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On the Composition of Distributions $x^{-s}\ln|x|$ and x^r

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Abstract: Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The composition of the distributions $x^{-s}\ln|x|$ and x^r is evaluated for $r, s = 1, 2, \ldots$.

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1 Introduction

In the following we let N be the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n: \qquad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

In the following, we define the distribution $x^{-1} \ln |x|$ by

$$x^{-1} \ln |x| = \tfrac{1}{2}(\ln^2 |x|)'$$

and we define the distribution $x^{-r-1} \ln |x|$ inductively by

$$x^{-r-1} \ln |x| = \frac{x^{-r-1} - (x^{-r} \ln |x|)'}{r}$$

for $r = 1, 2, \ldots$. It follows by induction that

$$\begin{aligned} x^{-r-1} \ln |x| &= \phi(r) x^{-r-1} + \frac{(-1)^r (x^{-1} \ln |x|)^{(r)}}{r!} \\ &= \phi(r) x^{-r-1} + \frac{(-1)^r (\ln^2 |x|)^{(r+1)}}{2r!}, \end{aligned} \tag{1}$$

where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} i^{-1}, & r = 1, 2, \ldots, \\ 0, & r = 0. \end{cases}$$

The following definition was given in [3].

Definition 1.1. Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\operatorname*{N-lim}_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle$$

for all test functions φ with compact support contained in (a, b).

The following theorem was proved in [7].

Theorem 1.2. *The distribution $(x^r)^{-s}$ exists and*

$$(x^r)^{-s} = x^{-rs}$$

for $r, s = 1, 2, \ldots$.

The following two lemmas can be proved easily by induction.

Lemma 1.3.

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\,dv = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r, \end{cases}$$

for $r = 0, 1, 2, \ldots$.

Lemma 1.4. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1, 1]$, *then*

$$\begin{aligned} \langle x^{-r}, \varphi(x)\rangle &= \int_{-1}^{1} x^{-r}\Big[\varphi(x) - \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\Big]\,dx \\ &\quad + \sum_{k=0}^{r-2}\frac{(-1)^{r-k-1}-1}{(r-k-1)k!}\varphi^{(k)}(0), \end{aligned}$$

for $r = 1, 2, \ldots$, *where the second sum is empty when* $r = 1$.

The next lemma can be proved easily by induction and the use of Lemma 1.4.

Lemma 1.5. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1, 1]$, *then*

$$\begin{aligned} \langle x^{-r}\ln|x|, \varphi(x)\rangle &= \int_{-1}^{1} x^{-r}\ln|x|\Big[\varphi(x) - \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\Big]\,dx \\ &\quad + \sum_{k=0}^{r-2}\frac{(-1)^{r-k-1}-1}{(r-k-1)^2 k!}\varphi^{(k)}(0), \end{aligned}$$

for $r = 1, 2, \ldots$, *where the second sum is empty when* $r = 1$.

2 The Main Theorem

We can now prove a theorem on the composition of distributions in the neutrix setting.

Theorem 2.1. *If* $F_s(x)$ *denotes the distribution* $x^{-s}\ln|x|$, *then the distribution* $F_s(x^r)$ *exists and*

$$F_s(x^r) = r\,F_{rs}(x) \tag{2}$$

for $r, s = 1, 2, \ldots$.

Proof. We first put

$$[(x^r)^{-s}]_n = (x^r)^{-s} * \delta_n(x) = \frac{(-1)^{s-1}}{(s-1)!}\int_{-1/n}^{1/n} \ln|x^r - t|\delta_n^{(s)}(t)\,dt,$$

and

$$\begin{aligned}[F_s(x^r)]_n &= F_s(x^r) * \delta_n(x) \\ &= \phi(s-1)[(x^r)^{-s}]_n - \frac{(-1)^s}{2(s-1)!}\int_{-1/n}^{1/n} \ln^2|x^r - t|\delta_n^{(s)}(t)\,dt. \qquad (3)\end{aligned}$$

Let us note that

$$\int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2|x^r - t|\delta_n^{(s)}(t)\,dt\,dx$$
$$= \begin{cases} 0, & rs-k \text{ odd}, \\ 2\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2|x^r - t|\delta_n^{(s)}(t)\,dt\,dx, & rs-k \text{ even}. \end{cases} \qquad (4)$$

Then

$$\begin{aligned}&\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2|x^r - t|\delta_n^{(s)}(t)\,dt\,dx \\ &= \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_0^{n^{-1/r}} x^k \ln^2|x^r - t|\,dx\,dt \\ &\times \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_{n^{-1/r}}^{1} x^k \ln^2|x^r - t|\,dx\,dt \\ &= \frac{n^{(rs-k-1)/r}}{r}\int_{-1}^{1}\rho^{(s)}(v)\int_0^1 u^{-(r-k-1)/r}\ln^2|(u-v)/n|\,du\,dv \\ &\quad + \frac{n^{(rs-k-1)/r}}{r}\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln^2|(u-v)/n|\,du\,dv \\ &= I_1 + I_2, \qquad (5)\end{aligned}$$

on using the substitutions $u = nx^r$ and $v = nt$.

