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Outermost-Strongly Solid Variety of Commutative Semigroups

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Abstract : Identities are used to classify algebras into collections called varieties, hyperidentities are used to classify varieties into collections called hypervarieties. Hyperidentities have an interpretation in the theory of switching circuits and are also closely related to clone theory. The tool used to study hyperidentities is the concept of a hypersubstitution, see [1]. The generalized concept of a hypersubstitution is a generalized hypersubstitution. Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language, which need not necessarily preserve the arities. Identities which are closed under generalized hypersubstitutions are called strong hyperidentities. A variety in which each of its identity is a strong hyperidentity is called strongly solid. In this paper we study a submonoid of the monoid of all generalized hypersubstitutions which is called the monoid of all outermost generalized hypersubstitutions and determine the greatest outermost-strongly solid variety of commutative semigroups.

Keywords : generalized hypersubstitution; outermost generalized hypersubstitution; outermost-strongly solid variety; commutative semigroup.
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1 Introduction

Let $n \geq 1$ be a natural number and let $X_n := \{x_1, x_2, ..., x_n\}$ be an n-element set which is called an *n-element alphabet* and its elements are called *variables*. Let

$X := \{x_1, x_2, ...\}$ be a countably infinite set of variables and let $\{f_i \mid i \in I\}$ be a set of n_i-ary operation symbols, which is disjoint from X, indexed by the set I. To every n_i-ary operation symbol f_i we assign a natural number $n_i \geq 1$, called the *arity* of f_i. The sequence $\tau = (n_i)_{i\in I}$ is called the *type*.

For $n \geq 1$, an *n-ary term* of type τ is defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, ..., t_{n_i}$ are n-ary terms of type τ then $f_i(t_1, ..., t_{n_i})$ is an n-ary term of type τ.

The smallest set which contains $x_1, ..., x_n$ and is closed under any finite number of applications of (ii) is denoted by $W_\tau(X_n)$. The set $W_\tau(X) := \cup_{n=1}^{\infty} W_\tau(X_n)$ is called the set of all terms of type τ. An equation of type τ is a pair (s, t) where $s, t \in W_\tau(X)$. Such pairs are commonly written as $s \approx t$. Let $\underline{A} := (A, (f_i^A)_{i\in I})$ be an algebra of type τ. An *equation* $s \approx t$ is an *identity* of an algebra $\underline{A}$, denoted by $\underline{A} \models s \approx t$ if $s^{\underline{A}} = t^{\underline{A}}$ where $s^{\underline{A}}$ and $t^{\underline{A}}$ are the corresponding induced term functions on $\underline{A}$. A *generalized hypersubstitution* of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which does not necessarily preserve arity. The set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we define first the concept of a *generalized superposition* of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Every generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i.

Then we can define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$.

Then we have the following proposition.

Proposition 1.1. ([2]) *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

It turns out that $\underline{Hyp_G(\tau)} := (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid and the monoid $\underline{Hyp(\tau)} := (Hyp(\tau), \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$.

Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)}$ and V be a variety of algebras of type τ. An identity $s \approx t$ of V is called an *M-strong hyperidentity* of V if for every $\sigma \in M, \hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V. A variety V is called *M-strongly solid* if every identity of V is satisfied as an M-strong hyperidentity. In case of $M = Hyp_G(\tau)$ we will call *strong hyperidentity* and *strongly solid* instead of M-strong hyperidentity and M-strongly solid, respectively.

Let V be a variety of algebras of type τ. To test whether an identity $s \approx t$ of V is a strong hyperidentity of V, our definition requires to check, for each generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ that $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V. We can restrict our testing to certain special generalized hypersubstitutions σ those which correspond to V-normal form generalized hypersubstitutions.

2 *V*-Proper and *V*-Normal Form Generalized Hypersubstitutions

The concept of a V-proper hypersubstitution was introduced by J. Płonka in [3] and the concept of a normal form hypersubstitution was introduced by Sr. Arworn and K. Denecke in [4]. In [5], the author and S. Phatchat generalized these concepts to V-proper generalized hypersubstitution and V-normal form generalized hypersubstitution.

