Thai Journal of Mathematics
Volume 14 (2016) Number 1 : 191–201
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Monoid of Cohypersubstitutions of Type $\tau = (n)$[1]

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Abstract : A mapping σ which assigns to every n_i-ary cooperation symbol f_i an n_i-ary coterm of type $\tau = (n_i)_{i\in I}$ is said to be a cohypersubstitution of type τ. The concepts of cohypersubstitutions were introduced in [1]. Every cohypersubstition σ of type τ induces a mapping $\hat{\sigma}$ on the set of all coterms of type τ. The set of all cohypersubstitutions of type τ under the binary operation $\hat{\circ}$ which is defined by $\sigma_1 \hat{\circ} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in \mathit{Cohyp}(\tau)$ forms a monoid which is called the monoid of cohypersubstitution of type τ. In this research, we characterize all idempotent and regular elements of $\mathit{Cohyp}(n)$ and characterize some Green's relations L and R on $\mathit{Cohyp}(n)$.

Keywords : cohypersubstitutions; coterms; superpositions; idempotent elements; regular elements.
2010 Mathematics Subject Classification : 16W30; 20M12.

1 Introduction

Let A be a non-empty set and n be a positive integer. The n-th copower $A^{\sqcup n}$ of A is the union of n disjoint copies of A; formally, we define $A^{\sqcup n}$ as the cartesian product $A^{\sqcup n} := \underline{n} \times A$, where $\underline{n} := \{1, \ldots, n\}$. An element (i, a) in this copower corresponds to the element a in the i-th copy of A, for $1 \leq i \leq n$.

[1]This research was supported by the Faculty of Science of Mahasarakham University Thailand

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A co-operation on A is a mapping $f^A : A \to A^{\sqcup n}$ for some $n \geq 1$; the natural number n is called the arity of the co-operation f^A. We also need to recall that any n-ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A : A \to \underline{n}$ and $f_2^A : A \to A$; the first mapping gives the labelling used by f^A in mapping elements to copies of A, and the second mapping tells us what element of A any element is mapped to.
We shall denote by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$ the set of all $n - ary$ co-operations defined on A, and by $cO_A := \cup_{n\geq 1} cO_A^{(n)}$ the set of all finitary co-operations defined on A. An indexed coalgebra is a pair $(A; (f_i^A)_{i\in I})$, where f_i^A is an n_i-ary cooperation defined on A, and $\tau = (n_i)_{i\in I}$ for $n_i \geq 1$ is called the type of the coalgebra. Coalgebras were studied by Drbohlav [2]. In [3], the following superposition of cooperations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A, \ldots, g_{n-1}^A \in cO_A^{(k)}$, then the k-ary co-operation $f^A[g_0^A, \ldots, g_{n-1}^A] : A \to A^{\sqcup k}$ is defined by

$$a \mapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)), (g^A_{f_1^A(a)})_2(f_2^A(a)))$$

for all $a \in A$. The co-operation $f^A[g_0^A, \ldots, g_{n-1}^A]$ is called the *superposition* of f^A and $g_0^A, \ldots, g_{n-1}^A$. It will also be denoted by $comp_k^n(f^A, g_0^A, \ldots, g_{n-1}^A)$.

The *injection co-operations* $\iota_i^{n,A} : A \to A^{\sqcup n}$ are special cooperations which are defined for each $0 \leq i \leq n-1$ by $\iota_i^{n,A} : A \to A^{\sqcup n}$ with $a \mapsto (i, a)$ for all $a \in A$. Then we get a multi-based algebra

$$((cO_A^{(n)})_{n\geq 1}, (comp_k^n)_{k,n\geq 1}, (\iota_i^{n,A})_{0\leq i\leq n-1}),$$

called the *clone of co-operations* on A. In [3] it is mentioned that this algebra is a clone, i.e. it satisfies the three clone axioms $(C1), (C2), (C3)$. In [4], K.Denecke and K.Saengsura gave a full proof of this fact. In [4], the following coterms of type $\tau = (n_i)_{i\in I}$ were introduced. Let $(f_i)_{i\in I}$ be an indexed set of co-operation symbols such that for each $i \in I$, f_i has arity n_i. Let $\bigcup\{e_j^n \mid n \geq 1,\ n \in \mathbb{N},\ 0 \leq j \leq n-1\}$ be a set of symbols which is disjoint from the set $\{f_i \mid i \in I\}$ such that for each $0 \leq j \leq n-1$, e_j^n has arity n. Then coterms of type τ are defined as follows:

(i) For every $i \in I$ the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $n \geq 1$ and $0 \leq j \leq n-1$.

