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On Mathematical Modeling and Analysis of Co-Movement and Optimal Portfolios of Stock Markets

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Abstract : This paper proposes to use the concept of time-varying copulas in probability theory as an appropriate mathematical modeling tool for investigating an important problem in economics, namely the co-movement of stock markets as well as optimal portfolio constructions on them. In the sense of expected shortfall, a coherent risk measure widely used in risk management of financial markets, we show that our time-varying copula models for GARCH perform better than the conventional DCC-GARCH model. We exhibit also various advantages of this approach in investment decisions. An application to G7 stock markets is given.

Keywords : time-varying copulas; expected shortfall; backtesting; G7.
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1 Introduction

Financial data often exhibit the possible skewness of their potential distribution as well as variations in dependence structures during different economic regimes such as downturns or upturns. The linear correlation is not a good ap-

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proach to handle non-normal and heavy-tailed data to measure risk in financial markets. Fortunately, copula theory can overcome the above issues. The copula models become attractive tools to completely capture the type of dependence and can also be treated as time dynamic. There are many important applications in econometrics using copula methodology that were published by Thai Journal of Mathematics, such as Autchariyapanitkul et al. [1] and Nguyen et al. [2] etc. This study is an application of time-varying copulas to co-movement and optimal portfolios of G7 stock markets.

Economic downturn or crises in some of these G7 countries can be contagious to other members of the group and the rest of the world. The recent 2008-2009 financial crisis had severely affected the United States and other developed countries proving the market linkage and contagion effects. Many studies define contagion as the spread of market downside risk from one country to the others and the effect of crises due to market linkages. The cross-market correlations are significantly greater during crisis periods compared to normal periods. The larger contagion reflects the stronger correlations (see Forbes and Rigobon[3], [4] and Hwang et al. [5], Celik [6], Jayech and Zina [7]). In recent years, there have been abundant studies that examine the volatility, long memory, and spillover effect in G7 stock markets, such as Bilel and Nadhem [8], Liow [9], Bentes [10], Chiang and Wang [11] etc. However, there are relatively few studies that have focused on the portfolio and relationship among the G7's major stock market indices. For example, Bhar and Hamori [12] examined the issue of co-movement in G7 equity markets. They found that the USA and Canada remain in phase with each other for most of the time, and the three main European markets also usually move in phase. Lee et al. [13] studied the relationship and portfolio construction between stock price of G7 and oil prices using DCC models. Chen et al. [14] used semiparametric T copula models to estimate the VaR and ES risks of stock-bond portfolios of G7 stock market indices. Most of the studies have resulted in diversification and hedging benefits among G7 equity markets. As a result, modeling co-movement and forecasting optimal portfolio allocation of G7 stock markets are of considerable interest to help investors, portfolio managers, or risk managers in their decision making to diversify portfolios as well as avoid the loss and riskiness. Many studies found that in many equity markets, the impacts of negative price movement volatility are different from the positive one. Various works on multivariate financial returns revealed significant increase and much stronger correlation between international equity returns when market trend is downturn than in a time of normal situation (Longin and Solnik [15], Ang and Bekaert [16], Ang and Chen [17]). Specifically, we seek to explore and identify the models of high performance for studying the dynamic correlation in bull and bear market environments for risk management in international financial markets.

Many studies have been conducted to investigate volatility and co-movement between and among different financial assets using various tools and methodologies. Earlier studies have focused on use of the univariate generalized autoregressive conditional heteroscedasticity (GARCH) model proposed and developed by Engle [18] and Bollerslev [19] to estimate and forecast changing volatility of each asset in financial time series data. Recently, the class of multivariate GARCH model has been developed and extended into Dynamic Conditional Correlation (DCC) model introduced by Engle [20]. The DCC-GARCH model is capable of estimating large time-varying covariance matrices and is widely used to study co-movement and volatility between some assets or international stock markets. Multivariate GARCH is more flexible than the univariate GARCH for fitting portfolio's returns that are given weight because whenever the weight vector of univariate GARCH changes, the model has to be re-estimated. Some studies employed the DCC-GARCH model to estimate international asset portfolios such as Lee and Chinn [21] that used daily stock indexes of G7 to calculate the Value at Risk (VaR) with DCC-GARCH, simple moving average (SMA), and the Exponentially Weighted Moving Average (EWMA). The most effective model for measuring VaR was found to be DCC-GARCH$(1,1)-t$ followed by DCC-GARCH$(1,1)$ while SMA was the last adoptable model. Gupta and Donleavy [22] used DCC-GARCH model to estimate time-varying correlations and estimate the portfolio optimization model for emerging markets and Australia. They found that, in spite of increasing correlations, the Australian investors still gain potential benefits from diversifying into international emerging markets.

Nevertheless, many previous works on multivariate models were with normal distribution assumptions of data and thus are not efficient enough to estimate VaR and construct a portfolio of financial assets. The limitation of the correlation analysis especially in financial data assuming linear correlation is not a good approach to handle the case of non-normal and heavy-tailed [23]. Patton [24] verified and extended the constant copula to time-varying copula to analyze the dependence structure between financial and exchange rate markets. After that, the copula began to be used in finance. For example, Hyde et al. [25] studied time-varying conditional correlations among equity markets in the Asian-Pacific counties, Europe, and the US. Ane and Labidi [26] and Bartram et al. [27] applied the copula model to measure dependences among some European stock indices. Wang et al. [28] used time-varying copula model to study the dependence structure between the Chinese market and other international stock markets.

This study has the first objective of analyzing the co-movement between pair G7 stock markets thereby capturing the dynamic characteristics of nonlinear correlation and tail dependences. Second, the optimal portfolios of G7 stock markets are calculated by minimizing expected shortfall framework. The third objective is to test whether the time-varying copula-GARCH model has a better performance than DCC-GARCH model in risk management. The main contribution of this paper is that the time-varying copula-GARCH is proposed to examine the optimal portfolios for G7 stock markets. This study applied 17 static copulas and several time-varying copulas to capture inter-dependences in G7 stock markets. We found the time-varying copulas to have a better performance than the static copulas. Therefore, the time-varying copula-GARCH model was used to measure optimal portfolios thereby improving the accuracy of predicted extreme loss. In addition, this study compared DCC-GARCH with the time-varying copula-GARCH models in terms of optimal portfolios. According to the empirical analysis, we have confidence in the time-varying copula-GARCH model, while the DCC-GARCH model maybe shows that linear correlation has some drawbacks to measure risk and optimal portfolio. Last, we proposed some suggestions to investors and institutions regarding portfolio diversification, risk hedging, dynamic asset allocation, and portfolio rebalancing measures.