Definition 2.1. Let V be a variety of algebras of type τ. A generalized hypersubstitution σ of type τ is called a *V-proper generalized hypersubstitution* if for every identity $s \approx t$ of V, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in V.

The set of all V-proper generalized hypersubstitutions of type τ is denoted by $P_G(V)$. It turns out that $\underline{P_G(V)}$ forms a submonoid of $\underline{Hyp_G(\tau)}$, see [5].

Definition 2.2. Let V be a variety of algebras of type τ. Two generalized hypersubstitutions σ_1 and σ_2 of type τ are called *V-generalized equivalent* if $\sigma_1(f_i) \approx \sigma_2(f_i)$ is an identity of V for all $i \in I$. In this case we write $\sigma_1 \sim_{VG} \sigma_2$.

Theorem 2.3. ([5]) *Let V be a variety of algebras of type τ, and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following statements are equivalent:*

(i) $\sigma_1 \sim_{VG} \sigma_2$.

(ii) For all $t \in W_\tau(X)$, the equations $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in V.

(iii) For all $\underline{A} \in V$, $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$ where $\sigma_k[\underline{A}] = (A, (\sigma_k(f_i)^A)_{i\in I})$, for $k = 1, 2$.

Proposition 2.4. ([5]) *Let V be a variety of algebras of type τ. Then the following statements hold:*

(i) For all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then σ_1 is a V-*proper generalized hypersubstitution iff σ_2 is a* V- *proper generalized hypersubstitution.*

(ii) For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V.

The relation $\sim_{VG}$ is an equivalence relation on $Hyp_G(\tau)$, but it is not neccessarily a congruence relation. So the structure $Hyp_G(\tau)_{|\sim_{VG}}$ is not necessarily a monoid. (Recall that the quotient set gives a monoid if and only if the equivalence relation used to factor it is a congruence.) We factorize $Hyp_G(\tau)$ by $\sim_{VG}$ and then consider the submonoid $\underline{P_G(V)}$ of $\underline{Hyp_G(\tau)}$ is the union of equivalence classes generated by $\sim_{VG}$. This can be done for a submonoid $\underline{M}$ of $\underline{Hyp_G(\tau)}$ and the restricted relation $\sim_{VG_{|M}}$.

Lemma 2.5. ([5]) *Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)}$ and let V be a variety of algebras of type τ. Then the monoid $P_G(V) \cap M$ is the union of all equivalence classes of the restricted relation $\sim_{VG_{|M}}$.*

Definition 2.6. Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ, and let V be a variety of algebras of type τ. Let ϕ be a choice function which chooses from M one generalized hypersubstitution from each equivalence class generated by $\sim_{VG_{|M}}$, and let $N_\phi^M(V)$ be the set of generalized hypersubstitutions which are chosen. Thus $N_\phi^M(V)$ is a set of distinguished generalized hypersubstitutions from M, which we call *V-normal form generalized hypersubstitutions.* We will say that the variety V is $N_\phi^M(V)$*-strongly solid* if for every identity $s \approx t$ of V and for every generalized hypersubstitution $\sigma \in N_\phi^M(V)$, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V.

Theorem 2.7. ([5]) *Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ and let V be a variety of algebras of type τ. For any choice function ϕ, V is M-strongly solid if and only if V is $N_\phi^M(V)$-strongly solid.*

3 Outermost-Strongly Solid Varieties of Commutative Semigroups

In this section we give some examples of outermost-strongly solid varieties of commutative semigroups and then determine the greatest outermost-strongly solid variety of commutative semigroups. We recall first the definition of an outermost generalized hypersubstitution.

Definition 3.1. ([6]) A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called an *outermost generalized hypersubstitution* if for every $i \in I$, the first variable and the last variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ are x_1 and x_{n_i}, respectively.

