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Fixed Point Theorems for Modified (α-ψ-φ-θ)-Rational Contractive Mappings in α-Complete b-Metric Spaces[1]

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Abstract : In this paper, we introduce the notion of modified (α-ψ-φ-θ)-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible in α-complete b-metric spaces. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Our results extend the fixed point theorems in α-complete metric spaces proved by Hussian et al.[N. Hussain, M. A. Kutbi, P. Salimi, Fixed point theory in α-complete metric spaces with applications, Abstr. Appl. Anal. (2014) Article ID 280817] to α-complete b-metric spaces.

Keywords : triangular α-orbital admissible mappings; α-complete b-metric spaces; α-continuous mappings; modified (α-ψ-φ-θ)-rational contractive mappings; common fixed points.

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1 Introduction

Fixed point theory in metric spaces is one of the most important tools for proving the existence and uniqueness of the solutions to various mathematical models. Later in 1993 Czerwik [1], generalized the notion of metric spaces by introducing the notion of b-metric spaces. On the other hand, Samet et al. [2] proved the fixed point theorems for α-admissible mappings which are α-φ-contractive mappings in complete metric spaces. Salimi et al. [3] and Hussain et al. [4] modified these notions and assured the fixed point theorems. Recently, Hussain et al. [5] established fixed point theorems for modified α-φ-rational contractive mappings in α-complete metric spaces and proved the existence of solutions of integral equations.

In this paper, we extend the fixed point results in α-complete metric spaces proved by Hussian et al. [5] to α-complete b-metric spaces by introducing the notion of modified (α-ψ-φ-θ)-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g and is triangular g-α-admissible in the setting of α-complete b-metric spaces.

2 Preliminaries

We now recall some definitions and lemmas that will be used in the sequel.

In 2012, Samet et al. [2] introduced the notion of α-admissible mappings.

Definition 2.1 ([2]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *α-admissible* if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(Tx, Ty) \geq 1.$$

Recently Hussain et al. [5] introduced the concept of modified α-φ-rational contractive mappings and proved the fixed point theorems for such mappings in α-complete metric spaces.

Definition 2.2. A function $\varphi : [0, \infty) \to [0, \infty)$ is called a *Bianchini-Grandolfi gauge function* [6] if the following conditions hold:

(i) φ is nondecreasing;

(ii) $\sum_{k=1}^{\infty} \varphi^k(t)$ converges for all $t > 0$.

We denote by Φ the set of all Bianchini-Grandolfi gauge functions.

Lemma 2.3 ([7]). *If $\varphi \in \Phi$, then the following statements hold:*

(i) $\varphi(t) < t$ for all $t > 0$;

(ii) φ *is continuous at 0;*

(iii) $\varphi(0) = 0$.

Definition 2.4 ([5])**.** Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is a *modified α-ψ-rational contractive mapping* if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } d(Tx, Ty) \leq \varphi(M(x, y)), \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\},$$

and $\varphi \in \Phi$.

Theorem 2.5 ([5])**.** *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) X *is an α-complete metric space;*

(ii) T *is a modified α-φ-rational contractive mapping;*

(iii) T *is an α-admissible mapping;*

(iv) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(v) T *is an α-continuous mapping.*

Then T has a fixed point.

Recently, Popescu [8] studied the definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings.

Definition 2.6 ([8])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *α-orbital admissible* if

$$\alpha(x, Tx) \geq 1 \quad \text{implies} \quad \alpha(Tx, T^2x) \geq 1.$$

Definition 2.7 ([8])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *triangular α-orbital admissible* if

(a) T is α-orbital admissible;

(b) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

Lemma 2.8 ([8])**.** *Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Definition 2.9 ([1])**.** Let X be a nonempty set and let $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a *b-metric* if for all $x, y, z \in X$,

(i) $d(x, y) = 0$, if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then the pair (X, d) is called a *b-metric space*.

Note that a metric space is evidently a b-metric space but the converse is not generally true. For more details see [9].

In this paper, we use the following concepts in b-metric spaces.

Definition 2.10. Let (X, d) be a b-metric space and $\alpha : X \times X \to [0, +\infty)$. Then X is said to be an *α-complete b-metric space* if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Definition 2.11. Let (X, d) be a b-metric space, $\alpha : X \times X \to [0, +\infty)$ and $T : X \to X$. Then T is said to be an *α-continuous mapping on* (X, d) if for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies $Tx_n \to Tx$ as $n \to \infty$.

In 2014, Rosa and Vetro [10] introduced the notion of triangular g-α-admissible mappings.