2 Preliminaries

In this section, we recall some basic definitions and theorems concerning copulas that we will use in subsequent subject analysis of our work.

For volatility analysis, we first used the ARMA-GARCH to extract the standardized residuals which are the inputs for the DCC-GARCH model.

Definition 2.1 ([29]). A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is a *mixture autoregressive moving average model of order p and q*, ARMA(p,q), if it satisfies the following equation:

$$X_t = \mu + \phi_1 X_{t-1} + .. + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q}, \text{ for all } t \in \mathbb{Z}$$

$$\Phi(L)X_t = \mu + \Theta(L)\epsilon_t,$$

where $\phi_p \neq 0, \theta_q \neq 0$, μ is constant term, $(\epsilon_t)_{t\in\mathbb{Z}}$ is a weak white noise process with expectation zero and variance $\sigma_\varepsilon^2(\varepsilon_t \sim WN(0,\sigma_\varepsilon^2))$ AR and MA polynomial as follows

$$\Phi(L) = 1 - \phi_1 L - ...\phi_p L^p \quad and \quad \Theta(L) = 1 + \theta_1 L + ...\theta_q L^q.$$

We require that there are no common factors between the AR and MA polynomials; otherwise, the order (p,q) of the model can be reduced.

Definition 2.2 ([30, Definition 3.1.3]). Let $m,n \geq 0$ be given. A real-valued time series $\{Y\}_{t\in\mathbb{Z}}$ defined by $Y_t = \sigma_t\varepsilon_t$ with $\varepsilon_t \sim WN(0,1)$ iid and $\{\sigma_t^2\}_{t\in\mathbb{Z}}$ a stochastic process with coefficients $\alpha_0, \alpha_1, ..., \alpha_m,\ \beta_1, ...\beta_n \geq 0,\ \alpha_m, \beta_n \neq 0$ satisfying the difference equation

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i Y_{t-i}^2 + \sum_{k=1}^{n} \beta_k \sigma_{t-k}^2, \tag{2.1}$$

is called *GARCH(m,n) process* (stemming from Generalized Auto Regressive Conditionally Heteroscedastic). ${\sigma_t^2}_{t\in\mathbb{Z}}$ is called *volatility process.* If $n = 0$, $\{Y\}_{t\in\mathbb{Z}}$ is called an *ARCH(m) process.*

Definition 2.3 ([30, Definition 3.2.2]). Let $\{Y\}_{t\in\mathbb{Z}}$ be a vector-valued process with $Y_t \in \mathbb{R}^d$, $d > 2$ such that with $\mathbb{F}_t - t = \sigma(Y_{t-1}, Y_{t-2}...)$ representing the information set up to time $t-1$, we set

$$Y_t = \Gamma_t^{1/2}\varepsilon_t$$

$$\Gamma_t = Cov(Y_t \mid \mathbb{F}_{t-1}),$$

where ε_t is an uncorrelated zero-mean, identity-covariance d-variate with white noise process $\varepsilon_t \sim WN(0, I_d)$ and conditional covariance matrices Γ_t decomposed into conditional standard deviations and correlations as

$$\sum_t = D_t \mathrm{P} D_t,$$

where $D_t = diag(\sigma_{1t}, ..., \sigma_{dt})$ and $\mathrm{P} = (\rho_{ij})_{i,j=1,..,d}$ a positive definite $(d \times d)$-matrix where $\rho_{ij} = 1$ for all $i = 1, ..., d$. This representation allows to model σ_{jt}^2, $j = 1, ..., d$ as volatility processes in univariate GARCH(m, n) according to equation (2.1). The vector $\sigma_t^2 = (\sigma_{1t}^2, ..., \sigma_{dt}^2)$ as the following form

$$\sigma_t^2 = \omega + \sum_{i=1}^{m} A_i Y_{t-i}^2 + \sum_{k=1}^{n} B_k \sigma_{t-k}^2,$$

where $\omega \in \mathbb{R}^d$ has positive components, $A_i, B_k \in \mathbb{R}^{d\times d}$ are diagonal matrices with positive entries and $Y_t^2 = Y_t \odot Y_t$, $\odot$ denotes the *Hadamard matrix product.* Moreover, the conditional covariance matrices Γ_t is under the respective condition.

1. $P \equiv P_0$ is constant and diagonal, Y_t is called a *constant conditional correlation (CCC) GARCH*(m, n) *process* if P is not diagonal,

2. or $P = P_t$ for every $t \in \mathbb{Z}$ such that
$P_t = (I_d \odot Q_)t^{-1/2} Q_t (I_d \odot Q_t)^{-1/2}$
$Q_t = (1 - a - b)S + a\varepsilon_{t-1}\varepsilon_{t-1}^T + bQ_{t-1}$

with $a > 0, b \geq 0, a + b < 1$, $S = Cov(\varepsilon_1 \varepsilon_1^T)$ and a positive definite $Q_0 \in \mathbb{R}^{d\times d}$ as starting value, then Y_t is called a *dynamic conditional correlation (DCC) GARCH*(m, n) *process* by Engle [20].

In the following, the copula approaches are reviewed so that we can use them to examine the dependence structure between stock returns.

Theorem 2.4 ([24, Theorem I.2]). *Let F be the distribution of X, G be the distribution of Y, and H be the joint distribution of (X, Y). Assume that F and G are continuous. Then there exists a unique copula C such that*

$$H(x, y) = C(F(x), G(y)), \text{ for all } (x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}. \tag{2.2}$$

Conversely, if we let F and G be a distribution function and C be a copula, then the function H defined by equation (2.2) *is a bivariate distribution function with marginal distributions F and G.*

Definition 2.5 ([24, Proposition I.3]). Let the joint distribution of (X, Y, W) be H_{xyw}, the marginal distribution of W be F_w, and the support of W be W. The conditional bivariate distribution of $(X, Y)|W$, denoted H, is defined as

$$H(x, y|w) \equiv f_w(w)^{-1} \cdot \frac{\partial H_{xyw}(x, y, w)}{\partial w}$$

and satisfies the following properties:

1. $H(x, -\infty)|w = H(-\infty, y|w) = 0$, and $H(\infty, \infty|w) = 1$ for all $(x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ and each $w \in W$,

2. $V_H([x_1, x_2] \times [y_1, y_2]|w) \equiv H(x_2, y_2|w) - H(x_1, y_2|w) - H(x_2, y_1|w) + H(x_1, y_1|w) \geq 0$ for all $x_1, x_2, y_1, y_2) \in \overline{\mathbb{R}}$, such that $x_1 \leq x_2, y_1 \leq y_2$ and each $w \in W$

The first condition simply provides the upper and lower bounds on the distribution function. The second condition ensures that the probability of observing a point in the region $[x_1, x_2] \times [y_1, y_2]$ is non-negative. Then define the conditional copula.