The set of all outermost generalized hypersubstitutions of type τ is denoted by $Out_G(\tau)$. $\underline{Out_G(\tau)}$ also forms a submonoid of $\underline{Hyp_G(\tau)}$, see [6].

Let $Alg(\tau)$ be the set of all algebras of type τ. For a class $K \subseteq Alg(\tau)$ and for a set Σ of equations of this type, we use the following notations as usual.

$IdK := \{s \approx t \mid \forall \underline{A} \in K(\underline{A} \models s \approx t\}$ - the set of all identities of K.

$HIdK := \{s \approx t \mid \forall \underline{A} \in K, \forall \sigma \in Hyp(\tau)(\underline{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t]\}$ - the set of all hyperidentities of K.

$Mod\Sigma := \{\underline{A} \in Alg(\tau) \mid \forall s \approx t \in \Sigma(\underline{A} \models s \approx t)\}$ - the variety defined by Σ.

$HMod\Sigma := \{\underline{A} \in Alg(\tau) \mid \forall s \approx t \in \Sigma(\underline{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t])\}$ - the hyperequational class defined by Σ.

Definition 3.2. Let V be a variety of algebras of type τ. $s \approx t \in IdV$ is called an *outermost-strong hyperidentity* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for all $\sigma \in Out_G(\tau)$. In this case we write $V \overset{\mathrm{Out_G}}{\models} s \approx t$.

We define:

$H_{Out_G}IdK := \{s \approx t \mid \forall \underline{A} \in K, \forall \sigma \in Out_G(\tau)(\underline{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t])\}$ - the set of all outermost-strong hyperidentities of K.

$H_{Out_G}Mod\Sigma := \{\underline{A} \in Alg(\tau) \mid \forall s \approx t \in \Sigma, \forall \sigma \in Out_G(\tau)(\underline{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t])\}$ - the outermost-strong hyperequational class defined by Σ.

$\chi^E_{Out_G}[s \approx t] := \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid \sigma \in Out_G(\tau)\}$ and

$\chi^E_{Out_G}[\Sigma] := \bigcup\limits_{s \approx t \in \Sigma} \chi^E_{Out_G}[s \approx t]$, this property is called *additive*.

Then $\chi^E_{Out_G} : \mathcal{P}(W_\tau(X) \times W_\tau(X)) \to P(W_\tau(X) \times W_\tau(X))$ is an operator defined on the power set of $W_\tau(X) \times W_\tau(X)$. Then we have the following proposition.

Proposition 3.3. *The operator $\chi^E_{Out_G}$ has the properties of an additive closure operator.*

Proof. Since the identity mapping belongs to $Out_G(\tau)$, so $\Sigma \subseteq \chi^E_{Out_G}[\Sigma]$ for all $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$. Let $\Sigma_1 \subseteq \Sigma_2 \subseteq W_\tau(X) \times W_\tau(X)$. By a consequence of additivity, $\chi^E_{Out_G}[\Sigma_1] \subseteq \chi^E_{Out_G}[\Sigma_2]$. Let $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$. By a consequence of monotonicity and the closedness of $Out_G(\tau)$ with respect to the product $\circ_G, \chi^E_{Out_G}[\chi^E_{Out_G}[\Sigma]] = \chi^E_{Out_G}[\Sigma]$. □

Using the operator $\chi^E_{Out_G}$ we define:

Definition 3.4. A variety V of algebras of type τ is called *outermost-strongly solid* if $\chi^E_{Out_G}[IdV] = IdV$, i.e., if every identity in V is an outermost-strong hyperidentity.

Clearly, every trivial variety is outermost-strongly solid.

Theorem 3.5. *Let V be a variety of algebras of type τ. Then the following statements are equivalent.*

(i) V *is outermost-strongly solid.*

(ii) $H_{Out_G}IdV = IdV$.

(iii) $V = H_{Out_G}ModH_{Out_G}IdV$.

