Convolution Conditions for Some Subclasses of Meromorphic Bounded Functions of Complex Order

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Abstract : Making use of the operator $\mathcal{H}_{q,s}(\alpha_1)$ for functions of the form $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1}$, which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathcal{C}$ and $0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$, we introduce two subclasses of meromorphic functions of complex order and investigate convolution properties, coefficient estimates and containment properties for these subclasses.

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1 Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1}, \tag{1.1}$$

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which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathcal{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^{k-1}, \tag{1.2}$$

then the Hadamard product (or convolution) of two power series $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^{k-1} = (g * f)(z). \tag{1.3}$$

We recall some definitions which we will be used in our paper.

Definition 1.1 ([1]). For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in $\mathbb{U}$, and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))(z \in \mathbb{U})$. Futhermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [1]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For real numbers

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \ \ (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; \ j = 1, 2, ..., s),$$

we consider the generalized hypergeometric function $_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [2, p. 19])

$$_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} z^k$$

$$(q \leq s+1; \ q, s \in \mathbb{N}_0; \ \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ \mathbb{N} = \{1, 2, ...\}; \ z \in \mathbb{U}), \tag{1.4}$$

where $(\lambda)_\nu$ is the Pochhammer symbol defined by

$$(\lambda)_\upsilon = \begin{cases} 1 & \text{if } \upsilon = 0, \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+\upsilon-1) & \text{if } \upsilon \in \mathbb{N}. \end{cases}$$

Corresponding to the function $\phi(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ given by

$$\phi(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-1} \ _qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \tag{1.5}$$

Liu and Srivatava [3] considered a linear operator $\mathcal{H}_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma \to \Sigma$ by

$$\mathcal{H}_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = \phi(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \ \ (f \in \Sigma; \ z \in \mathbb{U}^*).$$

(1.6)

For convenience, let

$$\mathcal{H}_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) = \mathcal{H}_{q,s}(\alpha_1).$$

If $f(z)$ is given by (1.1), then from (1.6), we deduce that

$$\mathcal{H}_{q,s}(\alpha_1)f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} a_k z^{k-1} \ (z \in \mathbb{U}^*). \qquad (1.7)$$

Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator $\mathcal{L}(a,c) = \mathcal{H}_{2,1}(a,1;c)$ $(a, \ c > 0)$ (studied among others by Liu and Srivastava [4, with $p = 1$]), the operator $\mathcal{D}^n = \mathcal{H}_{2,1}(n+1,1;1)$ which is analogous to the Ruscheweyh derivative operator (investigated by Ganigi and Uralegaddi [5] and generalized by Yang [6]) and the operator

$$\mathcal{J}_c = \frac{1}{z^{c+1}} \int\limits_0^z t^c f(t) dt = \mathcal{H}_{2,1}(c,1;c+1) \ (c > 0),$$

(studied by Uralegaddi and Somanatha [7, with $p = 1$]).

Definition 1.2 ([8])**.** For $b \in \mathcal{C}^* = \mathcal{C}\backslash\{0\}$, let $\mathcal{F}^*(b,M)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$-\frac{zf'(z)}{f(z)} \prec \frac{1 + [b(1+m)-m]z}{1-mz} \ \left(m = 1 - \frac{1}{M}; \ M \geq 1; \ z \in \mathbb{U}^*\right), \qquad (1.8)$$

or, equivalently,

$$\left| \frac{b - 1 - \dfrac{zf'(z)}{f(z)}}{b} - M \right| < M \ \left(m = 1 - \frac{1}{M}; \ M \geq 1; \ z \in \mathbb{U}^*\right). \qquad (1.9)$$

Also, let $\mathcal{G}^*(b,M)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:

$$-\frac{zf''(z)}{f'(z)} \prec 2 + \frac{b(1+m)z}{1-mz} \ \left(m = 1 - \frac{1}{M}; \ M \geq 1; \ z \in \mathbb{U}^*\right), \qquad (1.10)$$

or, equivalently,

$$\left| \frac{b - 2 - \dfrac{zf''(z)}{f'(z)}}{b} - M \right| < M \ \left(m = 1 - \frac{1}{M}; \ M \geq 1; \ z \in \mathbb{U}^*\right). \qquad (1.11)$$

It is easy to verify from (1.8) and (1.10) that,

$$f(z) \in \mathcal{G}^*(b,M) \Leftrightarrow -zf'(z) \in \mathcal{F}^*(b,M). \qquad (1.12)$$

The classes $\mathcal{F}^*(b, M)$ and $\mathcal{G}^*(b, M)$ were introduced and studied by Aouf [8].