Definition 2.6 ([24, Proposition I.4]). *A-two dimensional conditional copula* is a function $C : [0, 1] \times [0, 1] \times W \longrightarrow [0, 1]$ with the following properties:

1. $C(u, 0|w) = C(0, v|w) = 0$, and $C(u, 1|w) = u$ and $C(1, v|w) = v$ for every u,v in $[0, 1]$ and each $w \in W$

2. $V_c([u_1, u_2]) \times [v_1, v_2]|w) \equiv C(u_2, v_2|w) - C(u_1, v_2|w) - C(u_2, v_1|w) + C(u_1, v_1|w) \geq 0$ for all $u_1, u_2, v_1, v_2 \in [0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$) and each $w \in W$.

Theorem 2.7 ([24, Theorem I.3]). *Let F be the conditional distribution of $X|W, G$ be the conditional distribution of $Y|W$ and H be the joint conditional distribution of $(X, Y|W)$. Assume that F and G are continuous in x and y. Then there exists a unique conditional copula C such that*

$$H(x, y|w) = C(F(x|w), G(y|w)|w), \quad \text{for all } (x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \quad \text{and each } w \in W \tag{2.3}$$

Conversely, if let F be the conditional distribution of $X|W, G$ be the conditional distribution of $Y|W$, and C be a conditional copula, then function H defined by equation (2.3) *is a conditional bivariate distribution function with conditional marginal distributions F and G.*

The following is review of risk measures in finance econometrics (for more details, see Sriboonchita et al. [31])

Definition 2.8. Let $(\Omega, \mathbb{F}, P)$ be a probability space and V be a non-empty set of $\mathbb{F}$-measurable real-value random variables. Then any mapping $\rho : V \longrightarrow \mathbb{R} \cup \{\infty\}$ is called a *risk measure.*

Definition 2.9. The risk measure ρ is said to be *coherent* if it satisfies the following conditions;
(1) Monotonicity: $X, Y \in V, X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$.
(2) Positive homogeneity: $X \in V, h > 0, hX \in V \Rightarrow \rho(hX) = h\rho(X)$.
(3) Translation invariance: $X \in V, a \in \mathbb{R}, X + a \in V \Rightarrow \rho(X + a) = \rho(X) - a$.
(4) Sub-additivity: $X, Y \in V, X + Y \in V \Rightarrow \rho(X + Y) \leq \rho(X) + \rho(Y)$.

Definition 2.10. The value-at-Risk at level $\alpha \in (0, 1]$ of a the random variable X is defined to be it *α-quantile*, i.e.,

$$VaR_\alpha(X) = q_\alpha(X) = inf\{X \in \mathbb{R} : P(X \leq x) \geq \alpha\}.$$

Remark: *VaR_α is not sub-additive, and hence is not a coherent risk measure.*

3 Time Varying Copula-Based ARMA-GARCH Model

We adopted the step parametric estimation procedure (Joe and Xu [32]) to estimate the copula and marginal distribution parameters separately. To do so, we first used an ARMA-GARCH model with skewed student-t distribution to fit into each individual index. The standardized residuals of each margin are assumed to be i.i.d. We then transformed the standardized residuals into a uniform distribution between 0 and 1.

3.1 ARMA-GARCH Model

Bollerslev [19] proposed GARCH model which was developed from ARCH model by Engle [18] and which allows the conditional variance to be dependent upon previous lags. Then, on the margin of return series, the GARCH(1,1) model is sufficient to provide good estimates of the conditional volatility of financial variable by Bollerslev et al. [33]. The ARMA(p, q)-GARCH(m, n) is expressed as

$$r_t = \mu + \sum_{i=1}^{p} a_i r_{t-i} + \sum_{j=1}^{q} b_j \varepsilon_{t-j} + \varepsilon_t,$$

$$\varepsilon_t = \sigma_t \cdot z_t,$$

$$\sigma_t^2 = \omega_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{n} \beta_i \sigma_{t-i}^2,$$

The ARMA(p, q) process of autoregressive order p and moving average order q for conditional mean equation where r_t is the dependent variable at time t, μ is a constant term of the conditional mean equation, autoregressive coefficients a_i, moving average coefficients b_j, ε_t represents residuals, and z_t is a sequence of standardized residuals with zero mean and unit variance. In addition, σ^2 is the conditional variance of return series at time t, α is ARCH parameter associated with ε_{t-i}^2 representing to the short-run shock , β is a GARCH parameter associated with the volatility spillover effect from σ_{t-i}^2 to express the persistence of volatility, and the parameter restrictions in the variance impose the conditions $\sum_{i=1}^{p} a_i < 1$, $\omega_0 \geq 0, \alpha_i \geq 0, \beta_i \geq 0$, and $\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{n} \beta_i < 1$ to guarantee the conditional variance process to be positive and stationary.

3.2 Copulas

The earlier applications used the copulas with unconditional distribution to apply in various fields. Patton [24] employed the copulas for modeling the time-varying dependence by extension of the Sklar's theorem for conditional distribution.

Let $r_{1,t}$ and $r_{2,t}$ be the random variables that indicate a pair of G7 stock returns at period t. The marginal conditional distribution cumulative distribution functions are $u_{1,t} = G_1(r_{1,t} \mid \Omega_{t-1})$ and $u_{2,t} = G_2(r_{2,t} \mid \Omega_{t-1})$; the Ω_{t-1} is the past information. Then, the conditional copula function $C(u_{1,t}, u_{2,t} \mid \Omega_{t-1})$ using the bivariate time-varying cumulative distribution functions of random variables $r_{1,t}$ and $r_{2,t}$ can be expressed as

$$F(r_{1,t}, r_{2,t} \mid \Omega_{t-1}) = C(u_{1,t}, u_{2,t} \mid \Omega_{t-1}).$$

Considering the assumption that the cumulative distribution function is differentiable and the conditional joint density is given by:

$$\begin{aligned} f(r_{1,t}, r_{2,t} \mid \Omega_{t-1}) &= \frac{\partial^2 F(r_{1,t}, r_{2,t} \mid \Omega_{t-1})}{\partial r_{1,t} \partial r_{2,t}} \\ &= c(u_{1,t}, u_{2,t} \mid \Omega_{t-1}) \times g_1(r_{1,t} \mid \Omega_{t-1}) \times g_2(r_{2,t} \mid \Omega_{t-1}), \end{aligned}$$

where $g_i(\cdot)$ is the density function corresponding to $G_i(\cdot)$.