Let $cT_\tau^{(n)}$ be the set of all n-ary coterms of type τ and let $cT_\tau := \bigcup_{n\geq 1} cT_\tau^{(n)}$ be the set of all (finitary) coterms of type τ.

The superposition of coterms was introduced in [1] as follows: The operation $S_m^n : cT_\tau^{(n)} \times (cT_\tau^{(m)})^n \to cT_\tau^{(m)}$ is defined by induction on the complexity of coterm definition, as follows:

(i) $S^n_m(e^n_i, t_0, \ldots, t_{n-1}) := t_i$ for $0 \le i \le n-1$.

(ii) $S^{n_i}_{n_i}(f_i, e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}) := f_i$ for an n_i-ary co-operation symbol f_i .

(iii) $S^{n_j}_m(g_j, t_1, \ldots, t_{n_j}) := g_j[t_1, \ldots, t_{n_j}]$ if g_j is an n_j-ary co-operation symbol.

(iv) $S^n_m(f_i[s_1, \ldots, s_{n_i}], t_1, \ldots, t_n) := f_i[S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n)]$ where f_i is an n_i-ary co-operation symbol, $s_1, \ldots, s_{n_i}$ are n-ary coterms of type τ and $t_1, \ldots, t_n$ are m-ary coterms of type τ.

These operations give us a heterogeneous algebra

$$c\mathcal{T}_\tau := ((cT^{(n)}_\tau)_{n\ge1}, (S^n_m)_{m,n\ge1}, (e^n_j)_{1\le j\le n}).$$

We shall show that it is a clone, i.e., that it satisfies the clone axioms (C1),(C2),(C3).

Theorem 1.1. (Denecke and Saengsura [1, Proposition 2.3]) *The heterogeneous algebra $c\mathcal{T}_\tau$ satisfies the following identities:*

(C1) $\hat{S}^p_m(z, \hat{S}^n_m(y_1, x_1, \ldots, x_n), \ldots, \hat{S}^n_m(y_p, x_1, \ldots, x_n)) \approx \hat{S}^n_m(\hat{S}^p_n(z, y_1, \ldots, y_p), x_1, \ldots, x_n), \quad (m, n, p \in \mathbb{N}^+)$,

(C2) $\hat{S}^n_m(e^n_i, x_1, \ldots, x_n) \approx x_i \quad (m \in \mathbb{N}^+,\ 1 \le i \le n)$,

(C3) $\hat{S}^n_n(y, e^n_1, \ldots, e^n_n) \approx y, \quad (n \in \mathbb{N}^+)$.

(Here $\hat{S}^n_m, e^n_i$ are operation symbols corresponding to the clone type.)

A *cohypersubstitution* of type τ was introduced in [1] as a mapping $\sigma : \{f_i \mid i \in I\} \to CT_\tau$ from the set of all cooperation symbols to the set of all coterms which preserves the arities. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}[e^n_j] := e^n_j$ for every $n \ge 1$ and $0 \le j \le n-1$,

(ii) $\hat{\sigma}[f_i] := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}[f_i[t_1, \ldots, t_{n_i}]] := S^{n_i}_n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for $t_1, \ldots, t_{n_i} \in cT^{(n)}_\tau$.

