n(a) \subseteq n(a\gamma a)$. Since $a \in n(a), a\gamma a \in n(a)$. Hence $n(a\gamma a) \subseteq n(a)$, so $n(a\gamma a) = n(a)$. Therefore $(a\gamma a, a) \in n$. Since $a\gamma b \in n(a\gamma b)$, $b\gamma a \in n(a\gamma b)$. Thus $n(b\gamma a) \subseteq n(a\gamma b)$. Similarly, $n(a\gamma b) \subseteq n(b\gamma a)$. Hence $n(a\gamma b) = n(b\gamma a)$, so $(a\gamma b, b\gamma a) \in n$. Therefore $n \in SC(M)$.

(b) By (a) and Lemma 2.6 (b), $f(x)_n = t$ where t is the filter of M generated by $\bigcup_{y \in (x)_n} n(y)$. We note here that

$$\bigcup_{y \in (x)_n} n(y) = n(x).$$

Hence $t = n(x)$, so $f(x)_n = n(x)$.

(c) By (b) and Lemma 2.6 (a),

$$\begin{aligned} n(x) &= f(x)_n \\ &= \{a \in M : a \in (x)_n \text{ or } u\gamma a \in (x)_n \text{ for some } u \in f(x)_n \text{ and } \gamma \in \Gamma\} \\ &= \{a \in M : a \in (x)_n \text{ or } u\gamma a \in (x)_n \text{ for some } u \in n(x) \text{ and } \gamma \in \Gamma\}. \end{aligned}$$

Hence the proof is completed. □

Lemma 2.9. *If $x \in M$ and $\sigma \in OSC(M)$, then the following statements hold.*

(a) $F(x)_\sigma = \{a \in M : a \in [(x)_\sigma)$ *or* $u\gamma a \in [(x)_\sigma)$ *for some* $u \in F(x)_\sigma$ *and* $\gamma \in \Gamma\}$.

(b) $F(x)_\sigma = T$.

(c) *If* $b \in F(x)_\sigma$, *then* $F(b)_\sigma \subseteq F(x)_\sigma$.

(d) $\sigma = \{(x, y) \in M \times M : F(x)_\sigma = F(y)_\sigma\}$.

Proof. (a) Let

$$N := \{a \in M : u\gamma a \in [(x)_\sigma) \text{ or } a \in [(x)_\sigma) \text{ for some } u \in F(x)_\sigma \text{ and } \gamma \in \Gamma\}.$$

It is clear that $(x)_\sigma \subseteq N \subseteq F(x)_\sigma$. Conversely, to show that N is an ordered filter of M, let $a, b \in N$ and $\gamma \in \Gamma$. If $u_1\gamma_1 a, u_2\gamma_2 b \in [(x)_\sigma)$ for some $u_1, u_2 \in F(x)_\sigma$ and $\gamma_1, \gamma_2 \in \Gamma$, then $y_1 \leq u_1\gamma_1 a$ and $y_2 \leq u_2\gamma_2 b$ for some $y_1, y_2 \in (x)_\sigma$. Thus $y_1\gamma y_2 \leq u_1\gamma_1 a\gamma u_2\gamma_2 b$ and $y_1\gamma y_2 \in (x)_\sigma$ by Lemma 2.2 (a). Hence $(y_1\gamma y_2, y_1\gamma y_2\gamma u_1\gamma_1 a\gamma u_2\gamma_2 b) \in \sigma$ which implies that $(x, x\gamma u_1\gamma_1 a\gamma u_2\gamma_2 b) \in \sigma$. It follows from Lemma 2.2 (b) that

$$(x)_\sigma = (x\gamma u_1\gamma_1 a\gamma u_2\gamma_2 b)_\sigma = (x\gamma u_1\gamma_1 a\gamma b\gamma_2 u_2)_\sigma = (x\gamma u_1\gamma_1 u_2\gamma_2 a\gamma b)_\sigma.$$

Thus $a\gamma b \in N$ because $x\gamma u_1\gamma_1 u_2 \in F(x)_\sigma$. Similarly, it is easy to verify in the remain cases that $a\gamma b \in N$. Hence N is a sub-Γ-semigroup of M. We note here that for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in N$ implies $b\gamma a \in N$. Let $a, b \in M$ be such that $a\gamma b \in N$ and $\gamma \in \Gamma$. Since $N \subseteq F(x)_\sigma$, we have $a, b \in F(x)_\sigma$. Since $a\gamma b \in N$, $a\gamma b \in [(x)_\sigma)$ or $u\alpha a\gamma b \in [(x)_\sigma)$ for some $u \in F(x)_\sigma$ and $\alpha \in \Gamma$. Thus $b \in N$. Since $b\gamma a \in N, a \in N$. Hence N is a filter of M. Next, let $b \in M$ and $a \in N$ be such that $a \leq b$. Then $a \in [(x)_\sigma)$ or $u\alpha a \in [(x)_\sigma)$ for some $u \in F(x)_\sigma$ and $\alpha \in \Gamma$ which implies that $b \in [(x)_\sigma)$ or $u\alpha b \in [(x)_\sigma)$. Thus $b \in N$, so N is an ordered filter of M. Hence $F(x)_\sigma \subseteq N$, so $N = F(x)_\sigma$.

