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Existence Theorems for Cyclic-Prešić Operator in Product Spaces endowed with Directed Graphs

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Abstract : In this paper, we introduce a class of generalized cyclic contration mapping in product spaces endowed with directed graphs, called G-cyclic-Prešić operators. Some prove fixed point theorems for such introduced mappings are provided. Also, the examples are given to validate the results proved herein.

Keywords : graph; G-Prešić operator; fixed point.
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1 Introduction and Preliminaries

Indeed, it is well known that, in mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the Banach Contraction Principle which is one of the most important results of analysis and considered as the main source of metric fixed point theory. The theorem then yields existence and uniqueness theorem for differential and integral equation, namely, Picard's Existence and Uniqueness Theorem (Ordinary

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differential equations). Moreover, the Banach contraction principle has been generalized in many different directions (see [1]-[15]) and so on [1]. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting.

Afterwards, Kirk et al.[2] presented some extensions of the Banach contraction principle, in which the conclusion was obtained under mild modified conditions and which play important role in the development of metric fixed point theory. Next, these notions are the concepts that were introduced in [2].

Let X be a nonempty set and $f : X \to X$ be operator, m be a positive integer. Let $A_1, A_2, \dots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^m A_i$ is a *cyclic representation of* X *with respect to* f if

1. $A_i, i = 1, 2, \dots, m$ are nonempty sets;
2. $f(A_1) \subset A_2, f(A_2) \subset A_3, \dots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Also, they introduced the following concept.

Definition 1.1 ([2])**.** Let (X, d) be a metric space, $A_1, A_2, \dots, A_m$ be subsets of X and $X = \bigcup_{i=1}^m A_i$. An operator $f : X \to X$ is called a *cyclic contraction* if the following conditions hold:

1. $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;
2. there exists $\alpha \in [0, 1)$ such that, for each $i \in \{1, 2, \dots, m\}$, there holds, $x \in A_i, y \in A_{i+1},$ where $A_{m+1} = A_1$,
$$d(fx, fy) \leq \alpha d(x, y).$$

Under the setting of complete metric space (X, d) and the closedness of each A_i, they showed that a cyclic contraction f has a unique fixed point.

On the other hand, Prešić [3, 4] gave another generalization form of Banach contraction principle in the setting of product spaces, by proving the following result.

Theorem 1.2 ([3, 4])**.** *Let* (X, d) *be a complete metric space,* k *be a positive integer and* $f : X^k \to X$ *a mapping satisfying*

$$d(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \leq \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1}), \tag{1.1}$$

for every $x_1, x_2, \dots, x_{k+1} \in X$*, where* $\alpha_1, \alpha_2, \dots, \alpha_k$ *are nonnegative constants such that* $\sum_{i=1}^k \alpha_i < 1$*. Then there exists a unique point* $x \in X$ *such that* $f(x, x, \dots, x) = x$*.*

In this paper, for a mapping $f : X^k \to X$, where $k \geq 1$ is a positive integer, a point x in X such that $x = f(x, x, \dots, x)$ is also called a fixed point of f.

An operator satisfiying (1.1) is called a Prešić operator. Prešić type operator have applications in solving the nonlinear difference equations and in the convergence of sequences, for example, see [5, 6, 3].

Motivated by [2] and [3, 4] very recently, Shukla and Abbas[7] introduced the following notion of cyclic-Prešić operator on a generalized cyclic representation, and also investigated the existence and uniqueness of fixed point of such cyclic-Prešić operator.

Definition 1.3 ([7]). Let X be any nonempty set, k a positive integer, $f : X^k \to X$ an operator and $A_1, A_2, \ldots, A_m$ be the subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a *cyclic representation of* X *with respect to* f if

1. $A_i, i = 1, 2, \ldots, m$ are nonempty sets;
2. for each $i \in \{1, 2, \ldots, m\}$, $f(A_i \times A_{i+1} \times \cdots \times A_{i+k-1}) \subset A_{i+k}$, where $A_{m+j} = A_j$ for $j \in \mathbb{N}$.