Definition 2.12. Let $T, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *triangular g-α-admissible* if

1. $\alpha(gx, gy) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;
2. $\alpha(gx, gy) \geq 1$ and $\alpha(gy, gz) \geq 1$ imply $\alpha(gx, gz) \geq 1$.

Lemma 2.13 ([5]). *Let $T : X \to X$ be a triangular g-α-admissible. Assume that that there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Define a sequence $\{gx_n\}$ by $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(gx_m, gx_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Definition 2.14. Let $T, g : X \to X$. If $w = Tx = gx$ for some $x \in X$, then x is called a *coincidence point of T and g*, and w is called a point of coincidence of T and g.

Definition 2.15. Let $T, g : X \to X$. The pair $\{T, g\}$ is said to be *weakly compatible* if $Tgx = gTx$, whenever $Tx = gx$ for some x in X.

Abbas and Rhoades [11] proved the existence of the unique common fixed points of a pair of weakly compatible mappings by using the following proposition as a main tool.

Proposition 2.16 ([11]). *Let $T, g : X \to X$ and $\{T, g\}$ is weakly compatible. If T and g have a unique point of coincidence $w = Tx = gx$, then w is the unique common fixed point of T and g.*

3 Main results

In this section, unique fixed point theorems and unique common fixed point theorems in α-complete b-metric spaces and applications to integral equations are presented.

3.1 The unique fixed point theorems

We first introduce the concept of modified (α-ψ-φ-θ)-rational contractive mappings and prove the existence of fixed point theorems for such mappings.

Definition 3.1. Let (X, d) be a b-metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is a *modified (α-ψ-φ-θ)-rational contractive mapping* if there exists $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \psi(s^3 d(Tx, Ty)) \leq \varphi(\psi(M_b(x, y))) + L\theta(N_b(x, y)), \tag{3.1}$$

where

$$M_b(x, y) = \max\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2s}\},$$

$$N_b(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

and $\psi, \varphi, \theta : [0, \infty) \to [0, \infty)$ are continuous functions with $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$, φ is nondecreasing, $\theta(0) = 0$, $\psi(t) = 0$ if and only if $t = 0$ and ψ is increasing.

Theorem 3.2. *Let (X, d) be an α-complete b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ is a modified (α-ψ-φ-θ)-rational contractive mapping. Assume that the following conditions hold:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.8, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

If $x_N = x_{N+1}$ for some $N \in \mathbb{N}$, then T has a fixed point. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is a modified (α-ψ-φ-θ)-rational contraction and by (3.2), we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \psi(s^3 d(x_n, x_{n+1})) \\ &= \psi(s^3 d(Tx_{n-1}, Tx_n)) \\ &\leq \varphi(\psi(M_b(x_{n-1}, x_n))) + L\theta(N_b(x_{n-1}, x_n)) \qquad (3.3)\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}N_b(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}M_b(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, Tx_{n-1})}, \frac{d(x_n, Tx_n)}{1 + d(x_n, Tx_n)}, \\ &\quad \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\} \\ &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)}, \frac{d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\quad \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\}.\end{aligned}$$

Since

$$\frac{d(x_{n-1}, x_{n+1})}{2s} \leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2s},$$

it follows that

$$M_b(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \qquad (3.4)$$

By (3.3) and (3.4), we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(M_b(x_{n-1}, x_n))) + L\theta(N_b(x_{n-1}, x_n)) \\ &\leq \varphi(\psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})).\end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(d(x_n, x_{n+1}))) \\ &< \psi(d(x_n, x_{n+1})),\end{aligned}$$

which is a contradiction. This implies that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(d(x_{n-1}, x_n))) \\ &< \psi(d(x_{n-1}, x_n)),\end{aligned} \tag{3.5}$$

for each $n \in \mathbb{N}$. Since ψ is increasing, we get $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Therefore $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Assume that $r > 0$. Since ψ and φ are continuous, from (3.5), we have

$$\psi(r) \leq \varphi(\psi(r)) \leq \psi(r).$$

This implies that $\psi(r) = \varphi(\psi(r))$. Since $\varphi(t) < t$, for each $t > 0$, we obtain that

$$\psi(r) = \varphi(\psi(r)) < \psi(r),$$

which is a contradiction and therefore $r = 0$. It follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

Next we will prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exist two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{3.7}$$

Let $n(k)$ be the smallest number satisfying (3.7). Thus

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{3.8}$$

By triangle inequality, (3.7) and (3.8), we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq sd(x_{n(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{m(k)}) \\ &< sd(x_{n(k)}, x_{n(k)-1}) + s\varepsilon.\end{aligned}$$