(b) It is similar to the proof of Lemma 2.6 (b).

(c) Let $b \in F(x)_\sigma$ and $\gamma \in \Gamma$. By (a), we have $b \in [(x)_\sigma)$ or $u\alpha b \in [(x)_\sigma)$ for some $u \in F(x)_\sigma$ and $\alpha \in \Gamma$. Thus $(x)_\sigma = (x\gamma b)_\sigma$ or $(x)_\sigma = (x\gamma u\alpha b)_\sigma$ which implies that $(b)_\sigma \subseteq F(x)_\sigma$. Therefore $F(b)_\sigma \subseteq F(x)_\sigma$.

(d) Let

$$\tau := \{(x, y) \in M \times M : F(x)_\sigma = F(y)_\sigma\}.$$

It is clear that $\sigma \subseteq \tau$. Conversely, let $x, y \in M$ be such that $(x, y) \in \tau$ and $\gamma \in \Gamma$. Then $F(x)_\sigma = F(y)_\sigma$, so $x \in F(y)_\sigma$ and $y \in F(x)_\sigma$. By (a), it suffices to show that the following case is satisfied. If $u_1\gamma_1 x \in [(y)_\sigma)$ and $u_2\gamma_2 y \in [(x)_\sigma)$ for some $u_1, u_2 \in F(x)_\sigma = F(y)_\sigma$ and $\gamma_1, \gamma_2 \in \Gamma$, then $(y, y\gamma u_1\gamma_1 x) \in \sigma$ and $(x, x\gamma u_2\gamma_2 y) \in \sigma$. It follows from Lemma 2.2 (b) that

$$(u_2\gamma_2 y)_\sigma = (u_2\gamma_2 y\gamma u_1\gamma_1 x)_\sigma = (u_1\gamma_1 x\gamma u_2\gamma_2 y)_\sigma = (u_1\gamma_1 x)_\sigma.$$

Hence $(x)_\sigma = (x\gamma u_2\gamma_2 y)_\sigma = (u_2\gamma_2 y\gamma x)_\sigma = (u_1\gamma_1 x\gamma x)_\sigma = (u_1\gamma_1 x)_\sigma = (u_2\gamma_2 y)_\sigma = (u_2\gamma_2 y\gamma y)_\sigma = (y\gamma u_2\gamma_2 y)_\sigma = (y\gamma u_1\gamma_1 x)_\sigma = (y)_\sigma$, so $(x, y) \in \sigma$. Similarly, it is easy to verify in the remain cases that $(x, y) \in \sigma$. Therefore $\sigma = \tau$. □

Immediately from Lemma 2.9, we have

Corollary 2.10. *If $x \in M$ and $\sigma \in OSC(M)$, then $F(x)_\sigma = \{a \in M : a \in [(x)_\sigma)$ or $u\gamma a \in [(x)_\sigma)$ or $a\mu v \in [(x)_\sigma)$ or $u\gamma a\mu v \in [(x)_\sigma)$ for some $u, v \in F(x)_\sigma$ and $\gamma, \mu \in \Gamma\}$.*

Corollary 2.11. *If $x \in M$, then the following statements hold.*

(a) $\mathcal{N} \in OSC(M)$.

(b) $F(x)_\mathcal{N} = N(x)$.

(c) $N(x) = \{a \in M : a \in [(x)_\mathcal{N})$ *or* $u\gamma a \in [(x)_\mathcal{N})$ *for some* $u \in N(x)$ *and* $\gamma \in \Gamma\}$.

Proof. (a) By the similarity of the proof of Corollary 2.8 (a), we have $\mathcal{N} \in SC(M)$. Now, let $a, b \in M$ be such that $a \leq b$ and $\gamma \in \Gamma$. Then $a \in N(a\gamma b)$ because $a\gamma b \in N(a\gamma b)$, so $N(a) \subseteq N(a\gamma b)$. Since $a \in N(a), b \in N(a)$. Thus $a\gamma b \in N(a)$, so $N(a\gamma b) \subseteq N(a)$. Hence $N(a) = N(a\gamma b)$, so $(a, a\gamma b) \in \mathcal{N}$. Therefore $\mathcal{N} \in OSC(M)$.