Let $Cohyp(\tau)$ be the set of all cohypersubstitutions of type τ. On the set $Cohyp(\tau)$ of all cohypersubstitutions of type τ we may define a binary operation $\hat{\circ}$ by $\sigma_1 \hat{\circ} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the cohypersubstitution defined by $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then we have that $(Cohyp(\tau); \hat{\circ}, \sigma_{id})$ forms a monoid which is called the monoid of cohypersubstitution of type τ where the cohypersubstitution σ_{id} satisfies the equation $\hat{\sigma}_{id}[t] = t$ for all $t \in cT_\tau$ (see e.g. [1]).

2 Main results

2.1 Idempotent and regular of monoid cohypersubstitutions of type $\tau = (n)$

In 2012, M. Kapeedaeng and K. Saengsura were studied the idempotent and the regular of cohypersubstitutions of type $\tau = (2)$ and D. Boonchari and K. Saengsura were studied the idempotent and regular of cohypersubstitutions of type $\tau = (3)$ (see [5],[6]). In this Section, we characterize all idempotent and regular cohypersubstitutions of type $\tau = (n)$, where n is the positive integer. Let S be a semigroup, an element a of S is called idempotent if $aa = a$, and called regular if there exists $x \in S$ such that $axa = a$. We denote by $E(S)$ and $R(S)$ the set of all idempotent elements and the set of all regular elements of S, respectively (see [7]). For any $\sigma \in Cohyp(\tau)$ and $\tau = (n)$, if $\sigma(f) = t$, we denote σ by σ_t. For any positive integer n we call the symbol e_j^n the injection symbol, for all $0 \leq j \leq n-1$ and for each coterm t, let $E(t)$ be the set of all injection symbols which occur in t.

Lemma 2.1. *Let $t, s_1, s_2, \ldots, s_n \in CT_\tau$, where $\tau = (n)$. If $E(t) = \{e_{j-1}^n \mid \forall j \in J$ where $J \neq \emptyset$ and $J \subseteq \{1, \ 2, \ldots, n\}\}$ and $s_j = e_{j-1}^n$ for all $j \in J$, then $t[s_1, s_2, \ldots, s_n] = t$*

Proof. We give a proof by induction on the complexity of the coterm t.
If $t = e_{j-1}^n$, for some $j \in J$, then

$$\begin{aligned} e_{j-1}^n[s_1, s_2, \ldots, s_n] &= s_j \\ &= e_{j-1}^n. \end{aligned}$$

Assume that $t = f[t_1, t_2, \ldots, t_n]$ and that $t_i[s_1, s_2, \ldots, s_n] = t_i$ for all $i = 1, 2, \ldots, n$. Then we get that

$$\begin{aligned} t[s_1, s_2, \ldots, s_n] &= (f[t_1, t_2, \ldots, t_n])[s_1, s_2, \ldots, s_n] \\ &= f[t_1[s_1, s_2, \ldots, s_n], t_2[s_1, s_2, \ldots, s_n], \ldots, t_n[s_1, s_2, \ldots, s_n]] \\ &= f[t_1, t_2, \ldots, t_n] \\ &= t. \end{aligned}$$

□

The next result is a condition for an element of $Cohyp(n)$ to be idempotent.

Theorem 2.2. *If $\sigma_t \in Cohyp(n)$, then σ_t is an idempotent if and only if $\hat{\sigma_t}(t) = t$.*

Proof. Assume that σ_t is an idempotent.

$$\text{Then} \quad \begin{aligned} \hat{\sigma_t}(t) &= \hat{\sigma_t}(\sigma_t(f)) \\ &= \sigma_t \hat{\circ} \sigma_t(f) \\ &= \sigma_t(f) \\ &= t. \end{aligned}$$

Conversely, assume that $\hat{\sigma_t}(t) = t$.

$$\text{Then} \quad \begin{aligned} \sigma_t \hat{\circ} \sigma_t(f) &= \hat{\sigma_t}(\sigma_t(f)) \\ &= \hat{\sigma_t}(t) \\ &= t \\ &= \sigma_t(f). \end{aligned}$$

Therefore, σ_t is an idempotent. □

Corollary 2.3. *For every $i \in \{0,1,2,\ldots,n-1\}$, $\sigma_{e_i^3}$ is an idempotent and σ_{id} is an idempotent.*