Proof. (i)⇒(ii) Since V is outermost-strongly solid, $\chi^E_{Out_G}[IdV] = IdV$. By the definition of $H_{Out_G}IdV$ we have

$$\begin{aligned} H_{Out_G}IdV &= \{s \approx t \in W_\tau(X) \times W_\tau(X) \mid \forall \underline{A} \in V(\underline{A} \overset{\mathrm{Out_G}}{\models} s \approx t)\} \\ &= \{s \approx t \in W_\tau(X) \times W_\tau(X) \mid \forall \underline{A} \in V(\underline{A} \models \chi^E_{Out_G}[s \approx t])\} \\ &= \chi^E_{Out_G}[IdV]. \end{aligned}$$

By a consequence of $\chi^E_{Out_G}[IdV] = IdV$ we have $H_{Out_G}IdV = IdV$.
(ii)⇒(iii) Consider

$$\begin{aligned} H_{Out_G}ModH_{Out_G}IdV &= H_{Out_G}ModIdV \\ &= \{\underline{A} \in Alg(\tau) \mid \forall s \approx t \in IdV(\underline{A} \overset{\mathrm{Out_G}}{\models} s \approx t)\} \\ &= \{\underline{A} \in Alg(\tau) \mid \forall s \approx t \in \chi^E_{Out_G}[IdV](\underline{A} \models s \approx t)\} \\ &= Mod\chi^E_{Out_G}[IdV] \\ &= ModH_{Out_G}IdV \\ &= ModIdV \\ &= V. \end{aligned}$$

(iii)⇒(i) Since $V = H_{Out_G}ModH_{Out_G}IdV$, we have

$$\begin{aligned} IdV &= IdH_{Out_G}ModH_{Out_G}IdV \\ &= IdMod\chi^E_{Out_G}[H_{Out_G}IdV] \\ &= IdMod\chi^E_{Out_G}[\chi^E_{Out_G}[IdV]] \\ &= IdMod\chi^E_{Out_G}[IdV] \\ &= \chi^E_{Out_G}[IdV]. \end{aligned}$$

This finishes the proof. □

Proposition 3.6. *Every strongly solid variety* V *of commutative semigroups is an outermost-strongly solid variety.*

Proposition 3.7. *The variety of zero semigroups* $Z = Mod\{xy \approx uv\}$ *is the least non-trivial outermost-strongly solid variety of commutative semigroups.*

From now on, we fix our type to be $\tau = (2)$. So we have only one binary operation symbol, and we shall denote the binary operation of our variety simply by juxtaposition, and omit brackets where convenient due to associativity. Furthermore, the generalized hypersubstitution σ which maps f to the term t is denoted by σ_t.

Proposition 3.8. *The variety* $V^C_{Rec} := Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3) \approx x_1x_3, x_1x_2 \approx x_2x_1\}$ *is an outermost-strongly solid variety of commutative semigroups.*

Proof. To show that V^C_{Rec} is an outermost-strongly solid variety of commutative semigroups, we have to check that each of its identity is an outermost-strong hyperidentity. By Theorem 2.7 with the identities of V^C_{Rec}, we can restrict our checking to a single outermost generalized hypersubstitution $\sigma_{x_1x_2}$. It suffices to check this generalized hypersubstitution only an equational basis for V^C_{Rec}. Obviously, if we apply $\sigma_{x_1x_2}$ on each side of $(x_1x_2)x_3 \approx x_1(x_2x_3) \approx x_1x_3$ and $x_1x_2 \approx x_2x_1$ we get the same identities because $\sigma_{x_1x_2}$ is the identity element of $Out_G(2)$. □

Lemma 3.9. *Let* $V \subseteq Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2\}$. *Then*

(i) $x_1^nx_2 \approx x_1x_2^n \in IdV$, *for* $n \geq 3$.

(ii) $x_1^nx_2 \approx x_1^{n+1}x_2 \in IdV$, *for* $n \geq 3$.