We note that :

(*i*) $\mathcal{F}^*(b, \infty) = \mathcal{F}^*(b)$ and $\mathcal{G}^*(b, \infty) = \mathcal{G}^*(b)$ ($b \in \mathcal{C}^*$) (see Aouf [8]), where $\mathcal{F}^*(b)$ is the class of meromorphic starlike functions of complex order b and $\mathcal{G}^*(b)$ is the class of meromorphic convex functions of complex order b.

(*ii*) $\mathcal{F}^*(1 - a, M) = \mathcal{F}_M^*(a)$ ($0 \leq a < 1$) (see Kaczmarski [9]) and $\mathcal{G}^*(1 - a, M) = \mathcal{G}_M^*(a)$ ($0 \leq a < 1$) (see Aouf [8]), where $\mathcal{F}_M^*(a)$ is the class of meromorphic bounded starlike functions of order a and $\mathcal{G}_M^*(a)$ is the class of meromorphic bounded convex functions of order a.

(*iii*) $\mathcal{F}^*(1, \infty) = \mathcal{F}^*(1)$ ($0 \leq a < 1$) (see Clunie [10]) and $\mathcal{G}^*(1, \infty) = \mathcal{G}^*(1)$ ($0 \leq a < 1$) (see Aouf [8]), where $\mathcal{F}^*(1)$ is the class of meromorphic starlike functions and $\mathcal{G}^*(1)$ is the class of meromorphic convex functions.

(*iv*) $\mathcal{F}^*(1 - a, \infty) = \mathcal{F}^*(1 - a)$ ($0 \leq a < 1$) (see Kaczmarski [9] and Pommeremke [11]) and $\mathcal{G}^*(1 - a, \infty) = \mathcal{G}^*(1 - a)$ ($0 \leq a < 1$) (see Aouf [8]), where $\mathcal{F}^*(1 - a)$ is the class of meromorphic starlike function of order a and $\mathcal{G}^*(1 - a)$ is the class of meromorphic convex function of order a.

(*v*) $\mathcal{F}^*((1 - a)e^{-i\beta} \cos \beta, M) = \mathcal{F}_M^*(a, \beta)$ ($0 \leq a < 1$, $|\beta| < \frac{\pi}{2}$) (see Kaczmarski [9]) and $\mathcal{G}^*((1 - a)e^{-i\beta} \cos \beta, M) = \mathcal{G}_M^*(a, \beta)$ ($0 \leq a < 1$, $|\beta| < \frac{\pi}{2}$) (see Aouf [8]), where $\mathcal{F}_M^*(a, \beta)$ is the class of meromorphic bounded β−spirallike function of order a and $\mathcal{G}_M^*(a, \beta)$ is the class of meromorphic bounded β-Robertson function of order a.

(*vi*) $\mathcal{F}^*((1 - a)e^{-i\beta} \cos \beta, \infty) = \mathcal{F}^*(a, \beta)$ ($0 \leq a < 1$, $|\beta| < \frac{\pi}{2}$) (see Kaczmarski [9]) and $\mathcal{G}^*((1 - a)e^{-i\beta} \cos \beta, \infty) = \mathcal{G}^*(a, \beta)$ ($0 \leq a < 1$, $|\beta| < \frac{\pi}{2}$) (see Aouf [8]), where $\mathcal{F}^*(a, \beta)$ is the class of meromorphic β−spirallike function of order a and $\mathcal{G}^*(a, \beta)$ is the class of meromorphic β-Robertson function of order a.

Definition 1.3. For real numbers $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$, $M \geq 1$ and $b \in \mathcal{C}^*$, let

$$\mathcal{F}^*([\alpha_1]; b, M) = \{f(z) \in \Sigma : \mathcal{H}_{q,s}(\alpha_1)f(z) \in \mathcal{F}^*(b, M)\}, \qquad (1.13)$$

and

$$\mathcal{G}^*([\alpha_1]; b, M) = \{f(z) \in \Sigma : \mathcal{H}_{q,s}(\alpha_1)f(z) \in \mathcal{G}^*(b, M)\}. \qquad (1.14)$$

It is easy to show that

$$f(z) \in \mathcal{G}^*([\alpha_1]; b, M) \Leftrightarrow -zf'(z) \in \mathcal{F}^*([\alpha_1]; b, M). \qquad (1.15)$$

We note that :

(i) $\mathcal{F}^*([a, 1; c]; b, M) = \mathcal{F}^*_{a,c}(b, M) = \{f(z) \in \Sigma : \mathcal{L}(a, c)f(z) \in \mathcal{F}^*(b, M)\}$ and
$\mathcal{G}^*([a, 1; c]; b, M) = \mathcal{G}^*_{a,c}(b, M) = \{f(z) \in \Sigma : \mathcal{L}(a, c)f(z) \in \mathcal{G}^*(b, M)\}$.