An important characteristic of copulas is a capability to capture tail dependences. In financial market, the dependence structure is used to examine both upper and lower tails of the returns; that is very useful for measuring tail dependence of the asset returns in financial market with a tendency to crash together. For the joint distribution [34], the upper and lower tail dependence are defined as

$$\lambda_U = \lim P[r_1 > G_1^{-1}(u) \mid r_2 > G_2^{-1}(u)] = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}$$

$$\lambda_L = \lim P[r_1 \leq G_1^{-1}(u) \mid r_2 \leq G_2^{-1}(u) = \lim_{u \to 0^+} \frac{C(u,u)}{u},$$

where the λ_U and $\lambda_L \in [0,1]$ are the coefficients of upper and lower tail dependence, respectively. There is a symmetric tail dependence between two variables when the lower and upper tail dependence coefficient values are not equal. This approach uses the tail dependence coefficient for ordering copula where the copula C_1 is more concordant than copula C_2 if λ_U of C_1 is greater than λ_U of C_2 (see Nelsen [34]).

In our analysis, we employed several candidates of the time-varying copulas. For description of dependence, the time-varying copulas are considered as the dynamic generalizations of a Pearson correlation (ρ) or Kendall's tau (τ). Patton [24] and Wu et al. [35] proposed the evolution equations for some time-varying copulas, respectively. However, it is difficult to specify how the parameters evolve over time lies in defining covariates for the evolution equation. Therefore, we applied the two evolution equations to capture time-varying dependences in this study, and selected the better one in terms of Akaike information criterion (AIC). The evolution equation of Patton [24] for Gaussian and T copulas is expressed as

$$\rho_t = \Lambda(\alpha + \beta\rho_{t-1} + \gamma \mid u_{1,t-1} - u_{2,t-1} \mid),$$

and Wu et al. [35] defined the time-varying Gaussian and T copulas as

$$\rho_t^* = \alpha + \beta\rho_{t-1}^* + \gamma(u_{1,t-1} - 0.5)(u_{2,t-1} - 0.5),$$

where $\Lambda(x) = (1 + e^{-x})^{-1}$ and $\rho_t^* = -ln[(1 - \rho_t)/(\rho_t + 1)]$ are to guarantee the dependence parameter with the interval (-1, 1), and ρ_t is the Pearson's correlation. In addition, we proposed an evolution equation for rotated BB1 copula which might help us capture the time-varying dependences of G7 stock markets. The evolution equation for rotated BB1 copula is written as:

$$\theta_t = \hat{\Lambda}(\alpha + \beta \cdot \theta_{t-1} + \gamma \mid u_{1,t-1} - u_{2,t-1} \mid),$$

where we define $\hat{\Lambda}(x) = exp(x)$, which can guarantee the parameter θ is in the range. There are two parameters θ, and δ, in rotated BB1 copula. Actually, both of parameters can be considered time-varying characteristics. According to the performance of empirical results of this study, we just proposed the evolution equation for θ in rotated BB1 copula. Among all the time-varying copulas, the preferable time-varying copula is selected in terms of AIC.

In this study, we applied various 17 families both static and time-varying copulas for comparison and selected the best one. For example, Gaussian copula is symmetric and does not have tail dependence. Student-t copula reflects symmetric tail dependence; Gumbel and Clayton copulas can measure right and left tail dependence, respectively. Moreover, we included rotate Clayton and Gumbel copulas as a mirror image of the density Clayton and Gumbel copulas and BB1, BB6, BB7, BB8, etc. Additional for confirmation of the goodness-of-fit, we compared and chose the most appropriate static with time-varying copula models by using the criteria of AIC. Moreover, Kendall's tau is a concept of concordance which provides nonparametric measurement of dependence between variables.

4 Inference for Margins

The estimation of the parameters of copula-based GARCH model used the alternative computation to the two-stage estimation method. This technique is

called *inference for margins* (IFM) by Joe and Xu [32]. The advantage of IFM is the reduction of numerical complexity that this estimator efficiency is close to consistency and asymptotic normality to the maximum likelihood under the regularity conditions (see proof in Theorem of Lehmann and Casella [36]). The multivariate model proceeds as follows:
(1) The first stage, every margin log-likelihood is maximized with respect to the marginal parameter to obtain

$$\hat{\Theta}_i = argmax \sum_{t=1}^{T} logg_i(r_{it}; \Theta_i),$$

where $i = 1, 2..., 7$ and Θ is a parameter vector that represents from $(\mu, a, b, \omega_0, \alpha, \beta, \gamma, v)$ ARMA-GARCH models.
(2) The second stage, given the marginal estimators, we performed maximizing over the copula parameter leading to

$$\hat{\theta} = argmax \sum_{t=1}^{T} logc(G_1(r_{it}; \hat{\Theta}_i), G_2(r_{jt}; \hat{\Theta}_j); \theta),$$

where θ represents copula parameters, $i \neq j$ = 1,2,...,7.

5 Portfolio Optimization Model

For measuring financial risk, the VaR is widely mentioned and has become the standard benchmark. The risk measurement techniques have been developed and used to manage the portfolios that help the investors and portfolio managers to decide on the best trade-off between risk and return. However, VaR does not satisfy the subadditivity condition. Thus, Expected Shortfall (ES) is an alternative method which satisfies the property of subadditivity and provides a more conservative measure of losses relative to VaR. We followed the method proposed by Rockafellar and Uryasea [37], the optimization approach-based on minimum ES with simulation can be expressed as

$$\min ES_\beta(W) = min\{VaR_\beta(W) + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} \left[-W^T r_k - VaR_\beta(W)\right]^*\}, \quad (5.1)$$

where $[t]^* = max(t, 0), \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} w_i E(r_i) \geq \frac{1}{n} \sum_{i=1}^{n} E(r_i)$, q denotes the number of samples generated by Monte Carlo simulation. Following [37] showed that the equation (5.1) is a suitable approximation to the minimum ES of integral form. $VaR_\beta(W)$ is the VaR under the β confidence level and the W portfolio allocations, and r_k is the K^{th} vector of simulated returns.