Proof. We give the proofs by mathematical induction on n and using $(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2 \in IdV$.
(i) First, if $n = 3$, then our assertion is that $x_1^3x_2 \approx x_1x_1^2x_2 \approx x_1x_1x_2^2 \approx x_1^2x_2x_2 \approx x_1x_2^2x_2 \approx x_1x_2^3$. So $x_1^3x_2 \approx x_1x_2^3 \in IdV$. Suppose that for $k \geq 3, x_1^kx_2 \approx x_1x_2^k \in IdV$. Consider $x_1^{k+1}x_2 \approx x_1^{k-1}x_1^2x_2 \approx x_1^{k-1}x_1x_2^2 \approx x_1^kx_2x_2 \approx x_1x_2^kx_2 \approx x_1x_2^{k+1}$. So $x_1^{k+1}x_2 \approx x_1x_2^{k+1} \in IdV$. By mathematical induction, we get $x_1^nx_2 \approx x_1x_2^n \in IdV$ for $n \geq 3$.
(ii) First, if $n = 3$, then our assertion is that $x_1^3x_2 \approx x_1x_1^2x_2 \approx x_1x_1x_2^2 \approx x_1^2x_2^2 \approx (x_1^2)^2x_2 \approx x_1^4x_2$. So $x_1^3x_2 \approx x_1^4x_2 \in IdV$. Suppose that for $k \geq 3, x_1^kx_2 \approx x_1^{k+1}x_2 \in IdV$. Consider $x_1^{k+1}x_2 \approx x_1x_1^kx_2 \approx x_1x_1^{k+1}x_2 \approx x_1^{k+2}x_2$. So $x_1^{k+1}x_2 \approx x_1^{k+2}x_2 \in IdV$. By mathematical induction, we get $x_1^nx_2 \approx x_1^{n+1}x_2 \in IdV$ for $n \geq 3$. □

Let $V^C_{Out_G}$ be the variety of commutative semigroups defined by the identity $x_1^2x_2 \approx x_1x_2^2$, i.e. $V^C_{Out_G} = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2\}$.

Theorem 3.10. $V^C_{Out_G}$ *is the greatest outermost-strongly solid variety of commutative semigroups.*

Proof. The greatest outermost-strongly solid variety of commutative semigroups is the model class of all semigroups for which the associative law and the commutative law are satisfied as outermost-strong hyperidentities, i.e. the class $H_{Out_G}Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}$. We will show that $V^C_{Out_G} = H_{Out_G}Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}$. Under the application of $\sigma_{x_1x_2} \in Out_G(2)$ to the associative law and the commutative law, we obtain $(x_1x_2)x_3 \approx x_1(x_2x_3)$ and $x_1x_2 \approx x_2x_1$. Applying $\sigma_{x_1^2x_2} \in Out_G(2)$ to the commutative law provides $x_1^2x_2 \approx x_1x_2^2$. That means $(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2 \in Id(H_{Out_G}Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\})$. Hence $H_{Out_G}Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}$ satisfies all identities of $V^C_{Out_G}$, i.e., $H_{OutG}Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\} \subseteq V^C_{OutG}$. To prove the converse inclusion we have to check the associative law and the commutative law using all outermost generalized hypersubstitutions. By using Theorem 2.7 together with the identities of V^C_{OutG}, we can restrict our checking to the following outermost generalized hypersubstitutions σ_t where $t \in \{x_1^m x_2^n \mid m,n \in \mathbb{N}\} \cup \{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n \mid m,n,k,i_k,j_k \in \mathbb{N} \text{ and } i_k > 2\}$.

If we apply $\sigma_{x_1^m x_2^n}$, for $m,n \in \mathbb{N}$ on both sides of the associative law and the commutative law, we have

$$\begin{aligned} \hat{\sigma}_{x_1^m x_2^n}[(x_1x_2)x_3] &= S^2(x_1^m x_2^n, S^2(x_1^m x_2^n, x_1, x_2), x_3) \\ &= (x_1^m x_2^n)^m x_3^n \\ &= (x_1^m)^m (x_2^n)^m x_3^n, \end{aligned}$$

and $\hat{\sigma}_{x_1^m x_2^n}[x_1(x_2x_3)] = S^2(x_1^m x_2^n, x_1, S^2(x_1^m x_2^n, x_1, x_2), x_2, x_3) = x_1^m (x_2^m x_3^n)^n = x_1^m (x_2^m)^n (x_3^n)^n$. Also, $\hat{\sigma}_{x_1^m x_2^n}[x_1x_2] = S^2(x_1^m x_2^n, x_1, x_2) = x_1^m x_2^n$ and $\hat{\sigma}_{x_1^m x_2^n}[x_2x_1] = S^2(x_1^m x_2^n, x_2, x_1) = x_2^m x_1^n$.