By taking the upper limit as $k \to \infty$ and (3.6), we have

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}) \leq s\varepsilon. \tag{3.9}$$

Using triangle inequality again, we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq sd(x_{m(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{n(k)}) + s^2 d(x_{n(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{n(k)}) + (s^2 + s) d(x_{n(k)}, x_{n(k)+1}).\end{aligned}$$

From above inequality, we obtain that

$$\varepsilon \leq sd(x_{m(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \leq s^2 d(x_{m(k)}, x_{n(k)}) + (s^2 + s) d(x_{n(k)}, x_{n(k)+1}).$$

Taking the upper limit as $k \to \infty$, by (3.6) and (3.9), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon. \tag{3.10}$$

Similarly, we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq sd(x_{n(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{m(k)}) \\ &\leq s^2 d(x_{n(k)}, x_{m(k)}) + s^2 d(x_{m(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{m(k)}) \\ &\leq s^2 d(x_{n(k)}, x_{m(k)}) + (s^2+s) d(x_{m(k)}, x_{m(k)+1}).\end{aligned}$$

So from (3.6) and (3.9), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \leq s^2\varepsilon. \tag{3.11}$$

Since

$$d(x_{m(k)+1}, x_{n(k)}) \leq sd(x_{m(k)+1}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}),$$

and by using (3.6) and (3.11), we get that

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{3.12}$$

Using (3.6), (3.9), (3.10) and (3.11), we have

$$\begin{aligned}\limsup_{k\to\infty} M_b(x_{n(k)}, x_{m(k)}) &= \max\{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\to\infty} \frac{d(x_{n(k)}, Tx_{n(k)})}{1+d(x_{n(k)}, Tx_{n(k)})}, \\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, Tx_{m(k)})}{1+d(x_{m(k)}, Tx_{m(k)})}, \\ &\quad \frac{\limsup_{k\to\infty} d(x_{n(k)}, Tx_{m(k)}) + \limsup_{k\to\infty} d(x_{m(k)}, Tx_{n(k)})}{2s}\} \\ &= \max\{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\to\infty} \frac{d(x_{n(k)}, x_{n(k)+1})}{1+d(x_{n(k)}, x_{n(k)+1})}, \\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, x_{m(k)+1})}{1+d(x_{m(k)}, x_{m(k)+1})}, \\ &\quad \frac{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) + \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})}{2s}\} \\ &\leq \max\{s\varepsilon, 0, 0, \frac{s^2\varepsilon + s^2\varepsilon}{2s}\} = s\varepsilon.\end{aligned}$$

This implies that

$$\limsup_{k\to\infty} M_b(x_{n(k)}, x_{m(k)}) \leq s\varepsilon. \tag{3.13}$$

By using the same argument as above, we have

$$\limsup_{k\to\infty} N_b(x_{n(k)}, x_{m(k)}) = 0. \tag{3.14}$$

Since T is a modified (α-ψ-φ-θ)-rational contraction, by using Lemma 2.8 and (3.12), we have

$$\begin{aligned}
\psi(s\varepsilon) = \psi(s^3 \cdot \frac{\varepsilon}{s^2}) &\leq \psi(s^3 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})) \\
&= \limsup_{k\to\infty} \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) \\
&= \limsup_{k\to\infty} \psi(s^3 d(Tx_{m(k)}, Tx_{n(k)})) \\
&\leq \limsup_{k\to\infty} [\varphi(\psi(M_b(x_{m(k)}, x_{n(k)}))) + L\theta(N_b(x_{m(k)}, x_{n(k)}))] \\
&= \varphi(\psi(\limsup_{k\to\infty} M_b(x_{m(k)}, x_{n(k)}))) + L\theta(\limsup_{k\to\infty} N_b(x_{m(k)}, x_{n(k)})) \\
&\leq \varphi(\psi(s\varepsilon)) \\
&< \psi(s\varepsilon),
\end{aligned}$$

which is a contradiction. Then we can conclude that $\{x_n\}$ is a Cauchy sequence. From (3.2) and since X is an α-complete b-metric space, we have $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Since T is α-continuous, we obtain that $\lim_{n\to\infty} Tx_n = Tx$. This implies that $\lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0$. Then T has a fixed point. □

Example 3.3. Let $X = [0, 6)$ and $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|^2$. Then d is a b-metric on X with $s = 2$. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{\sqrt{2}}{6}x, & \text{if } x \in [0, 1]\ ; \\ \frac{1}{2}x, & \text{if } x \in (1, 6), \end{cases}$$

and define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1]\ ; \\ 0, & \text{if otherwise.} \end{cases}$$