In this study, the multi-period ahead forecasts of portfolio strategies were performed by using the principle of the daily rolling window forecasting of returns.

The summarized process can be shown in four steps. First, use the estimation results of the preferred time-varying copula to generate the random number 10,000. Second, use the inverse function of the corresponding marginal distribution of each variable to get the standardized residuals. Third, forecast the value of each variable at the $t+1$ period by using the GARCH model; thus, 10,000 possible values are generated at the $t+1$ period for each variable, which can be expressed as

$$r^n_{m,t+1} = \hat{c} + \sum_{i=1}^{p} \hat{\phi}_{m,i} r_{m,t-i+1} + \sum_{j=1}^{q} \hat{\psi}_{m,i} \varepsilon_{m,t-i+1} + \hat{\sigma}_{m,t+1} \eta^n_{m,t+1},$$

where $n = 1, 2, ..., 10,000, m$ equals the number of variables, $\eta^n_{m,t+1} = G^{-1}_m(u^n_{m,t+1})$, and u_m is from the simulation of the preferred time-varying copula. Last, by giving an unknown weight to each variable, the optimal portfolio weights of the selected assets are estimated under the minimum ES framework (equation (5.1)). We used these weights to compute the returns at the $t+1$ period. Rate of portfolio returns is given as

$$\begin{aligned} PR &= \hat{w_1}[\exp(r_{1,t+1}) - 1] + \hat{w_2}[\exp(r_{2,t+1}) - 1], ..., \hat{w_m}[\exp(r_{m,t+1}) - 1] \\ &= \hat{w}^T[exp(R-1)], \end{aligned}$$

where $\widehat{W}$ is the vector of w_i and R is the vector of $r_{m,t+1}$.

6 Empirical Results

In this paper, the data set consists of G7 countries stock markets during the period from 1 January 2008 to 31 December 2014. The G7 indices including:(1) S&P/TSX Composite index of Canada; (2) CAC 40 index of France; (3) DAX index of Germany; (4) FTSE MIB index of Italy; (5) NIKKEI 225 index of Japan; (6) FTSE 100 index of the United Kingdom; and (7) S&P500 index of the United States [US]. We computed for all the daily returns on G7 indices by $r_t = lnP_t - lnP_{t-1}$, where P_t is the closing price of G7 stock indexes at time t. The data set was divided into two parts that are in sample and out-of-sample. The in-sample data are from 1 January 2008 to 31 December 2012, with 1,282 observations to be used for estimating the parameters of the marginal models. Subsequently, the 460 observations in the out-of-sample were used to estimate the VaR and ES by using the principle of the daily rolling window forecasting of returns. The constant correlation estimates in Table 1 show that the returns of each pair of indices are closely related to each other. France and Germany stock market indices exhibit the highest correlation, and the smallest correlation is between France and Japan. These results also observed that geographical and cultural are important determinants of the correlation in the international stock market. In addition, we found that France has close relationships with most of the counties except Canada and USA, while the correlation between Canada and USA is the biggest for each other. Therefore, we selected the pairs of Canada and USA, France and UK, France and Japan, France and Italy, and Germany and France, as the research objects among 21 pairs of G7.

Table 1: Constant correlation estimates of G7 stock returns

	Canada	*France*	*Germany*	*Italy*	*Japan*	*UK*	*USA*
Canada	1	0.588	0.583	0.542	0.307	0.602	0.762
France	0.588	1	0.928	0.919	0.419	0.922	0.639
Germany	0.583	0.928	1	0.860	0.402	0.878	0.672
Itary	0.542	0.919	0.860	1	0.383	0.834	0.588
Japan	0.307	0.419	0.402	0.383	1	0.422	0.208
UK	0.602	0.922	0.878	0.834	0.422	1	0.616
USA	0.762	0.639	0.672	0.588	0.209	0.616	1

Table 2: KS and LM test for normal , student-t and skewed student-t distribution

	Pair1		Pair2		Pair3		Pair4		Pair5	
	Germany	France	France	Italy	France	UK	France	Japan	Canada	USA
Normal										
KS test statistics	0.972	0.962	0.963	0.949	0.962	0.965	0.964	0.960	0.970	0.969
P value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
P value of LM test	0.134	0.636	0.424	0.276	0.540	0.895	0.537	0.870	0.064	0.554
Student-t										
KS test statistics	0.033	0.024	0.023	0.025	0.023	0.037	0.025	0.024	0.052	0.052
P value	0.131	0.445	0.488	0.415	0.497	0.058	0.443	0.487	0.002	0.002
P value of LM test	0.134	0.422	0.264	0.178	0.335	0.783	0.423	0.896	0.064	0.487
skewed student-t										
KS test statistics	0.024	0.022	0.023	0.031	0.024	0.024	0.024	0.028	0.028	0.033
P value	0.476	0.593	0.522	0.179	0.438	0.452	0.492	0.316	0.263	0.140
P value of LM test	0.365	0.824	0.468	0.229	0.572	0.975	0.727	0.896	0.253	0.840

We first selected the suitable ARMA(1,1)-GARCH(1,1) distribution to model the margin return series of each pair country of G7. This paper mainly focused on the three types of distribution: normal distribution, student-t distribution, and skewed student-t distribution, and compared to determine which one has the best performance due to the stylized fact of financial series having fat tails. Then, we conducted the copula analysis with the specification of the marginal distribution model. Thus, we had to test the marginal distribution for each pair of the return series to satisfy the serial independence assumption and distribution as uniform (0,1). The misspecification of the marginal distribution can cause incorrect fit to the copula if any of these assumptions is rejected. Therefore, the testing of these two assumptions to choose the most appropriate specification for the marginal distribution model is the important step in constructing multivariate distribution models using copula [24]. Table 2 summarizes the probability values for both KS (Kolmogorov-Smirnov) and Lagrange Multiplier (LM) tests for uniform (0,1) distribution; that is to test the marginal distribution and the margin satisfies the i.i.d (independently and identically distributed) assumptions, respectively. For each pair of stocks with the three distributions, such as normal, student-t and skewed student-t distributions the model with normal distribution of errors is rejected, and none for student-t and skewed student-t distribution at 5% significance level. It implies that student-t and skewed student-t distribution fit the data better than the normal distribution. However, the ARMA(1,1)-GARCH(1,1) with skewed student-t distribution has the p-values of LM test greater than the case of student-t distribution. From these results, the models with skewed student-t distribution seem to outperform others which indicates that the margins satisfy both uniformly distribution and the i.i.d assumptions.