Proof. Since $\hat{\sigma}_t(e_i^n) = e_i^n$ for all $i \in \{0,1,2,\ldots,n-1\}$ and $t \in CT_\tau^{(n)}$, then $\sigma_{e_i^n}(e_i^n) = e_i^n$ for all $i \in \{0,1,2,\ldots,n-1\}$. By Theorem 2.1, $\sigma_{e_i^n}$ is an idempotent for all $i \in \{0,1,2,\ldots,n-1\}$. Also σ_{id} is an idempotent because it is the identity cohypersubstitution of type $\tau = (n)$. □

Theorem 2.4. *If $t = f[t_1,\ldots,t_n]$ and $E(t) = \{e_{j-1}^n\}$ for some $j \in \{1,\ldots,n\}$, then σ_t is an idempotent if and only if $t_j = e_{j-1}^n$.*

Proof. Assume that σ_t is an idempotent.

$$\begin{aligned}\text{Then}\quad f[t_1,\ldots,t_j,\ldots,t_n] &= \sigma_t(f)\\ &= \sigma_t \hat{\circ} \sigma_t(f)\\ &= \hat{\sigma}_t(\sigma_t(f))\\ &= \sigma_t(f)[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)]\\ &= f[t_1,\ldots,t_j,\ldots,t_n][\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)]\\ &= f[t_1[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)],\ldots,\\ &\qquad t_j[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)],\ldots,\\ &\qquad t_n[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)]].\end{aligned}$$

Therefore, $t_j = t_j[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)]$.
Suppose that $t_j \neq e_{j-1}^n$. This implies that the number of cooperation symbols f which occur in coterm t_j is greater than or equal to 1 and hence $\hat{\sigma}_t(t_j) \neq e_{j-1}^n$. It follows that $t_j \neq t_j[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)]$. Then

$$\begin{aligned}f[t_1,\ldots,t_j,\ldots,t_n] \neq\ & f[t_1[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_n),\ldots,\hat{\sigma}_t(t_n)],\ldots,\\ & t_j[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)],\ldots,\\ & t_n[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_j),\ldots,\hat{\sigma}_t(t_n)]],\end{aligned}$$

which is a contradiction.
Conversely, let $t_j = e_{j-1}^n$.

$$\begin{aligned}\text{Then}\quad \hat{\sigma}_t(t) &= \hat{\sigma}_t(f[t_1,\ldots,t_{j-1},t_j,t_{j+1},\ldots,t_n])\\ &= \hat{\sigma}_t(f[t_1,\ldots,t_{j-1},e_{j-1}^n,t_{j+1},\ldots,t_n])\\ &= \sigma_t(f)[\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_{j-1}),\hat{\sigma}_t(e_{j-1}^n),\hat{\sigma}_t(t_{j+1}),\ldots,\hat{\sigma}_t(t_n)]\\ &= (f[t_1,\ldots,t_{j-1},e_{j-1}^n,t_{j+1},\ldots,t_n])\\ &\qquad [\hat{\sigma}_t(t_1),\ldots,\hat{\sigma}_t(t_{j-1}),e_{j-1}^n,\hat{\sigma}_t(t_{j+1}),\ldots,\hat{\sigma}_t(t_n)]\\ &= f[t_1,\ldots,t_{j-1},e_{j-1}^n,t_{j+1},\ldots,t_n] \qquad \text{(by Lemma 2.1)}\\ &= t.\end{aligned}$$

□

Now we give a characterization of a cohypersubstitution σ_t such that $E(t) > 1$, first of all we need the following lemma:

Lemma 2.5. *Let $t \in cT^{(n)}$. If $E(t) = \{e_{j-1}^n \mid \forall j \in J$ where $J \subseteq \{1,\ldots,n\}$ and $|J| > 1\}$ and $s_1,\ldots,s_n \in cT^{(n)}$ such that $s_j \neq e_{j-1}^n$ for some $j \in J$, then $t[s_1,\ldots,s_n] \neq t$.*

Proof. We give a proof by induction on the complexity of the coterm t. If $t = e_{j-1}^n$, then $e_{j-1}^n[s_1,\ldots,s_n] = s_j \neq e_{j-1}^n$.