Definition 1.4 ([7]). Let $A_1, A_2, \ldots, A_m$ be a subsets of a metric space (X, d), k a positive integer, and $X = \bigcup_{i=1}^{m} A_i$. An operator $f : X^k \to X$ is called a *cyclic-Prešić operator* if following conditiona are met:

1. $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f;
2. there exists nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i < 1$ and

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1}) \quad (1.2)$$

for all $x_1 \in A_i, x_2 \in A_{i+1}, \ldots, x_{k+1} \in A_{i+k}, (i = 1, 2, \ldots, m$ where $A_{m+j} = A_j$ for all $j \in \mathbb{N})$.

If we take $k = 1$, then above definition reduce to the concepts which were introduced in [6].

They also introduced notion of $m-$cyclic sequence in a metric space.

Definition 1.5 ([7]). Let X be a nonempty set and $A_1, A_2, \ldots, A_m$ nonempty subsets of X. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is called $m-$*cyclic sequence* if:

1. there exists $i \in \{1, 2, \ldots, m\}$ such that $x_1 \in A_i$;
2. $x_n \in A_i$ for some $n \in \mathbb{N}, i \in \{1, 2, \ldots, m\}$ implies that $x_{n+1} \in A_{i+1}$, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$.

Shukla and Abbas[7] proved the existence and uniqueness theorems for cyclic-Prešić operators on a generalization of a cyclic contractions, that is $X = \bigcup_{i=1}^{m} A_i$ where A_i is a closed subset of complete metric space (X, d).

Spacially, Shukla and Abbas[7] showed that the completeness of underlying space and closedness of sets A_i are replaced by imposing an additional condition on mapping f. In other words, supppose there exists $u \in \bigcap_{i=1}^{m} A_i$ such that

$$d(u, f(u, u, ..., u)) \leq d(x, f(x, x, ..., x)) \text{ for all } x \in X.$$

Then, we get u is a fixed point of f.

Proposition 1.6 ([7]). *Let $A_1, A_2, \ldots, A_m$ be closed subsets of a complete metric space (X, d). Suppose that $\{x_n\}_{n=1}^{\infty}$ is an $m-$cyclic sequence in $X = \bigcup_{i=1}^{m} A_i$. If $\{x_n\}_{n=1}^{\infty}$ converges to some $u \in X$, then $u \in \bigcap_{i=1}^{m} A_i$.*

Next, we recall the concept of fixed point theorem in metric spaces endowed with graphs, which was firstly studied by Jachymski[8].

Let (X, d) be a metric space. Let Δ denotes the diagonal of the Cartesian product $X \times X$, this is,

$$\Delta = \{(x, x) : x \in X\}.$$

Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is $\Delta \subseteq E(G)$. Moreover, a graph G is weakly connected if there is an undirected path between any pair of vertrices.

In 2007, Jachymski [8] introduced the following concept.

Definition 1.7 ([8]). A mappimg $f : X \to X$ is said to be a *Banach G-contraction* if f preserves edges of G, i.e.,

$$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$$

and f decreases weights of edges of G in the following way:

$$\exists \delta \in (0, 1), \forall x, y \in X, (x, y) \in E(G) \Rightarrow d(fx, fy) \leq \delta d(x, y).$$

Jachymski [8] proved existence and uniqueness fixed point theorem for $G-$contraction mapping in the complete metric space setting. He also showed some applications of the results to Kelisky-Rivlin theorem on iterates of Bernstein operators on the space $C[0, 1]$.

In this paper, motivated by above literatures, we will introduce a type of Prešić operator endowed with a graph on a generalization of cyclic contractions. Some examples and fixed point theorems of such a mapping in the setting of generalized completed metric space, as $f-$orbitally complete metric space, will be discussed and provided. Reasonably, we shall gives a concept of $f-$orbitally complete metric space type.

Let (X, d) be a metric space with a graph G, k a positive integer and $f : X^k \to X$ be a mapping. Let $\{x_i\}_{i=1}^{k}$ be any path in G. An orbit of f with respect to $\{x_i\}_{i=1}^{k}$ is the set

$$O(\{x_1, x_2, ..., x_k\}, f) = \{x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ...\},$$

where $x_{k+n} = f(x_n, x_{n+1}, ..., x_{k+(n-1)})$ for each $n \in \mathbb{N}$.

A metric space (X, d) with a graph G is said to be $f-$orbitally complete with respect to G if for every path $\{x_i\}_{i=1}^k$ in G, any Cuachy subsequence of orbit $O(\{x_1, x_2, ..., x_k\}, f)$ converges in X.