Table 3: Estimates of the dependence parameters of copula model for G7 countries

Country	Copula	Parameters	Values	Standard error	Kendall tau	AIC
Germany and France	Student-t	ρ	0.943***	0.003	0.784	-2828
		d	6.514***	1.266		
France and Italy	Student-t	ρ	0.912***	0.004	0.731	-2282
		d	7.496***	1.629		
France and UK	Student-t	ρ	0.908***	0.005	0.724	-2251
		d	6.056***	1.144		
France and Japan	Gaussian	ρ	0.352***	0.024	0.229	-158
Canada and USA	rotated BB1	q	0.221***	0.057	0.536	-1060
		d	1.941***	0.064		

Note : *, **, *** indicates statistical significance at the 5%, 1%, 0.1% level, respestively.

Table 4: the estimation of time-varying Copula pattern of G7 markets

Country	Copula	ω	β	γ	v	LL	AIC
Germany and France	Student-t	0.499 (0.337)	0.911*** (0.064)	-1.843 (1.086)	8.861*** (0.957)	1432.941	-2858
France and Italy	Student-t	0.343*** (0.111)	0.929**** (0.025)	-1.141*** (0.316)	7.975*** (0.527)	1160.729	-2313
France and UK	Student-t	0.428*** (0.124)	0.914*** (0.026)	-1.472*** (0.403)	8.962*** (0.862)	1159.072	-2310
France and Japan	Gaussian	0.349 (0.200)	0.703*** (0.206)	-0.459** (0.225)		83.362	-161
Canada and USA	rotated BB1	0.075* (0.044)	0.945*** (0.031)	-1.051 (0.559)		541.551	-1077

Note : *, **, *** indicates statistical significance at the 5%, 1%, 0.1% level,respestively

After we chose the most appropriate form of marginal density of multivariate ARMA-GARCH(1,1) with skewed student-t distribution to capture the time-varying volatility structures of each pair of G7 indices returns, we employed the many families of both static and time-varying copula functions to describe the dependence structures between each pair of G7 returns. Table 3 presents the dependence parameter estimates of static copula. We can observe all the best copula families among the candidates of each pair of G7 stock in terms of the smallest AIC value. For each pair of stock returns of Germany and France, France and Italy, as well as France and UK, Student-t copula is suitable for to capturing the extreme dependence between variables. The dependence between France and Japan stock returns is the best candidate for fitting with Gaussian copula. And the pair of Canada and USA stock returns can be described as the best by the rotated BB1 copula. The Kendall's tau values show the rank dependence of copula model; we may notice that the Kendall's tau has the same sign and is consistent with the Pearson (linear) correlation in Table 1. Table 4 presents the parameter estimates for different time-varying copula models of G7 markets. The autocorrelation parameter β of the pairs between Germany and France, France and Italy, France and UK with student-t time-varying copula as well as France and Japan with Gaussian time-varying copula are 0.911, 0.929, 0.914 and 0.703, respectively. Moreover, the β value of Canada and USA with rotated BB1 time-varying is 0.945. This implies a high degree of persistence concerning the dependence structure between each pair of stock returns. The latent parameter γ exhibits that the latest return information is an important measure. We can observe that γ in France and Japan paired with Gaussian time-varying copula has a larger difference than others, implying that it has a greater short-run response than other copula functions. For each pair of G7 stocks, the best candidate time-varying copula is likely the constant copula, according to the maximized log-likelihood values and AIC. To compare Table 3 with Table 4, the time-varying copula models outperform the constant copula models in terms of AIC for constructing the optimum portfolios.