Assume that $t = f[t_1, \ldots, t_n]$ and $t_i[s_1, \ldots, s_n] \neq t_i$ for all t_i where $e^n_{j-1} \in E(t_i)$. Then

$$\begin{aligned} t[s_1, \ldots, s_n] &= (f[t_1, \ldots, t_n])[s_1, \ldots, s_n] \\ &= f[t_1[s_1, \ldots, s_n], \ldots, t_n[s_1, \ldots, s_n]] \\ &\neq f[t_1, \ldots, t_n]. \end{aligned}$$

□

We obtain the following result:

Theorem 2.6. *Let $\sigma_t \in Cohyp(n)$ such that $E(t) = \{e^n_{j-1} \mid \forall j \in J$ where $J \subseteq \{1, \ldots, n\}$ and $|J| > 1\}$. Then σ_t is an idempotent if and only if $t_j = e^n_{j-1}$ for all $j \in J$.*

Proof. Assume that σ_t is an idempotent. Similar to the proof of Theorem 2.2, we have that

$$f[t_1, \ldots, t_n] = f[t_1[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_n)], \ldots, t_n[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_n)]].$$

Suppose that $t_j \neq e^n_{j-1}$ for some $j \in J$. Then $\hat{\sigma}_t(t_j) \neq e^n_{j-1}$. Since $e^n_{j-1} \in E(t)$, then by Lemma 2.2 there is $k \in \{1, \ldots, n\}$ such that $t_k[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_j), \ldots, \hat{\sigma}_t(t_n)] \neq t_k$ and $e^n_{j-1} \in E(t_k)$. Therefore,

$$f[t_1, \ldots, t_n] \neq f[t_1[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_n)], \ldots, t_n[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_n)]].$$

This yields a contradiction. Hence, $t_j = e^n_{j-1}$ for all $j \in J$.
Conversely, Assume that $t = f[t_1, \ldots, t_n]$ and $t_j = e^n_{j-1}$ for all $j \in J$. Then $\hat{\sigma}_t(t_j) = e^n_{j-1}$ for all $j \in J$. Since $E(t) = \{e^n_{j-1} \mid \forall j \in J$ where $J \subseteq \{1, \ldots, n\}$ and $|J| > 1\}$, then by Lemma 2.1 we get that

$$\begin{aligned} \hat{\sigma}_t(t) &= \hat{\sigma}_t(f[t_1, \ldots, t_n]) \\ &= \sigma_t(f)[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_n)] \\ &= (f[t_1, \ldots, t_n])[\hat{\sigma}_t(t_1), \ldots, \hat{\sigma}_t(t_n)] \\ &= f[t_1, \ldots, t_n]. \end{aligned}$$

□

Now, we characterize all regular elements of $Cohyp(n)$. By using the injection symbols which occur in the coterm t, we obtain the following result:

Theorem 2.7. *Let $t \in CT_{(n)}$ and $E(t) = \{e^n_{j-1} \mid \forall j \in J$ where $J \subseteq \{1, \ldots, n\}\}$. Then σ_t is a regular if and only if for each $j \in J$, $e^n_{j-1} = t_i$ for some $i \in \{1, \ldots, n\}$.*

Proof. Assume that σ_t is regular. Let $s = f[s_1, \ldots, s_n] \in cT_{(n)}$ and $\sigma_t \hat{\circ} \sigma_s \hat{\circ} \sigma_t = \sigma_t$.
Suppose that $t_i \neq e^n_{j-1}$ for all $i = 1, \ldots, n$. Then $\hat{\sigma}_s(t_i) \neq e^n_{j-1}$ for all $i = 1, \ldots, n$. Therefore,

$$\begin{aligned} \hat{\sigma}_s(t) &= \hat{\sigma}_s(f[t_1, \ldots, t_n]) \\ &= \sigma_s(f)[\hat{\sigma}_s(t_1), \ldots, \hat{\sigma}_s(t_n)] \\ &= (f[s_1, \ldots, s_n])[\hat{\sigma}_s(t_1), \ldots, \hat{\sigma}_s(t_n)] \\ &= f[s_1[\hat{\sigma}_s(t_1), \ldots, \hat{\sigma}_s(t_n)], \ldots, s_n[\hat{\sigma}_s(t_1), \ldots, \hat{\sigma}_s(t_n)]]. \end{aligned}$$