(ii) $\mathcal{F}^*([n + 1, 1; 1]; b, M) = \mathcal{F}^*_n(b, M) = \{f(z) \in \Sigma : \mathcal{D}^n f(z) \in \mathcal{F}^*(b, M)\}$ and
$\mathcal{G}^*([a, 1; c]; b, M) = \mathcal{G}^*_n(b, M) = \{f(z) \in \Sigma : \mathcal{D}^n f(z) \in \mathcal{G}^*(b, M)\}$.

(iii) $\mathcal{F}^*([c, 1; c + 1]; b, M) = \mathcal{F}^*_c(b, M) = \{f(z) \in \Sigma : \mathcal{J}_c f(z) \in \mathcal{F}^*(b, M)\}$ and
$\mathcal{G}^*([a, 1; c]; b, M) = \mathcal{J}_c f(z) = \{f(z) \in \Sigma : \mathcal{L}(a, c)f(z) \in \mathcal{G}^*(b, M)\}$.

The object of the present paper is to investigate some convolution properties, coefficient estimates and containment properties for the subclasses $\mathcal{F}^*([\alpha_1]; b, M)$ and $\mathcal{G}^*([\alpha_1]; b, M)$.

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $M \geq 1$, $b \in \mathcal{C}^*$ and $\alpha_1, ..., \alpha_q$, $\beta_1, ..., \beta_s$ are real numbers.

Theorem 2.1. *If $f(z) \in \Sigma$, then $f(z) \in \mathcal{F}^*(b, M)$ if and only if*

$$z \left[f(z) * \frac{1 + (C - 1)z}{z(1 - z)^2} \right] \neq 0 \text{ for } z \in \mathbb{U}, \tag{2.1}$$

where $C = C_\theta = \frac{e^{-i\theta} - m}{b(1+m)}$, $\theta \in [0, 2\pi)$.

Proof. It is easy to verify that

$$f(z) * \frac{1}{z(1 - z)} = f(z) \text{ and } f(z) * \left[\frac{1}{z(1 - z)^2} - \frac{2}{(1 - z)^2} \right] = -zf'(z). \tag{2.2}$$

(i) In view of (1.8), $f(z) \in \mathcal{F}^*(b, M)$ if and only if (1.8) holds. By using the principle of subordination, we can write (1.8) as

$$-\frac{zf'(z)}{f(z)} = \frac{1 + [b(1 + m) - m] w(z)}{1 - mw(z)}, \tag{2.3}$$

where $w(z)$ is Schwarz function, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$, hence

$$-\frac{zf'(z)}{f(z)} \neq \frac{1 + [b(1 + m) - m] e^{i\theta}}{1 - me^{i\theta}}. \tag{2.4}$$

Thus

$$-zf'(z)(1 - me^{i\theta}) - \left\{ 1 + [b(1 + m) - m] e^{i\theta} \right\} f(z) \neq 0 \ (z \in \mathbb{U}^*),$$

or, equivalently,

$$z \left[\; -zf'(z)(1 - me^{i\theta}) - \left\{ 1 + [b(1+m) - m]\, e^{i\theta} \right\} f(z) \; \right] \neq 0 \text{ for } z \in \mathbb{U}, \; \theta \in [0, 2\pi).$$

(2.5)

Using (2.2), Eq. (2.5) may be written as

$$z \left[f(z) * \frac{1 + \left(\frac{e^{-i\theta} - m}{b(1+m)} - 1 \right) z}{z(1 - z)^2} \right] \neq 0 \text{ for } z \in \mathbb{U}.$$

(2.6)

Thus the first part of Theorem 2.1 was proved.

(ii) Reversely, since, it was shown in the first part of the proof that the assumption (2.5) is equivalent to (2.1), we obtain that

$$-\frac{zf'(z)}{f(z)} \neq \frac{1 + [b(1+m) - m]\, e^{i\theta}}{1 - me^{i\theta}} \text{ for } z \in \mathbb{U}, \; \theta \in [0, 2\pi).$$

(2.7)

Assume that

$$\varphi(z) = -\frac{zf'(z)}{f(z)}, \;\; \psi(z) = \frac{1 + [b(1+m) - m]\, e^{i\theta}}{1 - me^{i\theta}}.$$

The relation (2.7) means that $\varphi(\mathbb{U}) \cap \psi(\partial \mathbb{U}) = \emptyset$. Thus, the simply connected domain is included in a connected component of $\mathcal{C} \backslash \psi(\partial \mathbb{U})$. From this, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, this implies that $f(z) \in \mathcal{F}^*(b, M)$. Thus the proof of Theorem 2.1 is completed. □

Remark 2.2.   (*i*) *Putting* $m = 1$ *in Theorem 2.1, we obtain the result obtained by Bulboacă et al.* [12, *Theorem 1, with* $A = 1$ *and* $B = -1$] *and Aouf et al.* [13, *Theorem 4, with* $\lambda = 0, A = 1$ *and* $B = -1$].