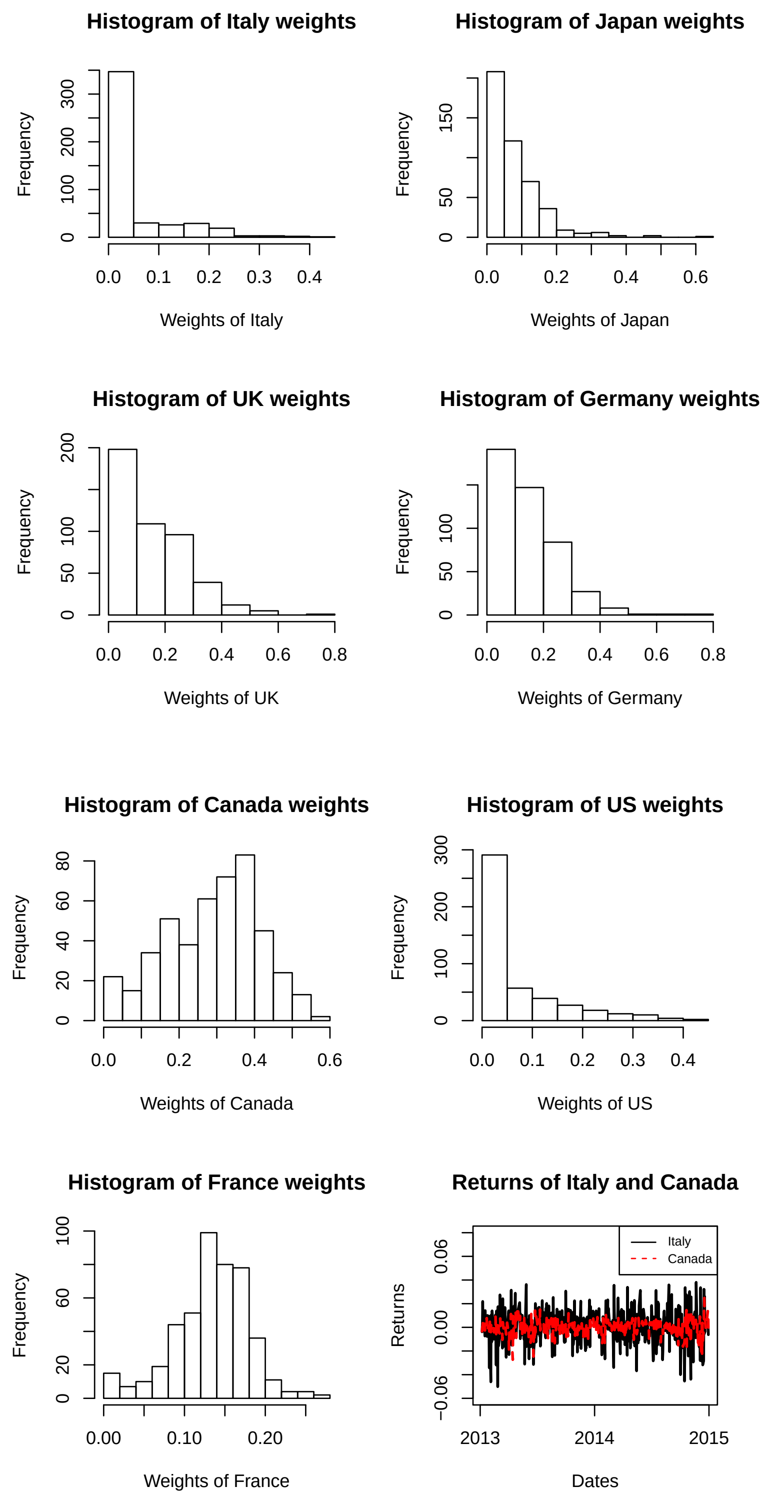

Figure 1: Histograms of G7 weights and the returns of Italy and Canada in 2013-2014

Table 5: Number of violations of the VaR estimation and backtesting

Panel A: Violations			
α	1%	5%	10%
expected no.	5	23	46
DCC-N	0	0	5
DCC-T	0	2	11
copula-GARCH	4	21	46
Panel B: Backtesting			
DCC-T model			
POF	9.25	33.22	41.4
CCLR	9.79	33.76	41.92
DCC-N model			
POF	9.25	47.19	63.73
CCLR	9.25	47.19	63.85
copula-GARCH model			
POF	0.08	0.19	0
CCLR	0.15	2.2	0.48

We now turn to evaluate the performance of the estimates in the copula-GARCH model. The Percentage of Failure Likelihood Ratio (PoFLR) and Conditional Coverage Likelihood Ratio (CCLR) tests are performed to judge whether the model is correct and accurate. Table 5 shows the number of violations of the VaR estimation and VaR backtests. The DCC-GARCH with normal marginal distribution and DCC-GARCH with student-t marginal distribution are selected as the benchmark. Comparing them with the copula-GARCH with skewed student-t distribution model, the copula-GARCH model takes into full account asymmetric tail dependence and high kurtosis. If the number of violations are closer to the expected numbers, then this model should be more appropriate. As it can be seen in Table 5, the numbers of violations in the copula-GARCH model equals to 4, 21 and 46 at confidence levels 99%, 95%, and 90%, respectively. They are very close to the expected numbers while the number of violations in DCC-N and DCC-T models are so much smaller than the expected numbers that they underestimate VaR. The backtesting statistics of PoFLR and CCLR for DCC-N and DCC-T models reject the null hypotheses while they do not reject the null hypotheses for the copula-GARCH model. Therefore, we conclude that the copula-GARCH method of estimating VaR is accurate and correct for G7 returns. This may be because the copula-GARCH with skewed student-t distribution model successfully captures asymmetric tail dependences and extreme losses for G7 returns.

Figure 1 shows the optimal weighted histograms of G7 indexes by minimum ES at 95% confidence level and the returns of Italy and Canada from 2013 to 2014. The weight distributions of Italy index share rarely, while Canada shares a great many. The returns of Italy and Canada from 2013 to 2014 have a large difference. Italy stock market showed a stronger volatility than Canada stock market. In other words, Italy stock market is more risky than Canada. To avoid risk, the time-varying copula-GARCH model suggests that we distribute a tiny amount to Italy, and invest a lot in Canada stock market. France, Germany, and UK have similarly distributed weights. This may be because they all belong to Europe Union, and there exists strong correlation between them. Figure 2 shows the forecasted expected shortfall of G7 stock markets during 2013-14. There are two periods, June-July 2013 and October-December 2014, with a high risk performance. In the last quarter 2014, the G7 feared that Europe's economy was slipping back into a recession, and also worried about plunging oil prices and concerns of possible weakness in the U.S. economy. The Europe markets in June and July, 2013 hung in balance. For example, Europe shares closed at the lowest since January on 21 June. The Dow bounced above the psychologically important 15,000 level, and European shares closed higher on 1 July 2013.

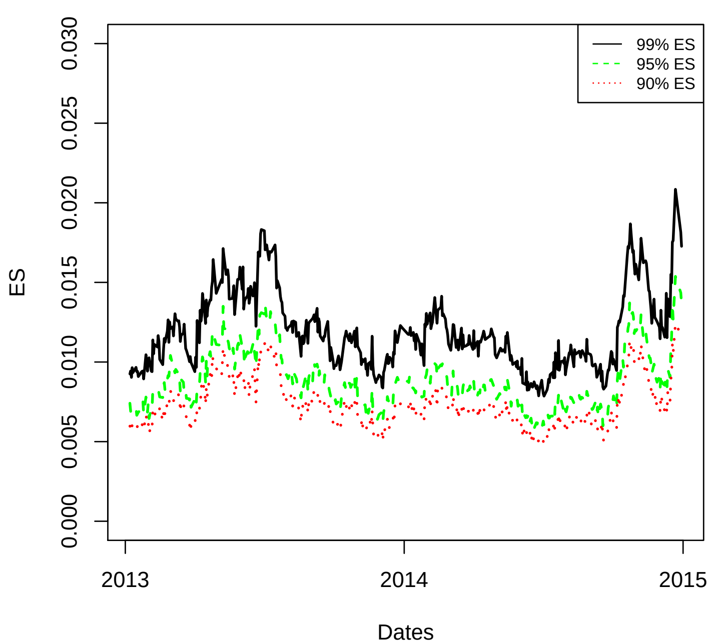

Figure 2: The estimated ES at level 90%, 95% and 99% by copula-GARCH model

7 Conclusions

In this study, we analyzed the co-movement and the optimum portfolio for investors and risk managers in financial markets of the G7 group. From the recent 2008-2009 crisis, the G7 countries have been impacted from the global financial system and the catastrophic event could easily trigger and spread to the global financial market. We examined the dependence structure of each pair of G7 stock markets, and the results indicated that the time-varying copula GARCH model has performance superior to the constant copula model in terms of AIC. In addition, the out-of-sample observations were used to estimate the VaR and ES by using the principle of the daily rolling window forecasting of returns. For the evaluation of the performance of the estimates in the copula-GARCH model, the results reported that the number of violations in the copula-GARCH model are equal to 4, 21, and 46 at confidence levels 99%, 95%, and 90%, respectively, and are very close to the expected numbers. In terms of the out of sample forecasting, Italy stock market showed a stronger volatility than Canada stock market. To avoid risk, the time-varying copula-GARCH model suggested that we should distribute a tiny amount to Italy, and invest a lot in Canada stock market. Finally, the time-varying copula GARCH model was supported by backtesting.