2 Main Result

We start by introducing a new class of generalized Prešić type mappings.

Definition 2.1. Let (X, d) be a metric space and $f : X^k \to X$ be a mapping, where k is a positive integer. Let $A_1, A_2, \ldots, A_m$ be subsets of a metric space such that $X = \bigcup_{i=1}^m A_i$. A mapping f is said to be *G-cyclic-Prešić operator* if following conditions are met:

1. $X = \cup_{i=1}^m A_i$ is a cyclic repersentation of X with respect to f;
2. there is a graph G such that if $\{x_i\}_{i=1}^{k+1}$ be a path in G for which $(x_1, x_2, \ldots, x_k, x_{k+1}) \in \prod_{j=i}^{i+k} A_j$ where $A_{m+j} = A_j$ for $j \in \mathbb{N}$, then

$$(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \in E(G),$$

and there are nonnegative constants $q_i's$ with $\sum_{i=1}^k q_i < 1$ such taht

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^k q_i d(x_i, x_{i+1}). \quad (2.1)$$

The next example shows that class of $G-$cyclic-prešić operator is a genuine generalization of the class of cyclic-Prešić operator, that was introduced [12].

Example 2.2. Let $X = [-1, 1]$ and d be the usual metric on X. Let us consider a mapping, $f : X \times X \to X$, which defined by

$$f(x, y) = \begin{cases} -1, & \text{if } (x, y) \in [-1, -\frac{1}{2}) \times (0, \frac{1}{2}); \\ 1, & \text{if } (x, y) \in (0, 1] \times (0, 1] \cup [-1, 0) \times [-1, 0); \\ 0, & \text{if otherwise.} \end{cases}$$

Let us take $A_1 = [-1, 0], A_2 = [0, 1]$ and choose a directed graph G defined by

$$E(G) = \{(x, y) \in X \times X : x, y \in X \text{ with } |x| \leq |y|\}.$$

Obviously, $f(A_1 \times A_2) \subset A_1$ and $f(A_2 \times A_1) \subset A_2$. Therefore $X = \bigcup_{i=1}^2 A_i$ is a cyclic representation of X with respect to f, therefore f satisfied condition (i) of G-cyclic Prešić operator.

Next, we will show that f is satisfies condition (ii) of G-cyclic Prešić operator. Let $\{x_i\}_{i=1}^3$ be a path in graph G. So, we have two cases is possible, $x_1 \in A_1, x_2 \in A_2, x_3 \in A_1$ or $x_1 \in A_2, x_2 \in A_1, x_3 \in A_2$, where $|x_1| \leq |x_2| \leq |x_3|$.

Case 1: If $x_1 \in A_1, x_2 \in A_2, x_3 \in A_1$, then $x_1 \in \left[-1, -\frac{1}{2}\right] \cup \left[-\frac{1}{2}, 0\right], x_2 \in \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right], x_3 \in \left[-1, -\frac{1}{2}\right] \cup \left[-\frac{1}{2}, 0\right]$.

Case 1.1: If $x_1 \in \left[-\frac{1}{2}, 0\right], x_2 \in \left[0, \frac{1}{2}\right], x_3 \in \left[-\frac{1}{2}, 0\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 1.2: If $x_1 \in \left[-\frac{1}{2}, 0\right], x_2 \in \left[0, \frac{1}{2}\right], x_3 \in \left[-1, -\frac{1}{2}\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 1.3: If $x_1 \in \left[-\frac{1}{2}, 0\right], x_2 \in \left[\frac{1}{2}, 1\right], x_3 \in \left[-1, -\frac{1}{2}\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 1.4: If $x_1 \in \left[-1, -\frac{1}{2}\right], x_2 \in \left[\frac{1}{2}, 1\right], x_3 \in \left[-1, -\frac{1}{2}\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 2: If $x_1 \in A_2, x_2 \in A_1, x_3 \in A_2$, then $x_1 \in \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right], x_2 \in \left[-1, -\frac{1}{2}\right] \cup \left[-\frac{1}{2}, 0\right], x_3 \in \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$.