(*ii*) *Putting* $b = m = 1$ *and* $e^{i\theta} = x$ *in Theorem 2.1, we obtain the result obtained by Ponnusamy* [14, *Theorem 4, with* $\lambda = 0, A = 1$ *and* $B = -1$].

(*iii*) *Putting* $m = 1$, $b = (1 - \alpha)e^{-i\mu} \cos \mu$ ($\mu \in \mathcal{R}$, $|\mu| < \frac{\pi}{2}$, $0 \leq \alpha < 1$) *and* $e^{i\theta} = x$ *in Theorem 2.1, we obtain the result obtained by Ravichandran et al.* [15, *Theorem 1.2 with* $p = 1$].

Theorem 2.3. *If* $f(z) \in \Sigma$, *then* $f(z) \in \mathcal{G}^*(b, M)$ *if and only if*

$$z \left[f(z) * \frac{1 - 3z - 2(C - 1)z^2}{z(1 - z)^3} \right] \neq 0 \;\; \text{for } z \in \mathbb{U},$$

(2.8)

where $C = C_\theta = \frac{e^{-i\theta} - m}{b(1+m)}, \theta \in [0, 2\pi)$.

Proof. Putting

$$g(z) = \frac{1 + (C-1)z}{z(1-z)^2}.$$

Then

$$-zg'(z) = \frac{1 - 3z - 2(C-1)z^2}{z(1-z)^3}.$$

From (1.12) and using the identity

$$[-zf'(z)] * g(z) = f(z) * [-zg'(z)],$$

we obtain the required result from Theorem 2.1. □

Remark 2.4. (i) *Putting $m = 1$ in Theorem 2.3, we obtain the result obtained by Bulboacă et al. [12, Theorem 2, with $A = 1$ and $B = -1$] and Aouf et al. [13, Theorem 6, with $\lambda = 0, A = 1$ and $B = -1$].*

(ii) *Putting $b = m = 1$ and $e^{i\theta} = x$ in Theorem 2.3, we obtain the result obtained by Ponnusamy [14, Theorem 2.2, with $A = 1$ and $B = -1$].*

Theorem 2.5. *If $f(z) \in \Sigma$, then $f(z) \in \mathcal{F}^*([\alpha_1]; b, M)$ if and only if*

$$1 + \sum_{k=1}^{\infty} \frac{\left[k(e^{-i\theta} - m) + b(1+m)\right]}{b(1+m)} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} a_k z^k \neq 0, \qquad (2.9)$$

for all $\theta \in [0, 2\pi)$.

Proof. If $f(z) \in \Sigma$, from Theorem 2.1, we have $f(z) \in \mathcal{F}^*([\alpha_1]; b, M)$ if and only if

$$z \left[\mathcal{H}_{q,s}(\alpha_1)f(z) * \frac{1 + (C-1)z}{z(1-z)^2} \right] \neq 0 \text{ for } z \in \mathbb{U}, \qquad (2.10)$$

where $C = C_\theta = \frac{e^{-i\theta} - m}{b(1+m)}$, $\theta \in [0, 2\pi)$. *Since*

$$\frac{1 + (C-1)z}{z(1-z)^2} = \frac{1}{z} + \sum_{k=1}^{\infty} (kC + 1) z^{k-1} \text{ for } z \in \mathbb{U}^*.$$

It is easy to show that (2.10) holds if and only if (2.9) holds. This completes the proof of Theorem 2.5. □

Theorem 2.6. *If $f(z) \in \Sigma$, then $f(z) \in \mathcal{G}^*([\alpha_1]; b, M)$ if and only if*

$$1 - \sum_{k=1}^{\infty} \frac{(k-1)\left[k(e^{-i\theta} - m) + b(1+m)\right]}{b(1+m)} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} a_k z^k \neq 0, \qquad (2.11)$$

for all $\theta \in [0, 2\pi)$.