Define $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \frac{4}{9}t$. For all $x, y \in X$ and $\alpha(x, y) \geq 1$, we have $x, y \in [0, 1]$ and then

$$\begin{aligned}
\psi(s^3 d(Tx, Ty)) &= \frac{s^3 d(Tx, Ty)}{2} \\
&= \frac{2^3 |\frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y|^2}{2} \\
&= 4\Big|\frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y\Big|^2 \\
&= 4 \cdot \frac{2}{36}|x - y|^2 \\
&= \frac{4}{9}\frac{|x - y|^2}{2}
\end{aligned}$$

$$\begin{aligned} &= \frac{4}{9}\frac{d(x,y)}{2} \qquad (3.15)\\ &= \frac{4}{9}\psi(d(x,y))\\ &= \varphi(\psi(d(x,y))) \le \varphi(\psi(M_b(x,y))). \end{aligned}$$

Then T is a modified (α-ψ-φ-θ)-rational contractive mapping. We next show that (X,d) is an α-complete b-metric. If $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0,1]$. Now, since $([0,1],d)$ is a complete b-metric space, then the sequence $\{x_n\}$ converges in $[0,1]$. We will show that T is α-continuous. If $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, then $x_n \in [0,1]$ for all $n \in \mathbb{N}$ and so

$$d(Tx_n,Tx) = |\frac{\sqrt{2}}{6}x_n - \frac{\sqrt{2}}{6}x|^2 = \frac{1}{18}|x_n - x|^2 = \frac{1}{18}d(x_n,x) \to 0 \text{ as } n \to \infty.$$

Let $\alpha(x,Tx) \ge 1$. Thus $x \in [0,1]$ and $Tx \in [0,1]$ and so $T^2x = T(Tx) \in [0,1]$. Then $\alpha(Tx,T^2x) \ge 1$. Thus T is α-orbital admissible. Let $\alpha(x,y) \ge 1$ and $\alpha(y,Ty) \ge 1$. We have $x,y,Ty \in [0,1]$. This implies that $\alpha(x,Ty) \ge 1$. Hence T is triangular α-orbital admissible. It is clear that condition(ii) of Theorem 3.2 is satisfied with $x_0 = 0$ since $\alpha(x_0,Tx_0) = \alpha(0,T(0)) = \alpha(0,0) = 1$. Thus all assumptions of Theorem 3.2 are satisfied and so T has a fixed point which is $x = 0$.

We next replace the α-continuity of the mapping T by some appropriate conditions.

Theorem 3.4. *Let (X,d) be an α-complete b-metric space and $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a modified (α-ψ-φ-θ)-rational contractive mapping. Assume that the following conditions hold:*

- *(i) T is triangular α-orbital admissible;*
- *(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$;*
- *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \ge 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point.

Proof. As in Theorem 3.2, we can construct the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. From condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)},x) \ge 1 \text{ for all } k \in \mathbb{N}. \qquad (3.16)$$

We claim that x is a fixed point of T. Assume that $d(x,Tx)>0$. By triangle inequality, we obtain that

$$\begin{aligned} d(x,Tx) &\leq sd(x,x_{n(k)+1})+sd(x_{n(k)+1},Tx)\\ &= sd(x,x_{n(k)+1})+sd(Tx_{n(k)},Tx). \end{aligned}$$

Taking limit $k\to\infty$ in above inequality, we have

$$d(x,Tx)\leq \lim_{k\to\infty} sd(Tx_{n(k)},Tx). \tag{3.17}$$

Since T is a modified (α-ψ-φ-θ)-rational contractive mapping, using (3.16) and (3.17), we have

$$\begin{aligned} \psi(s^2d(x,Tx)) &\leq \lim_{k\to\infty} \psi(s^3d(Tx_{n(k)},Tx))\\ &\leq \lim_{k\to\infty}[\varphi(\psi(M_b(x_{n(k)},x)))+L\theta(N_b(x_{n(k)},x))]\\ &\leq \varphi(\psi(\lim_{k\to\infty} M_b(x_{n(k)},x)))+L\theta(\lim_{k\to\infty} N_b(x_{n(k)},x)), \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} M_b(x_{n(k)},x) &= \max\{d(x_{n(k)},x),\frac{d(x_{n(k)},Tx_{n(k)})}{1+d(x_{n(k)},Tx_{n(k)})},\frac{d(x,Tx)}{1+d(x,Tx)},\\ &\quad \frac{d(x_{n(k)},Tx)+d(x,Tx_{n(k)})}{2s}\}\\ &= \max\{d(x_{n(k)},x),\frac{d(x_{n(k)},x_{n(k)+1})}{1+d(x_{n(k)},x_{n(k)+1})},\frac{d(x,Tx)}{1+d(x,Tx)},\\ &\quad \frac{d(x_{n(k)},Tx)+d(x,x_{n(k)+1})}{2s}\}\\ &\leq \max\{d(x_{n(k)},x),d(x_{n(k)},x_{n(k)+1}),d(x,Tx)\\ &\quad \frac{d(x_{n(k)},Tx)+d(x,x_{n(k)+1})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} N_b(x_{n(k)},x) &= \min\{d(x_{n(k)},Tx_{n(k)}),d(x_{n(k)},Tx),d(x,Tx_{n(k)})\}\\ &= \min d(x_{n(k)},x_{n(k)+1}),d(x_{n(k)},Tx),d(x,x_{n(k)+1})\}. \end{aligned}$$