Case 2.1: If $x_1 \in \left[0, \frac{1}{2}\right], x_2 \in \left[-\frac{1}{2}, 0\right], x_3 \in \left[0, \frac{1}{2}\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 2.2: If $x_1 \in \left[0, \frac{1}{2}\right], x_2 \in \left[-\frac{1}{2}, 0\right], x_3 \in \left[\frac{1}{2}, 1\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 2.3: If $x_1 \in \left[0, \frac{1}{2}\right], x_2 \in \left[-1, -\frac{1}{2}\right], x_3 \in \left[\frac{1}{2}, 1\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Case 2.4: If $x_1 \in \left[\frac{1}{2}, 1\right], x_2 \in \left[-1, -\frac{1}{2}\right], x_3 \in \left[\frac{1}{2}, 1\right]$, then $(f(x_1, x_2), f(x_2, x_3)) = (0, 0) \in E(G)$.

Clearly from above case, we obtain that

$$d(f(x_1, x_2), f(x_2, x_3) \leq \sum_{i=1}^{2} q_i d(x_i, x_{i+1}),$$

for $q_1, q_2 \in \left[\frac{3}{8}, \frac{1}{2}\right)$.

Thus, f is satisfied condition (ii) of G-cyclic-Prešić operator. Hence, f is G-cyclic-Prešić operator with respect $q_1, q_2 \in \left[\frac{3}{8}, \frac{1}{2}\right)$.

However, for $x_1 = -\frac{1}{3} \in \left[-1, -\frac{1}{2}\right), x_2 = \frac{1}{3} \in \left(0, \frac{1}{2}\right), x_3 = 0 \in \left(-\frac{1}{2}, 0\right)$, we see that $f(x_1, x_2) = -1$ and $f(x_2, x_3) = 0$. Thus, $d(f(x_1, x_2), f(x_2, x_3)) = 1$. Consider,

$$\begin{aligned} q_1 d(x_1, x_2) + q_2 d(x_2, x_3) &= q_1 d(-\frac{1}{3}, \frac{1}{3}) + q_2 d(\frac{1}{3}, 0) \\ &= q_1 \left| -\frac{1}{3} - \frac{1}{3} \right| + q_2 \left| \frac{1}{3} + 0 \right| \\ &\leq (q_1 + q_2)(1) \\ &< 1, \end{aligned}$$

where q_1, q_2 are nonnegative real numbers with $q_1 + q_2 < 1$. Thus,

$$d(f(x_1, x_2), f(x_2, x_3)) = 1 > q_1 d(x_1, x_2) + q_2 d(x_2, x_3)$$

where q_1, q_2 are nonnegative real numbers with $q_1 + q_2 < 1$. This shows that f is not satisfies condition (ii) of cyclic-Prešić operator and thus f is not cyclic-Prešić operator.

Additionaly, from now on, we will denoted P_f^k for the set of all paths $\{x_i\}_{i=1}^k$ of k vertices such that $(x_1, x_2, \ldots, x_k) \in \prod_{j=i}^{i+(k-1)} A_j$ for some $i \in \{1, 2, \ldots, m\}$ and $(x_k, f(x_1, x_2, \ldots, x_k)) \in E(G)$, that is,

$$\begin{aligned} P_f^k &= \{\{x_i\}_{i=1}^k : (x_1, x_2, \ldots, x_k) \in \prod_{j=i}^{i+(k-1)} A_j \text{ for some } i \in \{1, 2, ..., m\} \\ &\qquad \text{and } (x_i, x_{i+1}), (x_k, f(x_1, x_2, \ldots, x_k)) \in E(G) \text{ for } i = 1, 2, \ldots, k-1\} \end{aligned}$$

In our work, we must assume sufficient condition (MP) for verify existence of a fixed piont.

Property(MP): Let (X, d) be a metric space and $A_1, A_2, \ldots, A_m$ be subset of X such that $X = \bigcup_{i=1}^m A_i$, m is a positive integer. We will say that $X = \bigcup_{i=1}^m A_i$ has a *property (MP)* whenever an $m-$cyclic sequence $\{x_n\}_{n=1}^\infty$ in X such that $\{x_n\}_{n=1}^\infty$ converges to x in X and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n_p}\}_{p=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $(x_{n_p}, x) \in E(G)$ for all $p \in \mathbb{N}$.