Proof. If $f(z) \in \Sigma$, from Theorem 2.3, we have $f(z) \in \mathcal{G}^*([\alpha_1]; b, M)$ if and only if

$$z \left[\mathcal{H}_{q,s}(\alpha_1) f(z) * \frac{1 - 3z - 2(C-1)z^2}{z(1-z)^3} \right] \neq 0 \text{ for } z \in \mathbb{U}, \qquad (2.12)$$

where $C = C_\theta = \frac{e^{-i\theta} - m}{b(1+m)}$, $\theta \in [0, 2\pi)$. Since

$$\frac{1 - 3z - 2(C-1)z^2}{z(1-z)^3} = \frac{1}{z} - \sum_{k=1}^{\infty} (k-1)(kC+1) z^{k-1} \text{ for } z \in \mathbb{U}^*.$$

It is easy to show that (2.12) holds if and only if (2.11) holds. This completes the proof of Theorem 2.6. $\qquad \square$

Unless otherwise mentioned, we assume throughout the reminder of this section that $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ are positive real parameters.

Theorem 2.7. *If* $f(z) \in \Sigma$ *satisfies the inequality*

$$\sum_{k=1}^{\infty} (k + |b|) \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} |a_k| \leq |b|, \qquad (2.13)$$

then $f(z) \in \mathcal{F}^*([\alpha_1]; b, M)$.

Proof. Since

$$\left| 1 + \sum_{k=1}^{\infty} \frac{[k(e^{-i\theta} - m) + b(1+m)]}{b(1+m)} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} a_k z^k \right|$$

$$\geq 1 - \sum_{k=1}^{\infty} \left| \frac{[k(e^{-i\theta} - m) + b(1+m)]}{b(1+m)} \right| \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} |a_k| |z|^k$$

$$\geq 1 - \sum_{k=1}^{\infty} \frac{(k + |b|)}{|b|} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} |a_k| > 0, \ z \in \mathbb{U},$$

which implies that inequality (2.13). Thus the proof of Theorem 2.7 is completed. $\qquad \square$

Using similar arguments to those in the proof of Theorem 2.7, we obtain the following theorem.

Theorem 2.8. *If* $f(z) \in \Sigma$ *satisfies the inequality*

$$\sum_{k=1}^{\infty} (k-1)(k + |b|) \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} |a_k| \leq |b|, \qquad (2.14)$$

then $f(z) \in \mathcal{G}^*([\alpha_1]; b, M)$.

Now, Using the method due to Ahuja [16], we will prove the following theorem.

Theorem 2.9. *For* $\alpha_1 > 0$, *we have* $\mathcal{F}^*([\alpha_1 + 1]; b, M) \subset \mathcal{F}^*([\alpha_1]; b, M)$.

Proof. Since $f(z) \in \mathcal{F}^*([\alpha_1 + 1]; b, M)$, we see from Theorem 2.5 that

$$1 + \sum_{k=1}^{\infty} \frac{\left[k(e^{-i\theta} - m) + b(1 + m)\right]}{b(1 + m)} \frac{(\alpha_1 + 1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} a_k z^k \neq 0, \qquad (2.15)$$

We can write (2.15) as

$$\left[1 + \sum_{k=1}^{\infty} \frac{\alpha_1 + k}{\alpha_1} z^k\right] * \left[1 + \sum_{k=1}^{\infty} \frac{\left[k(e^{-i\theta} - m) + b(1 + m)\right]}{b(1 + m)} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} a_k z^k\right] \neq 0,$$
$$(2.16)$$

Since

$$\left[1 + \sum_{k=1}^{\infty} \frac{\alpha_1 + k}{\alpha_1} z^k\right] * \left[1 + \sum_{k=1}^{\infty} \frac{\alpha_1}{\alpha_1 + k} z^k\right] = 1 + \sum_{k=1}^{\infty} z^k. \qquad (2.17)$$

By using the property, if $f \neq 0$ and $g * h \neq 0$, then $f * (g * h) \neq 0$, (2.16) can be written as

$$1 + \sum_{k=1}^{\infty} \frac{\left[k(e^{-i\theta} - m) + b(1 + m)\right]}{b(1 + m)} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k} a_k z^k \neq 0, \qquad (2.18)$$

which means that $f(z) \in \mathcal{F}^*([\alpha_1]; b, M)$. This completes the proof of Theorem 2.9. $\square$

Using the same arguments as in the proof of Theorem 2.9, we obtain the following theorem.

Theorem 2.10. *For* $\alpha_1 > 0$, *we have* $\mathcal{G}^*([\alpha_1 + 1]; b, M) \subset \mathcal{G}^*([\alpha_1]; b, M)$.

Remark 2.11. *For different choices of* $(\alpha_i, \beta_j;\ i = 1, ..., q,\ j = 1, ..., s\ )$, *we obtain new results for different classes mentioned in the introduction.*

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