Taking limit as $k\to\infty$, we obtain that

$$\lim_{k\to\infty} M_b(x_{n(k)},x)\leq \max\{d(x,Tx),\frac{d(x,Tx}{2}\}=d(x,Tx)$$

and

$$\lim_{k\to\infty} N_b(x_{n(k)},x)=0.$$

By (3.18), we have

$$\begin{aligned} \psi(s^2 d(x,Tx)) &\leq \varphi(\psi(\lim_{k\to\infty} M_b(x_{n(k)},x))) + L\theta(\lim_{k\to\infty} N_b(x_{n(k)},x)) \\ &\leq \varphi(\psi(d(x,Tx))) \\ &< \psi(d(x,Tx)), \end{aligned}$$

which is a contradiction because $s \geq 1$. Then $d(x,Tx) = 0$ and hence x is a fixed point of T. □

For the uniqueness of a fixed point of a modified (α-ψ-φ-θ)-rational contractive mapping, we investigate some conditions introduced in [5].

Theorem 3.5. *Suppose that all hypotheses of Theorem 3.2 (respectively Theorem 3.4) hold. Assume that either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.*

Proof. Assume that w and z are fixed points of T with $w \neq z$. By assumption, we have

$$\alpha(w,z) \geq 1 \text{ or } \alpha(z,w) \geq 1.$$

Suppose that $\alpha(w,z) \geq 1$. Since T is a modified (α-ψ-φ-θ)-rational contractive mapping, we have

$$\begin{aligned} \psi(s^3(d(w,z))) &= \psi(s^3(d(Tw,Tz))) \\ &\leq \varphi(\psi(M_b(w,z))) + L\theta(N_b(w,z))), \end{aligned}$$

where

$$\begin{aligned} M_b(w,z) &= \max\{d(w,z), \frac{d(w,Tw)}{1+d(w,Tw)}, \frac{d(z,Tz)}{1+d(z,Tz)}, \frac{d(w,Tz)+d(z,Tw)}{2s}\} \\ &= \max\{d(w,z), \frac{d(w,w)}{1+d(w,w)}, \frac{d(z,z)}{1+d(z,z)}, \frac{d(w,z)+d(z,w)}{2s}\} \\ &= d(w,z) \end{aligned}$$

and

$$N_b(w,z) = \min\{d(w,Tw), d(w,Tz), d(z,Tw)\} = 0.$$

Then

$$\begin{aligned} \psi(s^3(d(w,z))) &= \varphi(\psi(d(w,z))) \\ &< \psi(d(w,z) \end{aligned}$$

which is a contradiction because $s \geq 1$. Thus $w = z$. Similarly, if $\alpha(z,w) \geq 1$, then we can prove that $w = z$. Hence T has a unique fixed point. □

In Theorem 3.5, if we take $\psi(t) = t$ for all $t \in [0,\infty)$, then we immediately obtain the following result.

Corollary 3.6. *Let (X, d) be an α-complete b-metric space where $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Assume that there exists $L \geq 0$ such that for all $x, y \in X$,*

$$\alpha(x, y) \geq 1 \text{ implies } s^3 d(Tx, Ty) \leq \varphi(M_b(x, y)) + L\theta(N_b(x, y)), \tag{3.19}$$

where

$$M_b(x, y) = \max\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2s}\},$$

$$N_b(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

and $\varphi, \theta : [0, \infty) \to [0, \infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$ and φ is nondecreasing. Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$. .

Then T has a fixed point. Moreover, either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

In Corollary 3.6, if $\varphi(t) = t - \varphi'(t)$ for all $t \in [0, \infty)$ where $\varphi' : [0, \infty) \to [0, \infty)$ is continuous such tha $\varphi'(t) < t$ for each $t > 0$ and φ' is nonincreasing and $L = 0$, then we obtain the following corollary.