The following theorem is sufficient condition for existence results.

Theorem 2.3. *Let (X, d) be a metric space endowed with a directed graph G and $f : X^k \to X$, k a positive integer. Let $A_1, A_2, \ldots A_m$ be closed subsets of metric space (X, d), with $X = \bigcup_{i=1}^m A_i$, and f is G-cyclic-Prešić operator with respect to G, such that $P_f^k \neq \emptyset$. Suppose that property (MP) holds and (X, d) is $f-$orbitally complete with respect to G, then f has a fixed point $u \in \bigcap_{i=1}^m A_i$.*

Moreover, if $i \in \{1, 2, \ldots, m\}$ *and* $x_1 \in A_i, x_2 \in A_{i+1}, \ldots, x_k \in A_{i+(k-1)}$, $(i = 1, 2, \ldots, m)$ *where* $A_{m+j} = A_j$ *for all* $j \in \mathbb{N}$ *and* $\{x_i\}_{i=1}^k$ *is a path in* P_f^k*, then the sequence* $\{x_n\}_{n=1}^\infty$ *defined by*

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \text{ for all } n \in \mathbb{N},$$

is an m*-cyclic sequence and converges to a fixed point of* f*.*

Proof. Let $i \in \{1, 2, \ldots, m\}$ and $x_1 \in A_i, x_2 \in A_{i+1}, \ldots, x_k \in A_{i+k-1}$ such that $\{x_i\}_{i=1}^k$ is a path in P_f^k, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$. We define a sequence $\{x_n\}_{n=1}^\infty$ in X by

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \text{ for all } n \in \mathbb{N}.$$

Then, by definition of P_f^k, we have

$$(x_i, x_{i+1}), (x_k, f(x_1, x_2, \ldots, x_k)) \in E(G) \text{ for } i = 1, 2, \ldots, k-1. \tag{2.2}$$

We see that

$$(x_k, x_{k+1}) = (x_k, f(x_1, x_2, \ldots, x_k)) \in E(G).$$

Since $f(x_1, x_2, \ldots, x_k) \in f\left(\prod_{j=i}^{i+(k-1)} A_j\right) \subset A_{i+k}$, it follows that $x_{k+1} \in A_{i+k}$. By (i) and (ii) of G-cyclic-Prešić operator , we obtain that

$$(x_{k+1}, x_{k+2}) = (f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \in E(G),$$

where $x_{k+1} \in A_{i+k}, x_{k+2} \in A_{i+k+1}$ and $\{x_i\}_{i=1}^{k+2}$ is a path of $k+2$ vertices in G.

In a similarly way, we find that, for any fixed $n > 1$, $\{x_i\}_{i=1}^n$ is a path of n vertices in G. As $X = \bigcup_{i=1}^m A_i$ is a cyclic representstion of X with respect to f, so we have $x_n \in A_{i+n-1}$ for all $n \in \mathbb{N}$ and so the sequence $\{x_n\}_{n=1}^\infty$ is a m-cyclic sequence and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, by (i), and (ii) of G-cyclic-Prešić operator.

Now we shall show that the sequence $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. For simplicity let $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and

$$\mu = \max\left\{\frac{d_1}{\theta}, \frac{d_2}{\theta^2}, \ldots, \frac{d_k}{\theta^k}\right\} \text{ where } \theta = \Big[\sum_{i=1}^k q_i\Big]^{\frac{1}{k}}.$$

We shall show that

$$d_n \le \mu\theta^n \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

By the definition of μ, it is clear that (2.3) is ture for $n = 1, 2, \ldots, k$.

Now let the following k inequalities:

$$d_n \le \mu\theta^n, d_{n+1} \le \mu\theta^{n+1}, \ldots, d_{n+k-1} \le \mu\theta^{n+k-1}$$

be the induction hypothesis.

