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A Simple Proof of Caristi's Fixed Point Theorem without Using Zorn's Lemma and Transfinite Induction

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Abstract : In this paper, we give a new and simple proof of Caristi's fixed point theorem without using Zorn's lemma, transfinite induction and any well-known principle.

Keywords : Caristi's fixed point theorem; transfinite induction; Zorn's lemma; Brézis-Browder order principle.
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1 Introduction and Preliminaries

In 1976, Caristi proved the following famous fixed point theorem (so-called Caristi's fixed point theorem [1]) by using transfinite induction:

Theorem 1 (Caristi [1])**.** *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ be a lower semicontinuous and bounded below function. Suppose that T is a Caristi type mapping on X dominated by f, that is, T satisfies*

$$d(x, Tx) \leq f(x) - f(Tx) \quad \text{for each } x \in X. \tag{$*$}$$

Then T has a fixed point in X.

It is known that Caristi's fixed point theorem is equivalent to the Ekeland's variational principle, to the Takahashi's nonconvex minimization theorem, to the

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Daneš' drop theorem, to the petal theorem, and to the Oettli-Théra's theorem, see [2–15] and references therein for more details. A large number of generalizations in various different directions of the Caristi's fixed point theorem have been investigated by several authors; see, for example, [2–19] and references therein. In fact, several elegant proofs of original Caristi's fixed point theorem (Theorem 1) were given. In [20], Wong gave a modification of Caristi's original transfinite induction argument. Kirk [21], Brøndsted [22] and Pasicki [23] proved Caristi's fixed point theorem by using Zorn's lemma. An interesting proof of Caristi's fixed point theorem was derived from a result of Brézis and Browder [24] (so-called Brézis-Browder order principle). Some other elegant proofs of the Caristi's fixed point theorem can be found in, for example, [4–18, 25] and references therein.

In this paper, we give a new and simple proof of Caristi's fixed point theorem without using Zorn's lemma, transfinite induction and any well-known principle.

2 A New Proof of Caristi's Fixed Point Theorem

Now, we present our new proof of original Caristis fixed point theorem as follows.

Proof of Caristi's fixed point theorem. On the contrary, suppose that $Tx \neq x$ for all $x \in X$. Define a set-valued mapping $\Gamma : X \to 2^X$ (the power set of X) by

$$\Gamma(x) = \{y \in X : y \neq x, d(x, y) \leq f(x) - f(y)\} \text{ for } x \in X.$$

From $(*)$, we know $Tx \in \Gamma(x)$ and hence $\Gamma(x) \neq \emptyset$ for all $x \in X$. We claim that for each $y \in \Gamma(x)$, we have $f(y) \leq f(x)$ and $\Gamma(y) \subseteq \Gamma(x)$. Let $y \in \Gamma(x)$ be given. Then $y \neq x$ and $d(x, y) \leq f(x) - f(y)$. So we get $f(y) \leq f(x)$. Since $\Gamma(y) \neq \emptyset$, let $z \in \Gamma(y)$. Then $z \neq y$ and $d(y, z) \leq f(y) - f(z)$. It follows that

$$f(z) \leq f(y) \leq f(x)$$

and

$$d(x, z) \leq d(x, y) + d(y, z) \leq f(x) - f(z).$$

Also we have $z \neq x$. Indeed, if $z = x$ then $f(z) = f(x)$. Since

$$d(x, y) \leq f(x) - f(y) \leq f(x) - f(z) = 0,$$

which implies $d(x, y) = 0$, that is $x = y$. This would imply $y = z$, a contradiction. Hence $z \in \Gamma(x)$. Therefore we prove $\Gamma(y) \subseteq \Gamma(x)$. We shall construct a sequence $\{x_n\}$ in X by induction, starting with any point $x_1 \in X$. Suppose that $x_n \in X$ is known. Then choose $x_{n+1} \in \Gamma(x_n)$ such that

$$f(x_{n+1}) \leq \inf_{z \in \Gamma(x_n)} f(z) + \frac{1}{n}, \quad n \in \mathbb{N}. \tag{2.1}$$