It is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = 0, \qquad (6)$$

for $k = 0, 1, \ldots, rs-2$.

Now,

$$\begin{aligned}I_2 &= \frac{n^{(rs-k-1)/r}}{r}\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}[\ln|1-v/u| + \ln u - \ln n]^2\,du\,dv \\ &= \frac{n^{(rs-k-1)/r}}{r}\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln^2|1-v/u|\,du\,dv \\ &\quad + \frac{2n^{(rs-k-1)/r}}{r}\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln u \ln|1-v/u|\,du\,dv \\ &\quad - \frac{2n^{(rs-k-1)/r}}{r}\ln n\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\ln|1-v/u|\,du\,dv \\ &= J_1 + J_2 + J_3, \qquad (7)\end{aligned}$$

since $\int_{-1}^{1}\rho^{(s)}(v)\,dv = 0$ for $s = 1, 2, \dots$, by Lemma 1.3.

It is easily seen that

$$\operatorname*{N-lim}_{n\to\infty} J_3 = 0. \tag{8}$$

Next we have

$$\begin{aligned}
J_1 &= \frac{n^{(rs-k-1)/r}}{r}\int_{-1}^{1}\rho^{(s)}(v)\int_1^n u^{-(r-k-1)/r}\left(\sum_{i=1}^{\infty}\frac{v^i}{iu^i}\right)^2 du\,dv \\
&= \frac{2n^{(rs-k-1)/r}}{r}\sum_{i=1}^{\infty}\frac{\phi(i)}{i+1}\int_{-1}^{1}v^{i+1}\rho^{(s)}(v)\int_1^n u^{(k+1)/r-i-2}\,du\,dv \\
&= \frac{2n^{(rs-k-1)/r}}{r}\sum_{i=1}^{\infty}\frac{\phi(i)}{i+1}\,\frac{r(n^{(k+1)/r-i-1}-1)}{k-r(i+1)+1}\int_{-1}^{1}v^{i+1}\rho^{(s)}(v)\,dv
\end{aligned}$$

and it follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} J_1 &= \frac{2\phi(s-1)}{s(rs-k-1)}\int_{-1}^{1}v^s\rho^{(s)}(v)\,dv \\
&= \frac{2(-1)^s\phi(s-1)(s-1)!}{rs-k-1},
\end{aligned} \tag{9}$$

on using Lemma 1.3, for $k = 0, 1, \dots, rs-2$.

Finally

$$\begin{aligned}
J_2 &= \frac{2n^{(rs-k-1)/r}}{r}\sum_{i=1}^{\infty}\frac{1}{i}\int_{-1}^{1}v^i\rho^{(s)}(v)\int_1^n u^{(k+1)/r-i-1}\ln u\,du\,dv \\
&= 2\sum_{i=1}^{\infty}\frac{1}{i}\left[\frac{n^{s-i}\ln n}{k-ri+1}-\frac{r(n^{s-i}-n^{(rs-k-1)/r})}{(k-ri+1)^2}\right]\int_{-1}^{1}v^i\rho^{(s)}(v)\,dv,
\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} J_2 = -\frac{2r}{s(rs-k-1)^2}\int_{-1}^{1}v^s\rho^{(s)}(v)\,dv = -\frac{2(-1)^s r(s-1)!}{(rs-k-1)^2} \tag{10}$$

on using Lemma 1.3, for $k = 0, 1, \dots, rs-2$.