Corollary 3.7. *Let (X, d) be an α-complete b-metric space where $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ is a mapping such that for all $x, y \in X$,*

$$\alpha(x, y) \geq 1 \text{ implies } s^3 d(Tx, Ty) \leq M_b(x, y) - \varphi'(M_b(x, y)), \tag{3.20}$$

where

$$M_b(x, y) = \max\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2s}\},$$

and $\varphi' : [0, \infty) \to [0, \infty)$ is continuous such that $\varphi'(0) = 0$, $\varphi'(t) < t$ for each $t > 0$ and φ' is nonincreasing Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point. Moreover, either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

3.2 The unique of common fixed point theorems

In this section, we introduce the concept of modified (α-ψ-φ-θ)-rational contractive mappings with respect to g and prove the the existence of unique common fixed point theorems in α-complete b-metric spaces.

Definition 3.8. Let (X, d) be a b-metric space, $\alpha : X \times X \to [0, \infty)$, and $T, g : X \to X$. We say that $T : X \to X$ is a *modified (α-ψ-φ-θ)-rational contractive mapping with respect to* g if there exists $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \psi(s^3 d(Tx, Ty)) \leq \varphi(\psi(M_b(x, y))) + L\theta(N_b(x, y)), \tag{3.21}$$

where

$$M_b(x, y) = \max\{d(gx, gy), \frac{d(gx, Tx)}{1 + d(gx, Tx)}, \frac{d(gy, Ty)}{1 + d(gy, Ty)}, \frac{d(gx, Ty) + d(gy, Tx)}{2s}\},$$

$$N_b(x, y) = \min\{d(gx, Tx), d(gx, Ty), d(gy, Tx)\}$$

and $\psi, \varphi, \theta : [0, \infty) \to [0, \infty)$ are continuous functions with $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$, φ is nondecreasing, $\theta(0) = 0$, $\psi(t) = 0$ if and only if $t = 0$ and ψ is increasing.

Definition 3.9. Let (X, d) be a b-metric space and $\alpha : X \times X \to [0, +\infty)$ and $T, g : X \to X$. Then T is said to be *α-continuous with respect to g*, if for each sequence $\{gx_n\}$ with $gx_n \to gx$ as $n \to \infty$, $\alpha(gx_n, gx_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $Tx_n \to Tx$ as $n \to \infty$.

Theorem 3.10. *Let (X, d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0, \infty)$ and T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Assume that the following conditions hold:*

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g.

Then T and g have a coincidence point.

Proof. Let $x_0 \in X$ be such that $\alpha(gx_0, Tx_0) \geq 1$. Since $TX \subseteq gX$, we can construct a sequence $\{gx_n\}$ such that

$$gx_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

By using Lemma 2.13, we have

$$\alpha(gx_n, gx_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.22}$$

By the analogous proof as in Theorem 3.2, we can prove that $\{gx_n\}$ is a Cauchy sequence. Since $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and X is an α-complete b-metric space, we have $\{gx_n\}$ converges to $z \in gX$. Thus there exists $x \in X$ such that $\lim_{n\to\infty} gx_n = gx$. Since T is α-continuous with respect to g, so $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_{n+1} = gx$. Then x is a coincidence point of T and g. $\square$

We replace the α-continuity of the mapping T with respect to g by some appropriate conditions.

Theorem 3.11. *Let (X, d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0, \infty)$ and T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Assume that the following conditions hold:*

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point.

Proof. As in the proof of Theorem 3.10, we can construct the sequence $\{gx_n\}$ with $Tx_n = gx_{n+1}$ for all $n \in \mathbb{N}$, $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} gx_n = gx$. By (iii), there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$, for all $k \in \mathbb{N}$. By the analogous proof as in Theorem 3.4, we obtain that T and g have a coincidence point. □

For the uniqueness of a common fixed point, we add some appropriate conditions to the hypotheses.