Since $\hat{\sigma}_s(t_i) \neq e^n_{j-1}$ for all $i \in \{1,\dots,n\}$, then $s_i[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)] \neq e^n_{j-1}$ for all $i \in \{1,\dots,n\}$, so $\hat{\sigma}_t(s_i[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]) \neq e^n_{j-1}$ for all $i \in \{1,\dots,n\}$.
Therefore,

$$\begin{aligned}
\hat{\sigma}_t(\hat{\sigma}_s(t)) &= \hat{\sigma}_t(f[s_1[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)],\dots,s_n[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]]) \\
&= \sigma_t(f)[\hat{\sigma}_t(s_1[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]),\dots,\hat{\sigma}_t(s_n[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)])] \\
&= (f[t_1,\dots,t_n])[\hat{\sigma}_t(s_1[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]),\dots,\hat{\sigma}_t(s_n[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)])] \\
&\neq f[t_1,\dots,t_n] \qquad \text{(by Lemma 2.2).}
\end{aligned}$$

This gives a contradiction. Hence $t_i = e^n_{j-1}$ for some $i \in \{1,\dots,n\}$. Conversely, let $t = f[t_1,\dots,t_n]$ and assume that for each $j \in J$, $e^n_{j-1} = t_i$ for some $i \in \{1,\dots,n\}$. Let $s = f[s_1,\dots,s_n]$ and for each $j \in J$, $s_j = e^n_{i-1}$ for some $i \in \{1,\dots,n\}$.
Then

$$\begin{aligned}
\hat{\sigma}_s(f[t_1,\dots,t_n]) &= \sigma_s(f)[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)] \\
&= (f[s_1,\dots,s_n])[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)] \\
&= f[s_1[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)],\dots,s_n[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)].
\end{aligned}$$

Since $t_i = e^n_{j-1}$ and $s_j = e^n_{i-1}$, then $\hat{\sigma}_s(t_i) = e^n_{j-1}$ and $s_j[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)] = e^n_{j-1}$ for all $j \in J$.
Then $\hat{\sigma}_t(s_j[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]) = e^n_{j-1}$ for all $j \in J$. Therefore,

$$\begin{aligned}
\hat{\sigma}_t(\hat{\sigma}_s(t)) &= \hat{\sigma}_t(f[s_1[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)],\dots,s_n[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]]) \\
&= (f[t_1,\dots,t_n])[s_1[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)],\dots,s_n[\hat{\sigma}_s(t_1),\dots,\hat{\sigma}_s(t_n)]] \\
&= f[t_1,\dots,t_n].
\end{aligned}$$

Hence, σ_t is regular. □

2.2 Green's relations of cohypersubstitutions of type $\tau = (n)$

In this section, we characterize Green's relations L and R on $Cohyp(n)$. First of all we define the equivalence coterms as follows: the coterms $s, t \in CT_{(n)}$ are said to be equivalence denoted by $s \equiv t$ if and only if $s = t$ or s and t are difference only the injection symbols which occurred in the coterms s and t, for instance $t = f[e^2_0, f[e^2_1, e^2_0]] \equiv s = f[e^2_1, f[e^2_0, e^2_1]]$, but $t = f[e^2_0, f[e^2_1, e^2_0]]$ is not equivalence to $r = f[e^2_0, e^2_1]$. Then we obtain the cohypersubstitutions σ_t and σ_s which are R-related as the following theorem:

Theorem 2.8. *If $t, s \in CT_{(n)}$, then $\sigma_t R \sigma_s$ if and only if the following are satisfied:*