Since $x_n \in A_{i+n-1}, x_{n+1} \in A_{i+n}, (x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, therefore we obtain from (ii) of G-cyclic-Prešić operator that

$$\begin{aligned}
d_{n+k} &= d(x_{n+k}, x_{n+k+1}) \\
&= d(f(x_n, x_{n+1}, \dots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \dots, x_{n+k})) \\
&\leq q_1 d(x_n, x_{n+1}) + q_2 d(x_{n+1}, x_{n+2}) + \cdots + q_k d(x_{n+k-1}, x_{n+k}) \\
&= q_1 d_n + q_2 d_{n+1} + \cdots + q_k d_{n+k-1} \\
&\leq q_1 \mu\theta^n + q_2 \mu\theta^{n+1} + \cdots + q_k \mu\theta^{n+k-1} \\
&\leq q_1 \mu\theta^n + q_2 \mu\theta^n + \cdots + q_k \mu\theta^n \\
&= \Big[\sum_{i=1}^{k} q_i\Big]\mu\theta^n \\
&= \theta^k \mu\theta^n \\
&= \mu\theta^{n+k}
\end{aligned}$$

and the inductive proof of (2.3) is complete.

Now, for $n, m \in \mathbb{N}$ with $n < m$, we obtain from (2.3) that

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&= d_n + d_{n+1} + \cdots + d_{m-1} \\
&\leq \mu\theta^n + \mu\theta^{n+1} + \cdots + \mu\theta^{m-1} \\
&= \mu\theta^n(1 + \theta + \cdots + \theta^{m-1-n}) \\
&\leq \mu\theta^n(1 + \theta + \cdots + \theta^{m-1-n} + \cdots) \\
&= \mu\theta^n \frac{1}{1-\theta}.
\end{aligned}$$

Since $\theta = [\sum_{i=1}^{k} q_i]^{\frac{1}{k}} < 1$, it follows from the above inequality that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is a $f-$orbitally complete space with respect to G, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

We shall show that u is a fixed point of f. By property (MP), there exists a subsequence $\{x_{n_p}\}_{p=1}^{\infty}$ of $m-$cyclic sequence $\{x_n\}_{n=1}^{\infty}$ such that $(x_{n_p}, u) \in E(G)$ for all $p \in \mathbb{N}$. Therefore for any $p \in \mathbb{N}$ with $n_p > k$,

$$\begin{aligned}
d(u, f(u, u, \dots, u)) &\leq d(u, x_{n_p+1}) + d(x_{n_p+1}, f(u, u, \dots, u)) \\
&= d(u, x_{n_p+1}) + d(f(x_{n_p-k+1}, x_{n_p-k+2}, \dots, x_{n_p}), f(u, u, \dots, u)) \\
&\leq d(u, x_{n_p+1}) + d(f(x_{n_p-k+1}, \dots, x_{n_p}), d(f(x_{n_p-k+2}, \dots, x_{n_p}, u)) \\
&\quad + d(f(x_{n_p-k+2}, \dots, x_{n_p}, u), f(x_{n_p-k+3}, \dots, x_{n_p}, u, u)) \\
&\quad + \cdots + d(f(x_{n_p}, u, \dots, u), f(u, u, \dots, u))
\end{aligned}$$

By Proposition(1.6), we know that $u \in \bigcap_{i=1}^{m} A_i$ and for every $n \in \mathbb{N}$, there exists $i \in \{1, 2, \dots, m\}$ such that $x_n \in A_i, x_{n+1} \in A_{i+1}, \dots, x_{n+k-1} \in A_{i+k-1}, \{x_i\}_{i=n}^{n+k-1}$

is a path in G, where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$, therefore we obtain from (ii) of G-cyclic-Prešić operator and previous inequality that

$$\begin{aligned}
&d(u, f(u, u, \ldots, u)) \\
&\leq d(u, x_{n_p+1}) + \{q_1 d_{n_p-k+1} + q_2 d_{n_p-k+2} + \cdots + q_{k-1} d_{n_p-1} + q_k d(x_{n_p}, u)\} \\
&+\{q_1 d_{n_p-k+2} + q_2 d_{n_p-k+3} + \cdots + q_{k-2} d_{n_p-1} + q_{k-1} d(x_{n_p}, u)\} \\
&+ \cdots + q_1 d(x_{n_p}, u).
\end{aligned}$$

Since $\lim_{p\to\infty} x_{n_p} = u$, it follows from previous inequality that

$$d(u, f(u, u, \ldots, u)) = 0,$$

that is, $f(u, u, \ldots, u) = u$. Hence u is a fixed piont of f □

In the above theorem the fixed point of f may be not unique. The next theorem we provide a sufficient condition for the uniqueness of a fixed point of G-cyclic prešić operator.