(b) It is similar to the proof of Corollary 2.8 (b).

(c) It is similar to the proof of Corollary 2.8 (c).

Hence the proof is completed. □

3 Main Results

In last section, we characterize the least semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups and show that $\mathcal{N}$ is not the least semilattice congruence on ordered Γ-semigroups in general.

Theorem 3.1.

(a) $n = \bigcap_{I \in SP(M)} \sigma_I$.

(b) $\mathcal{N} = \bigcap_{I \in OSP(M)} \sigma_I$.

(b) $n \subseteq \mathcal{N}$.

Proof. (a) Let

$$\tau := \bigcap_{I \in SP(M)} \sigma_I.$$

Let $x, y \in M$ be such that $(x, y) \in n$. Then $n(x) = n(y)$. Suppose that there exists $I \in SP(M)$ such that $(x, y) \notin \sigma_I$. By Corollary 1.2, $M \setminus I$ is a filter of M. Without loss of generality, we may assume that $x \in I$ and $y \in M \setminus I$. Then $x \in n(x) = n(y) \subseteq M \setminus I$, which is impossible. Hence $(x, y) \in \sigma_I$ for all $I \in SP(M)$, so $(x, y) \in \tau$. Conversely, let $x, y \in M$ be such that $(x, y) \in \tau$. Then $(x, y) \in \sigma_I$ for all $I \in SP(M)$. Suppose that $(x, y) \notin n$. Then $n(x) \neq n(y)$. By Corollary 2.8 (b), $f(x)_n = n(x) \neq n(y) = f(y)_n$. Without loss of generality, we may assume that $f(x)_n \not\subseteq f(y)_n$. By Lemma 2.6 (c), $x \notin f(y)_n$. Then $(x, y) \notin \sigma_{M \setminus f(y)_n}$. Since $M \setminus f(y)_n \neq \emptyset$, it follows from Corollary 1.2 that $M \setminus f(y)_n \in SP(M)$. This implies that $(x, y) \in \sigma_{M \setminus f(y)_n}$, which is impossible. Hence $(x, y) \in n$, this proves that $n = \bigcap\{\sigma_I : I \in SP(M)\}$.

(b) It is similar to the proof of (a).

(c) Since $OSP(M) \subseteq SP(M)$, it follows from (a) and (b) that $n \subseteq \mathcal{N}$. Hence the theorem is proved. □

Theorem 3.2. *If $\sigma \in SC(M)$, then the following statements hold.*

(a) $\sigma = \bigcap_{x \in M} \sigma_{M \setminus f(x)_\sigma}$.

(b) $n \subseteq \sigma$, *i.e.*, n *is the least element of* $SC(M)$.

Proof. (a) Let

$$\tau := \bigcap_{x \in M} \sigma_{M \setminus f(x)_\sigma}.$$

Let $x, y \in M$ be such that $(x, y) \in \sigma$. Then $f(x)_\sigma = f(y)_\sigma$ by Lemma 2.6 (d). Suppose that $(x, y) \notin \sigma_{M \setminus f(a)_\sigma}$ for some $a \in M$. Without loss of generality, we may assume that $x \in M \setminus f(a)_\sigma$ and $y \notin M \setminus f(a)_\sigma$. Then $y \in f(a)_\sigma$, it follows from Lemma 2.6 (c) that $x \in f(x)_\sigma = f(y)_\sigma \subseteq f(a)_\sigma$. It is impossible, so $(x, y) \in \sigma_{M \setminus f(a)_\sigma}$ for all $a \in M$. Conversely, let $x, y \in M$ be such that $(x, y) \in \tau$. Then $(x, y) \in \sigma_{M \setminus f(a)_\sigma}$ for all $a \in M$. Suppose that $(x, y) \notin \sigma$. By Lemma 2.6 (d), $f(x)_\sigma \neq f(y)_\sigma$. Without loss of generality, we may assume that $f(x)_\sigma \not\subseteq f(y)_\sigma$. By Lemma 2.6 (c), $x \notin f(y)_\sigma$. Then $(x, y) \notin \sigma_{M \setminus f(y)_\sigma}$, which is impossible. Hence $(x, y) \in \sigma$, this proves that

$$\sigma = \bigcap_{x \in M} \sigma_{M \setminus f(x)_\sigma}.$$

(b) By Corollary 1.2, $M \setminus f(x)_\sigma = \emptyset$ or $M \setminus f(x)_\sigma \in SP(M)$ for all $x \in M$. Thus

$$\{\sigma_{M \setminus f(x)_\sigma} : x \in M\} \subseteq \{\sigma_I : I \in SP(M)\}.$$

By (a) and Theorem 3.1 (a), $n \subseteq \sigma$. Therefore n is the least semilattice congruence on M. □

By the similarity of the proof of Theorem 3.2, we obtain

Theorem 3.3. *If $\sigma \in OSC(M)$, then the following statements hold.*

(a) $\sigma = \bigcap_{x \in M} \sigma_{M \setminus F(x)_\sigma}$.