For any $n \in \mathbb{N}$, since $x_{n+1} \in \Gamma(x_n)$, we have

$$x_{n+1} \neq x_n \tag{2.2}$$

and

$$d(x_n, x_{n+1}) \leq f(x_n) - f(x_{n+1}). \tag{2.3}$$

So $f(x_{n+1}) \leq f(x_n)$ for each $n \in \mathbb{N}$. Since f is bounded below,

$$\lambda := \lim_{n\to\infty} f(x_n) = \inf_{n\in\mathbb{N}} f(x_n) \quad \text{exists.} \tag{2.4}$$

For $m > n$ with $m, n \in \mathbb{N}$, by (2.3) and (2.4), we obtain

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq f(x_n) - \lambda.$$

Since $\lim\limits_{n\to\infty} f(x_n) = \lambda$, we obtain

$$\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. Since f is lower semicontinuous, by (2.4), we get

$$f(v) \leq \liminf_{n\to\infty} f(x_n) = \inf_{n\in\mathbb{N}} f(x_n) \leq f(x_j) \ \text{ for all } j \in \mathbb{N}. \tag{2.5}$$

We want to verify $v \neq x_n$ for all $n \in \mathbb{N}$. Arguing by contradiction, assume that there exists $k \in \mathbb{N}$ such that $v = x_k$. By (2.3) and (2.5), we have

$$d(x_k, x_{k+1}) \leq f(x_k) - f(x_{k+1}) \leq f(x_k) - f(v) = 0,$$

and hence deduces $x_{k+1} = x_k$ which contradicts (2.2). So it must be $v \neq x_n$ for all $n \in \mathbb{N}$.

Next, we prove that $\bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}$. For $m > n$ with $m, n \in \mathbb{N}$, by (2.3) and (2.5), we obtain

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \leq f(x_n) - f(v). \tag{2.6}$$

Since $x_m \to v$ as $m \to \infty$, the inequality (2.6) implies

$$d(x_n, v) \leq f(x_n) - f(v) \quad \text{for all } n \in \mathbb{N}. \tag{2.7}$$

Combining (2.7) with the fact that $v \neq x_n$ for all $n \in \mathbb{N}$, we have $v \in \bigcap_{n=1}^{\infty} \Gamma(x_n)$. Hence $\bigcap_{n=1}^{\infty} \Gamma(x_n) \neq \emptyset$, and moreover, we can see that

$$\Gamma(v) \subseteq \bigcap_{n=1}^{\infty} \Gamma(x_n).$$

For any $c \in \bigcap_{n=1}^{\infty} \Gamma(x_n)$, by (2.1), we have

$$\begin{aligned} d(x_n, c) &\leq f(x_n) - f(c) \\ &\leq f(x_n) - \inf_{z \in \Gamma(x_n)} f(z) \\ &\leq f(x_n) - f(x_{n+1}) + \frac{1}{n} \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n \to \infty} d(x_n, c) = 0$ or, equivalently, $x_n \to c$ as $n \to \infty$. By the uniqueness of the limit of a convergent sequence, we have $c = v$. So we show $\bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}$. Since $\Gamma(v) \neq \emptyset$ and

$$\Gamma(v) \subseteq \bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}$$

we get $\Gamma(v) = \{v\}$, which leads a contradiction. Therefore T must have a fixed point in X. The proof is completed. □

Remark 2.

(a) *Although the function f is lower semicontinuous, it do not deduce that $\Gamma(x)$ is a closed subset of X;*

(b) *Classic proofs of Caristi's fixed point theorem involve assigning a partial order $\precsim$ on X by setting*

$$x \precsim y \quad \Longleftrightarrow \quad d(x, y) \leq f(x) - f(y),$$

and then either using Zorn's lemma or the Brézis-Browder order principle with the set

$$S(x) = \{y \in X : x \precsim y\} \quad \text{for } x \in X.$$

By the reflexivity, $x \in S(x)$ for all $x \in X$. It is worth mentioning that the set $\Gamma(x)$ defined in our proof by

$$\Gamma(x) = \{y \in X : y \neq x, d(x, y) \leq f(x) - f(y)\} \quad \text{for } x \in X,$$

have the property $x \notin \Gamma(x)$ for all $x \in X$, so the Brézis-Browder order principle is not applicable here. Also, we do not define any partial order on X and hence Zorn's lemma does not be used in our proof.

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