Hence

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty}\int_0^1 x^k\int_{-1/n}^{1/n}\ln^2|x^r-t|\delta_n^{(s)}(t)\,dt\,dx \\
&\qquad = 2(-1)^s(s-1)!\Big[\frac{\phi(s-1)}{rs-k-1}-\frac{r}{(rs-k-1)^2}\Big]
\end{aligned} \tag{11}$$

for $k = 0, 1, \ldots, rs-2$, on using equations (5) to (10). Then using equations (4) and (11), we see that

$$\begin{aligned} \mathrm{N-}\lim_{n\to\infty} \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2|x^r - t|\delta_n^{(s)}(t)\,dt\,dx \\ = 2(-1)^s(s-1)!\Big[\frac{\phi(s-1)}{rs-k-1} - \frac{r}{(rs-k-1)^2}\Big][(-1)^{rs-k-1}-1], \end{aligned} \quad (12)$$

for $k = 0, 1, \ldots, rs-2$.

For the case $k = rs-1$, we have from equation (4)

$$\mathrm{N-}\lim_{n\to\infty} \int_{-1}^{1} x^{rs-1} F_s[(x^r)]_n\,dx = 0. \quad (13)$$

When $k = rs$ equation (5) still holds but now we have

$$I_1 = \frac{n^{-1/r}}{r} \int_{-1}^{1} \rho^{(s)}(v) \int_0^1 u^{-(r-rs-1)/r} \ln^2|(u-v)/n)|\,du\,dv,$$

and it follows that for any continuous function ψ

$$\lim_{n\to\infty} \int_0^{n^{-1/r}} x^{rs} \int_{-1/n}^{1/n} \ln|x^2 - t|\delta_n^{(s)}(t)\psi(x)\,dt\,dx = 0. \quad (14)$$

Similarly

$$\lim_{n\to\infty} \int_{-n^{-1/r}}^{0} x^{rs} \int_{-1/n}^{1/n} \ln|x^2 - t|\delta_n^{(s)}(t)\psi(x)\,dt\,dx = 0. \quad (15)$$

Next, when $x^r \geq 1/n$, we have

$$\begin{aligned} \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt &= n^s \int_{-1}^{1} \ln^2(x^r - v/n)\rho^{(s)}(v)\,dv \\ &= n^s \int_{-1}^{1} \Big[\ln x^r - \sum_{i=1}^{\infty} \frac{v^i}{in^i x^{ri}}\Big]^2 \rho^{(s)}(v)\,dv \\ &= -\sum_{i=s}^{\infty} \frac{2\ln x^r}{in^{i-s}x^{ri}} \int_{-1}^{1} v^i \rho^{(s)}(v)\,dv \\ &\quad + \sum_{i=s}^{\infty} \frac{2\phi(i-1)}{in^{i-s}x^{ri}} \int_{-1}^{1} v^i \rho^{(s)}(v)\,dv. \end{aligned}$$

It follows that

$$\left|\int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\right| \leq \sum_{i=s}^{\infty} \frac{4r|\ln x|K_s}{i^2 n^{i-s} x^{ri}} + \sum_{i=s}^{\infty} \frac{4\phi(i-1)K_s}{i^2 n^{i-s} x^{ri}},$$

for $s = 1, 2, \ldots$, where

$$K_s = \max\{|\rho^{(s)}(v)| : v \in [-1,1]\}.$$

If now $n^{-1/r} < \eta < 1$, then

$$\begin{aligned}
&(s-1)! \int_{n^{-1/r}}^{\eta} x^{rs} \left| \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt \right| dx \\
&\leq -\sum_{i=s}^{\infty} \frac{2rK_s}{i^2 n^{i-s}} \int_{n^{-1/r}}^{\eta} x^{rs-ri} \ln x\,dx + \sum_{i=s}^{\infty} \frac{2\phi(i-1)K_s}{i^2 n^{i-s}} \int_{n^{-1/r}}^{\eta} x^{rs-ri}\,dx \\
&= -\sum_{i=s}^{\infty} \frac{2rK_s}{i^2 n^{i-s}} \left[\frac{\eta^{rs-ri+1}\ln\eta - n^{-(rs-ri+1)/r}\ln n^{-1/r}}{rs-ri+1} - \frac{\eta^{rs-ri+1} - n^{-(rs-ri+1)/r}}{(rs-ri+1)^2} \right] \\
&\qquad + \sum_{i=s}^{\infty} \frac{2\phi(i-1)K_s(\eta^{rs-ri+1} - n^{-(rs-ri+1)/r})}{i^2 n^{i-s}(rs-ri+1)}.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \int_{n^{-1/r}}^{\eta} x^{rs} \left| \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt \right| dx = O(\eta|\ln\eta|),$$

for $r, s = 1, 2, \ldots$.