Using the associative law, the commutative law and identity $x_1^2x_2 \approx x_1x_2^2$ together with Lemma 3.9, we have $\hat{\sigma}_{x_1^m x_2^n}[(x_1x_2)x_3] = \hat{\sigma}_{x_1^m x_2^n}[x_1(x_2x_3)]$ and $\hat{\sigma}_{x_1^m x_2^n}[x_1x_2] = \hat{\sigma}_{x_1^m x_2^n}[x_2x_1]$.

If we apply $\sigma_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}$, where $m,n,k,i_k,j_k \in \mathbb{N}$ and $i_k > 2$, on both sides of the associative law and the commutative law, we have

$$\begin{aligned} \hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[(x_1x_2)x_3] &= S^2(x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n, S^2(x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n, x_1, x_2), x_3) \\ &= (x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n)^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_3^n \\ &= (x_1^m)^m (x_{i_1}^{j_1})^m \dots (x_{i_k}^{j_k})^m (x_2^n)^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_3^n \end{aligned}$$

and

$$\begin{aligned} \hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[x_1(x_2x_3)] &= S^2(x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n, x_1, S^2(x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n, x_2, x_3)) \\ &= x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} (x_2^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_3^n)^n \\ &= x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} (x_2^m)^n (x_{i_1}^{j_1})^n \dots (x_{i_k}^{j_k})^n (x_3^n)^n. \end{aligned}$$

Also, $\hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[x_1x_2] = S^2(x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n, x_1, x_2) = x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n$ and

$$\hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[x_2x_1] = S^2(x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n, x_2, x_1) = x_2^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_1^n.$$

Using the associative law, the commutative law and identity $x_1^2x_2 \approx x_1x_2^2$ together with Lemma 3.9, we have $\hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[(x_1x_2)x_3] = \hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[x_1(x_2x_3)]$ and $\hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[x_1x_2] = \hat{\sigma}_{x_1^m x_{i_1}^{j_1} \dots x_{i_k}^{j_k} x_2^n}[x_2x_1]$. This finishes the proof. $\square$

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References

[1] K. Denecke, D. Lau, R. Pöschel, D. Schweigert, Hypersubstitutions, Hyperequational classes and clones congruence, Contributions to General Algebras 7 (1991) 97-118.

[2] S. Leeratanavalee, K. Denecke, Generalized hypersubstitutions and strongly solid varieties, In General Algebra and Applications, Proc. of the "59 th Workshop on General Algebra", "15 th Conference for Young Algebraists Potsdam 2000", Shaker Verlag (2000), 135-145.

[3] J. Płonka, Proper and inner hypersubstitutions of varieties, General Algebra and Ordered Sets, Proc. of the International Conference: Summer School on General Algebra and Ordered Sets 1994, Palacky University Olomouc (1994), 106-155.

[4] Sr. Arworn, K. Denecke, Sets of hypersubstitutions and set solid varieites, In Algebras and Combinatorics, Proc. of the International Congress, ICAC'97, Hong Kong, Springer-Verlag, Singapore (1999), 9-33.

[5] S. Leeratanavalee, S. Phatchat, Pre-strongly solid and left-edge(right-edge)-strongly solid varieties of semigroups, International Journal of algebra 1 (5) (2007) 205-226.

[6] S. Leeratanavalee, Submonoids of generalized hypersubstitutions, Demonstratio Mathematica XL (1) (2007) 13-22.

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