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References

[1] K. Autchariyapanitkul, S. Chanaim, S. Sriboonchitta, Portfolio optimization of stock returns in high-dimensions, A copula-based approach, Thai Journal of Mathematics (2014) 11-23.

[2] H.T. Nguyen, V. Kreinovich, B. Wu, Using Second-Order Probabilities to Make Maximum Entropy Approach to Copulas More Reasonable, 2014.

[3] K. Forbes, R. Rigobon, Contagion in Latin America: Definitions, measurement, and policy implications, No. w7885, National Bureau of Economic Research, (2000).

[4] K. Forbes, R. Rigobon, No contagion, only interdependence: measuring stock market comovements, The journal of Finance 57 (5) (2002) 2223-2261.

[5] I. Hwang, F.H. In, T.S. Kim, Contagion effects of the US subprime crisis on international stock markets, In Finance and Corporate Governance Conference, (2010).

[6] S. Celik, The more contagion effect on emerging markets: The evidence of DCC-GARCH model, Economic Modelling 29 (5) (2012) 1946-1959.

[7] S. Jayech, N.B. Zina, Measuring financial contagion in the stock markets using a copula approach, International Journal of Data Analysis Techniques and Strategies 4 (2) (2012)154-180.

[8] T.M. Bilel, S. Nadhem, Long Memory in Stock Returns, Evidence of G7 Stocks Markets, Research Journal of International Studies 9 (1) (2009) 36-46.

[9] K.H. Liow, Volatility spillover dynamics and relationship across G7 financial markets, The North American Journal of Economics and Finance 33 (2015) 328-365.

[10] S.R. Bentes, Measuring persistence in stock market volatility using the FIGARCH approach, Physica A, Statistical Mechanics and its Applications 408 (2014) 190-197.

[11] M.H. Chiang, L.M. Wang, Volatility contagion, A range-based volatility approach, Journal of Econometrics 165 (2) (2011) 175-189.

[12] R. Bhar, S. Hamori, Co-movement in the price of risk of aggregate equity markets, Economic Systems 31 (3) (2007) 256-271.

[13] Y.H. Lee, Y.L Huang, C.Y. Wu, Dynamic Correlations and Volatility Spillovers between Crude Oil and Stock Index Returns: The Implications for Optimal Portfolio Construction, International Journal of Energy Economics and Policy 4 (3) (2014) 327-336.

[14] X.B. Chen, P. Silvapulle, M. Silvapulle, A semiparametric approach to value-at-risk, expected shortfall and optimum asset allocation in stockbond portfolios, Economic Modelling 42 (2014) 230-242.

[15] F. Longin, B. Solnik, Extreme correlation of international equity markets, Journal of Finance 56 (2) (2001) 649-676.

[16] A. Ang, G. Bekaert, International asset allocation with regime shifts, Review of Financial Studies 15 (4) (2002) 1137-1187.

[17] A. Ang, J. Chen, Asymmetric correlations of equity portfolios, Journal of Financial Economics 63 (3) (2002) 443-494.

[18] R.F. Engle, Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation, Econometrica: Journal of the Econometric Society (1982) 987-1007.

[19] T. Bollerslev, Generalized autoregressive conditional heteroskedasticity, Journal of Econometrics 31 (3) (1986) 307-327.

[20] R. Engle, Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models, Journal of Business & Economic Statistics 20 (3) (2002)339-350.

[21] J. Lee, M.D. Chinn, Current account and real exchange rate dynamics in the G7 countries, Journal of International Money and Finance 25 (2) (2006) 257-274.

[22] R. Gupta, G.D. Donleavy, Benefits of diversifying investments into emerging markets with time-varying correlations, an Australian perspective, Journal of Multinational Financial Management 19 (2) (2009) 160-177.

[23] P. Embrechts, A. McNeil, D. Straumann, Correlation and dependence in risk management, properties and pitfalls, Risk management: value at risk and beyond (2002) 176-223.

[24] A.J. Patton, Applications of copula theory in financial econometrics, Doctoral dissertation, University of California, San Diego, (2002).

[25] S. Hyde, N.T. Nguyen, D. Bredin, Correlation dynamics between Asia-Pacific, EU and US stock returns, EU and US Stock Returns, (2007).

[26] T. Ané, C. Labidi, Spillover effects and conditional dependence, International Review of Economics and Finance 15 (4) (2006) 417-442.

[27] S.M. Bartram, S.J. Taylor, Y.H. Wang, The Euro and European financial market dependence, Journal of Banking and Finance 31 (5) (2007) 1461-1481.

[28] K. Wang, Y.H. Chen, S.W. Huang, The dynamic dependence between the Chinese market and other international stock markets: A time-varying copula approach, International Review of Economics & Finance 20 (4) (2011) 654-664.

[29] R.S. Tsay, An introduction to analysis of financial data with R. John Wiley & Sons, 2014.

[30] M. Mroz, Time-varying copula models for financial time series, Doctoral dissertation, Ulm, Universitt Ulm, Diss, (2012).

[31] S. Sriboonchita, W.K. Wong, S. Dhompongsa, H.T. Nguyen, Stochastic dominance and applications to finance, risk and economics, CRC Press, (2009).

[32] H. Joe, J.J. Xu, The estimation method of inference functions for margins for multivariate models, Technical report, Department of statistics, University of British Columbia 166, (1996).

[33] T. Bollerslev, R.Y. Chou, K. F. Kroner, ARCH modeling in finance, A review of the theory and empirical evidence, Journal of Econometrics 52 (1) (1992) 5-59.

[34] R.B. Nelsen, An introduction to copulas, Springer, 2nd edition, (2006).

[35] C.C. Wu, H. Chung, Y.H. Chang, The economic value of co-movement between oil price and exchange rate using copula-based GARCH models, Energy Economics 34 (1) (2012) 270-282.

[36] E.L. Lehmann, G. Casella, Theory of point estimation, Springer Science & Business Media 31, (1998).

[37] R.T. Rockafellar, S. Uryasev, Conditional value-at-risk for general loss distributions, Journal of Banking & Finance 26 (7) (2002) 1443-1471.

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