Theorem 3.12. *Suppose that all hypotheses of Theorem 3.10 (respectively Theorem 3.11) hold. Assume that the following conditions hold:*

(i) the pair $\{T, g\}$ is weakly compatible;

(ii) either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Proof. Assume that $Tu = gu$ and $Tv = gv$. We will show that $gu = gv$. Suppose that $gu \neq gv$. Therefore $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$. Suppose that $\alpha(u, v) \geq 1$. It follows that

$$\psi(s^3 d(gu, gv)) = \psi(s^3 d(Tu, Tv)) \leq \varphi(\psi(M_b(u, v))) + L\theta(N_b(u, v)),$$

where

$$\begin{aligned} M_b(u,v) &= \max\{d(gu,gv), \frac{d(gu,Tu)}{1+d(gu,Tu)}, \frac{d(v,Tv)}{1+d(v,Tv)}, \frac{d(gu,Tv)+d(gv,Tu)}{2s}\} \\ &= \max\{d(gu,gv), \frac{d(gu,gu)}{1+d(gu,gu)}, \frac{d(gv,gv)}{1+d(gv,gv)}, \frac{d(gu,gv)+d(gv,gu)}{2s}\} \\ &= d(gu,gv) \end{aligned}$$

and

$$N_b(u,v) = \min\{d(gu,Tu), d(gu,Tv), d(gv,Tu)\} = 0.$$

This implies that

$$\begin{aligned} \psi(s^3(d(gu,gv))) &\leq \varphi(\psi(d(gu,gv))) \\ &< \psi(d(gu,gv) \end{aligned}$$

which is a contradiction because $s \geq 1$. Thus $gu = gv$. Similarly, if $\alpha(v,u) \geq 1$, we can prove that $gu = gv$. This implies that T and g have a unique point of coincidence. Since the pair $\{T,g\}$ is weakly compatible and by Theorem 2.16, we can conclude that T and g have a unique common fixed point. □

Corollary 3.13. *Let (X,d) be an α-complete b-metric space with respect to g and $T,g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $L \geq 0$ such that for all $x,y \in X$,*

$$\alpha(x,y) \geq 1 \textit{ implies } s^3 d(Tx,Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \tag{3.23}$$

where

$$M_b(x,y) = \max\{d(gx,gy), \frac{d(gx,Tx)}{1+d(gx,Tx)}, \frac{d(gy,Ty)}{1+d(gy,Ty)}, \frac{d(gx,Ty)+d(gy,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{\frac{d(gx,Tx)}{1+d(g,Tx)}, \frac{d(gx,Ty)}{1+d(gx,Ty)}, \frac{d(gy,Tx)}{1+d(gy,Tx)}\}$$

and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$. Assume that the following conditions hold:

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g or if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point. Moreover, assume that the following conditions hold:

(iv) the pair $\{T,g\}$ is weakly compatible;

(v) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

3.3 Applications to integral equations

In this section, we prove the existence of a solution of a nonlinear quadratic integral equation taken from Allahari et al. [12].

Let $C(I)$ be the set of all continuous functions defined on $I = [0,1]$ and $\rho : C(I) \times C(I) \to \mathbb{R}$ defined by

$$\rho(x,y) = \sup_{t\in I} |x(t) - y(t)| \text{ for } x, y \in C(I).$$

Let $p \geq 1$. We define $d : C(I) \times C(I) \to \mathbb{R}$ defined by

$$d(x,y) = (\rho(x,y))^p = (\sup_{t\in I} |x(t) - y(t)|)^p = \sup_{t\in I} |x(t) - y(t)|^p \text{ for all } x, y \in C(I).$$

It is well known that (X, d) is a complete b-metric space with $s = 2^{p-1}$ (see [13]). Let Γ be the set of functions $\gamma : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

(i) γ is nondecreasing and $(\gamma(t))^p \leq \gamma(t^p)$ for all $p \geq 1$;

(ii) There exists $\varphi : [0,+\infty) \to [0,+\infty)$ which is nonincreasing and continuous, $\varphi(t) < t$ for all $t > 0$ such that $\gamma(t) = t - \varphi(t)$ for all $t \in [0,+\infty)$.

Consider the nonlinear quadratic equation as follows:

$$x(t) = h(t) + \lambda \int_0^1 k(t,s) f(s, x(s)) ds, \; t \in I, \; \lambda \geq 0. \tag{3.24}$$

Suppose that the following conditions hold:

(A1) $h : I \to \mathbb{R}$ is continuous;

(A2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t,x) \geq 0$ and there exist $L \geq 0$, $\gamma \in \Gamma$ and a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$, for all $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$,

$$|f(t,a) - f(t,b)| \leq L\gamma(|a-b|);$$

(A3) $k : I \times I \to \mathbb{R}$ is continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $k(t,s) \geq 0$ and $\int_0^1 k(t,s)ds \leq K$;

(A4) $\lambda^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(A5) there exists $x_0 \in C(I)$ such that for all $t \in I$,

$$\xi(x_0(t), h(t) + \lambda \int_0^1 k(t,s) f(s, x_0(s)) ds) \geq 0;$$

(A6) for all $t \in I$ and for all $x, y, z \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), z(t)) \geq 0 \text{ imply } \xi(x(t), z(t)) \geq 0;$$

(A7) for all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi(h(t)+\lambda \int_0^1 k(t,s)f(s,x(s))ds, h(t)+\lambda \int_0^1 k(t,s)f(s,y(s))ds) \geq 0;$$

(A8) if $\{x_n\}$ is a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\xi(x_n(t), x_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in I$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in I$.