Thus, if ψ is a continuous function, then

$$\lim_{n\to\infty} \left| \int_{n^{-1/r}}^{\eta} x^{rs}\psi(x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dtdx \right| = O(\eta|\ln\eta|) \tag{16}$$

for $r, s = 1, 2, \ldots$.

Similarly,

$$\lim_{n\to\infty} \left| \int_{-\eta}^{-n^{-1/r}} x^{rs}\psi(x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dtdx \right| = O(\eta|\ln\eta|) \tag{17}$$

for $r, s = 1, 2, \ldots$.

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{rs-1} \frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^{rs}}{(rs)!}\varphi^{(rs)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\Big\langle \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt, \varphi(x)\Big\rangle &= \int_{-1}^{1} \varphi(x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx \\
&= \sum_{k=0}^{rs-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx \\
&+ \frac{1}{(rs)!} \int_{-n^{-1/r}}^{n^{-1/r}} x^{rs}\varphi^{(rs)}(\xi x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx \\
&+ \frac{1}{(rs)!} \int_{n^{-1/r}}^{\eta} x^{rs}\varphi^{(rs)}(\xi x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx \\
&+ \frac{1}{(rs)!} \int_{\eta}^{1} x^{rs}\varphi^{(rs)}(\xi x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx \\
&+ \frac{1}{(rs)!} \int_{-\eta}^{-n^{-1/r}} x^{rs}\varphi^{(rs)}(\xi x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx \\
&+ \frac{1}{(rs)!} \int_{-1}^{-\eta} x^{rs}\varphi^{(rs)}(\xi x) \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt\,dx.
\end{aligned}$$

Using equations (1) and (12) to (17) and noting from equation (4) that on the intervals $[-1, -\eta]$ and $[\eta, 1]$,

$$\lim_{n\to\infty} \frac{(-1)^s}{2(s-1)!} \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt \quad = \quad \phi(s-1)x^{-rs} - F_{rs}(x),$$

since x^{-r} and $F_s(x)$ are continuous functions on these intervals, it follows that

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty} \frac{(-1)^s}{2(s-1)!} \Big\langle \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt, \varphi(x)\Big\rangle \\
&= \sum_{k=0}^{rs-2} \Big[\frac{\phi(s-1)}{rs-k-1} - \frac{r}{(rs-k-1)^2}\Big] \frac{(-1)^{rs-k-1} - 1}{k!}\varphi^{(k)}(0) \\
&\quad + O(\eta|\ln\eta|) + \int_{\eta}^{1} \frac{[\phi(s-1) - r\ln x]\varphi^{(rs)}(\xi x)}{(rs)!}dx \\
&\quad + \int_{-1}^{-\eta} \frac{[\phi(s-1) - r\ln|x|]\varphi^{(rs)}(\xi x)}{(rs)!}\,dx \\
&= \sum_{k=0}^{rs-2} \Big[\frac{\phi(s-1)}{rs-k-1} - \frac{r}{(rs-k-1)^2}\Big] \frac{(-1)^{rs-k-1} - 1}{k!}\varphi^{(k)}(0) \\
&\quad + \int_{-1}^{1} \frac{[\phi(s-1) - r\ln|x|]\varphi^{(rs)}(\xi x)}{(rs)!}dx,
\end{aligned}$$

since η can be made arbitrarily small. It follows that

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} \frac{(-1)^s}{2(s-1)!}\Big\langle \int_{-1/n}^{1/n} \ln^2(x^r - t)\delta_n^{(s)}(t)\,dt, \varphi(x)\Big\rangle \\
&= \sum_{k=0}^{rs-2}\Big[\frac{\phi(s-1)}{rs-k-1} - \frac{r}{(rs-k-1)^2}\Big]\frac{(-1)^{rs-k-1}-1}{k!}\varphi^{(k)}(0) \\
&\quad +\phi(s-1)\int_{-1}^{1} x^{-rs}\left[\varphi(x) - \sum_{k=0}^{rs-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx \\
&\quad -r\int_{-1}^{1} x^{-rs}\ln|x|\left[\varphi(x) - \sum_{k=0}^{rs-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx \\
&= \phi(s-1)\langle x^{-rs}, \varphi(x)\rangle - r\langle x^{-rs}\ln|x|, \varphi(x)\rangle
\end{aligned}$$

on using Lemmas 1.4 and 1.5. This proves equation (2) on the interval $[-1,1]$. However, equation (2) clearly holds on any closed interval not containing the origin, and the proof is complete. For further related results, see [4], [6] and [8].

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