(b) $\mathcal{N} \subseteq \sigma$, *i.e.*, $\mathcal{N}$ *is the least element of* $OSC(M)$.

Immediately from Theorem 3.2 and Theorem 3.3, we have

Corollary 3.4.

(a) $n = \bigcap_{x \in M} \sigma_{M \setminus n(x)}$.

(b) $\mathcal{N} = \bigcap_{x \in M} \sigma_{M \setminus N(x)}$.

We shall give an example of an ordered Γ-semigroup M with $\mathcal{N}$ is not the least semilattice congruence on M.

Example 3.5. Let $M = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

$$x\gamma y = \begin{cases} b & \text{if } x, y \in \{a, b\}, \\ c & \text{otherwise.} \end{cases}$$

First to show that M is a Γ-semigroup, suppose not. Then there exist $x, y, z \in M$ such that $(x\gamma y)\gamma z \neq x\gamma(y\gamma z)$. If $(x\gamma y)\gamma z = b$, then $x, y, z \in \{a, b\}$. Thus $x\gamma(y\gamma z) = b$, which is impossible. If $x\gamma(y\gamma z) = b$, then $x, y, z \in \{a, b\}$. Thus $(x\gamma y)\gamma z = b$, which is impossible. Hence $(x\gamma y)\gamma z = x\gamma(y\gamma z)$ for all $x, y, z \in M$. Obviously, $x\gamma y = y\gamma x$ for all $x, y \in M$. Therefore M is a commutative Γ-semigroup.

Define a relation $\leq$ on M as follows:

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (b, c), (b, d), (c, d)\}.$$

Then $(M, \leq)$ is a partially ordered set. Let $x, y \in M$ be such that $x \leq y$. Since $x\gamma c = c = c\gamma x$ and $x\gamma d = c = d\gamma x$ for all $x, y \in M$ and $b \leq c$, $x\gamma z \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for all $z \in M$. Hence M is an ordered Γ-semigroup. We shall show that $SC(M) = \{n, \mathcal{N}\}$ and $n \subset \mathcal{N}$. Let

$$\begin{aligned} \sigma_1 &= M \times M, \\ \sigma_2 &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}. \end{aligned}$$

It is easy to see that $\sigma_1, \sigma_2 \in SC(M)$. Since $(a\gamma a, a) = (b, a)$ and $(d\gamma d, d) = (c, d), \sigma_2 \subseteq \sigma$ for all $\sigma \in SC(M)$. Let $\sigma \in SC(M)$. Then we have the following two cases:

Case 1: $(b, c) \in \sigma$. Since $(a, b) \in \sigma$, $(a, c) \in \sigma$. Thus $(a, d), (b, d) \in \sigma$ because $(c, d) \in \sigma$. Hence $\sigma = \sigma_1$.

Case 2: $(b, c) \notin \sigma$. If $(a, c) \in \sigma$, then $(b, c) \in \sigma$ because $(b, a) \in \sigma$, which is impossible. If $(a, d) \in \sigma$, then $(a, c) \in \sigma$ because $(d, c) \in \sigma$, which is impossible. If $(b, d) \in \sigma$, then $(b, c) \in \sigma$ because $(d, c) \in \sigma$, which is impossible. Hence $\sigma = \sigma_2$.

This proves that $SC(M) = \{\sigma_1, \sigma_2\}$. We shall show that $\sigma_1 = \mathcal{N}$ and $\sigma_2 = n$. We can easily get all ideals of M as follows:

$$P_1 = M, P_2 = \{c, d\}, P_3 = \{b, c\}, P_4 = \{c\}, P_5 = \{a, b, c\}, P_6 = \{b, c, d\}.$$

It is easy to see that $SP(M) = \{P_1, P_2\}$ and $OSP(M) = \{P_1\}$. By Theorem 3.1, we obtain that

$$\mathcal{N} = \bigcap_{I \in OSP(M)} \sigma_I = \sigma_{P_1} = M \times M = \sigma_1$$

and

$$n = \bigcap_{I \in SP(M)} \sigma_I = \sigma_{P_1} \cap \sigma_{P_2} = \sigma_{P_2}.$$

We note here that

$$\begin{aligned} \sigma_{P_2} &= \{(x, y) \in M \times M : x, y \in P_2 \text{ or } x, y \notin P_2\} \\ &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\} \\ &= \sigma_2. \end{aligned}$$

Hence $n = \sigma_2$, so $n \subset \mathcal{N}$.

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