Theorem 3.14. *Under assumptions (A1)-(A8), the integral equation (3.24) has a solution in $C(I)$.*

Proof. Let $T : C(I) \to C(I)$ be defined by

$$T(x)(t) = h(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds \text{ for } t \in I.$$

Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. Therefore

$$\begin{aligned} |T(x)(t) - T(y)(t)| &= |h(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds - h(t) - \lambda \int_0^1 k(t,s)f(s,y(s))ds| \\ &\leq \lambda \int_0^1 k(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leq \lambda \int_0^1 k(t,s)L\gamma(|x(s) - y(s)|)ds. \end{aligned}$$

Since γ is nondecreasing, we obtain that

$$\gamma(|x(s) - y(s)|) \leq \gamma(\sup_{s \in I} |x(s) - y(s)|) = \gamma(\rho(x,y)).$$

This implies that

$$|T(x)(t) - T(y)(t)| \leq \lambda K L \gamma(\rho(x,y)).$$

Therefore

$$\begin{aligned} d(Tx, Ty) &= \sup_{t \in I} |T(x)(t) - T(y)(t)|^p \\ &\leq [\lambda K L \gamma(\rho(x,y))]^p \\ &\leq \lambda^p K^p L^p \gamma(d(x,y)) \\ &\leq \lambda^p K^p L^p \gamma(M(x,y)) \\ &\leq \lambda^p K^p L^p [M(x,y) - \varphi(M(x,y))] \\ &\leq \frac{1}{2^{3p-3}}[M(x,y) - \varphi(M(x,y))], \end{aligned}$$

for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. We next define $\alpha : C(I) \times C(I) \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \geq 0,\ t \in I \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in C(I)$ be such that $\alpha(x, y) \geq 1$. It follows that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. This yields

$$s^3 d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y).$$

This implies that T satisfies the contractive condition in Corollary 3.7. Using (A7), for each $x \in C(I)$ such that $\alpha(x, Tx) \geq 1$ we obtain that $\xi(Tx(t), T^2x(t)) \geq 0$. This implies that $\alpha(Tx, T^2x) \geq 1$. Let $x, y \in C(I)$ be such that $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$. Thus

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), Ty(t)) \geq 0 \text{ for all } t \in I.$$

By applying (A6), we obtain that $\xi(x(t), Ty(t)) \geq 0$ and so $\alpha(x, Ty) \geq 1$. It follows that T is triangular α-orbital admissible. Using (A5), there exists $x_0 \in C(I)$ such that $\alpha(x_0, Tx_0) \geq 1$. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. By (A8), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$. This implies that $\alpha(x_{n(k)}, x) \geq 1$. Therefore all assumptions in Corollary 3.7 are satisfied. Hence T has a fixed point in $C(I)$ that is a solution of the integral equation (3.24). □

Corollary 3.15. *Assume that the following conditions hold:*

(i) $h : I \to \mathbb{R}$ *is a continuous;*

(ii) $f : I \times \mathbb{R} \to [0, \infty)$ *is continuous and nondecreasing and* $f(t, s) \geq 0$.

(iii) there exist $L \geq 0$ *and* $\gamma \in \Gamma$ *such that for all* $t \in I$, *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$, *we have*

$$|f(t, a) - f(t, b)| \leq L\gamma(|a - b|);$$

(iv) $k : I \times I \to \mathbb{R}$ *is continuous at* $t \in I$ *for every* $s \in I$ *and measurable at* $s \in I$ *for all* $t \in I$ *such that* $k(t, s) \geq 0$ *and* $\int_0^1 k(t, s)ds \leq K$;

(v) $\gamma^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(vi) there exists $x_0 \in C([0, 1])$ *such that for all* $t \in I$, *we have*

$$x_0(t) \leq h(t) + \lambda \int_0^1 k(t, s)f(s, x_1(s))ds.$$

Then (3.24) has a solution in $C(I)$.

Proof. Define a mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\xi(a, b) = b - a \quad \text{for all } a, b \in \mathbb{R}.$$

By the analogous proof as in Theorem 3.14, we obtain that (3.24) has a solution in $C(I)$. □

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