Theorem 2.4. *Let (X, d) be a metric space endowed with a directed graph G and $f : X^k \to X$, k is a positive integer. Let $A_1, A_2, \ldots, A_m$ be closed subsets of metric space (X, d) with $X = \bigcup_{i=1}^m A_i$ and f be a G-cyclic-Prešić operator with respect to G such that $P_f^k \neq \emptyset$. Suppose that all the conditions of Theorem2.3 are satisfied. If $\mathcal{F}(f) \subseteq \bigcap_{i=1}^m A_i$ and the subgraph $G_{\mathcal{F}(f)}$ is weakly connected, where $V(G_{\mathcal{F}(f)}) = \mathcal{F}(f)$ and $E(G_{\mathcal{F}(f)}) = E(G)$, then fixed point of f is unique.*

Proof. The existence of a fixed point follows from Theorem2.3. Suppose that $G_{\mathcal{F}(f)}$ is weakly connected. It is sufficient to show that for each $(u, v) \in E(G_{\mathcal{F}(f)})$, we get that $u = v$. Let u, v be fixed points of f such that $(u, v) \in E(G_{\mathcal{F}(f)})$. We known that $u, v \in \bigcap_{i=1}^m A_i$. From (2.1), we obtain that

$$\begin{aligned}
d(u, v) &= d(f(u, u, \ldots, u), f(v, v, \ldots, v)) \\
&\leq d(f(u, u, \ldots, u), f(u, u, \ldots, u, v)) + d(f(u, u, \ldots, u, v), f(u, u, \ldots, u, v, v)) \\
&\quad + \cdots + d(f(u, v, \ldots, v), f(v, v, \ldots, v)) \\
&\leq q_k d(u, v) + q_{k-1} d(u, v) + \cdots + q_1 d(u, v) \\
&= \sum_{i=1}^k q_i d(u, v) \\
&< d(u, v). \qquad (2.4)
\end{aligned}$$

Thus, $u = v$. Therefore, f has a unique fixed point. □

In next theorem the closedness of sets A_i are replaced by imposing an additional condition on mapping f.

Theorem 2.5. *Let $A_1, A_2, \ldots, A_m$ be subsets of a metric space (X, d), k a positive integer, and $X = \bigcup_{i=1}^m A_i$. Let $f : X^k \to X$ be a G-cyclic-Prešić operator. Supppose there exists $u \in \bigcap_{i=1}^m A_i$ such that $(u, f(u, u, \ldots, u)) \in E(G)$ and*

$$d(u, f(u, u, \ldots, u)) \le d(x, f(x, x, \ldots, x)) \text{ for all } x \in X,$$

then u becomes a fixed point of f.

Proof. Let $F(x) = d(x, f(x, x, \ldots, x))$ for all $x \in X$. Then by assumption we have $F(u) \le F(x)$ for all $x \in X$. Let $z = f(u, u, ..., u)$, then as $u \in \bigcap_{i=1}^m A_i$ and $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f, we obtain $z \in \bigcap_{i=1}^m A_i$. If $F(u) \neq 0$ then it follows from (2.1) that

$$\begin{aligned} F(z) &= d(z, f(z, z, \ldots, z)) \\ &= d(f(u, u, \ldots, u), f(z, z, \ldots, z)) \\ &\le d(f(u, u, \ldots, u), f(u, u, \ldots, u, z)) \\ &\quad + d(f(u, u, \ldots, u, z), f(u, u, \ldots, u, z, z)) \\ &\quad + \cdots + d(f(u, z, \ldots, z), f(z, z, \ldots, z)) \\ &\le \alpha_k d(u, z) + \alpha_{k-1} d(u, z) + \cdots + \alpha_1 d(u, z) \\ &= \sum_{i=1}^{k} \alpha_i d(u, z) \\ &< d(u, z) = d(u, f(u, u, \ldots, u)) = F(u). \end{aligned}$$

Thus we obtain $F(z) < F(u)$, where $z = f(u, u, \ldots, u) \in X$. This contradiction shows that $F(u) = d(u, f(u, u, \ldots, u)) = 0$, that is $f(u, u, ..., u) = u$. □

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