(i) $t \equiv s$,

(ii) *there is a uniquely bijection $\varphi : E(t) \to E(s)$ and if $e^n_j \in E(t)$, then e^n_j and $\varphi(e^n_j)$ are in the same position of the coterms t and s, respectively.*

Proof. Assume that $\sigma_t R \sigma_s$. Then there are $\sigma_r, \sigma_w \in Cohyp(n)$ such that $\sigma_t = \sigma_s \hat{\circ} \sigma_r$ and $\sigma_s = \sigma_t \hat{\circ} \sigma_w$. Let $t = f[t_1,\dots,t_n]$ and $s = f[s_1,\dots,s_n]$. Suppose that t is not equivalence to s. Then there is $i \in \{1,\dots,n\}$ such that t_i is not equivalence to s_i.
Case 1. If $opt(t_i) > opt(s_i)$, then $opt(s_i) < opt(t_i[l_1,\dots,l_n])$ for all $l_1,\dots,l_n \in CT_{(n)}$, so $s_i \neq t_i[l_1,\dots,l_n]$ for all $l_1,\dots,l_n \in CT_{(n)}$. Therefore, $f[s_1,\dots,s_n] \neq (f[t_1,\dots,t_n])[l_1,\dots,l_n]$ for all $l_1,\dots,l_n \in CT_{(n)}$. This means that there is no

$\sigma_w \in Cohyp(n)$ such that $\sigma_s = \sigma_t \hat{\circ} \sigma_w$. This gives a contradiction.
Case 2. If $opt(t_i) = opt(s_i)$ and the position of co-operation symbol f are different, then $opt(t_i)$ can be equal to $opt(s_i[l_1, \ldots, l_n])$ if $opt(l_j) = 0$ for all $j \in J_i$ such that $E(s_i) = \{e^n_{j-1} \mid j \in J_i$ and $J_i \subseteq \{1, \ldots, n\}$, so l_j are injections symbols for all $j \in J_i$. Therefore, the coterm $s_i[l_1, \ldots, l_n]$ have to change only injection symbols, but the positions of the co-operation symbols f have no changed. This shows that $t_i \neq s_i[l_1, \ldots, l_n]$ for all $l_1, \ldots, l_n \in CT_{(n)}$. There follows we get that $f[t_1, \ldots, t_n] \neq (f[s_1, \ldots, s_n])[l_1, \ldots, l_n]$ for all $l_1, \ldots, l_n \in CT_{(n)}$. This gives a contradiction. Hence $t \equiv s$.
To prove (ii), suppose that $|E(t)| > |E(s)|$. Since $t \equiv s$, then $t \equiv s[l_1, \ldots, l_n]$ if $opt(l_j) = 0$ for all $j \in J$ such that $E(s) = \{e^n_{j-1} \mid j \in J$ and $J \subseteq \{1, \ldots, n\}\}$, so the injection symbols of the coterm $s = f[s_1, \ldots, s_n]$ have to change at most $|E(s)|$. There follows $|E(t)| \neq |E(s[l_1, \ldots, l_n])|$ where $opt(l_j) = 0$ for all $j \in J$ such that $E(s) = \{e^n_{j-1} \mid j \in J$ and $J \subseteq \{1, \ldots, n\}\}$. This gives a contradiction. Then $|E(t)| \leq |E(s)|$. Similarly, one can shows that $|E(t)| \geq |E(s)|$. Therefore, $|E(t)| = |E(s)|$.
Hence there is a bijection between $E(t)$ and $E(s)$.
Suppose that there are $e^n_j, e^n_k \in E(t)$ such that the position of e^n_j and e^n_k in the coterm t have the same position with e^n_l in the coterm s in somewhere. Since $t \equiv s$, then $e^n_l[l_1, \ldots, l_n] = e^n_j$ and $e^n_l[l_1, \ldots, l_n] = e^n_k$ if and only if $e^n_j = e^n_k$. Therefore, for any $e^n_l \in E(s)$ there exists a uniquely $e^n_j \in E(t)$ such that the position of e^n_j and e^n_l in the coterm t and s are the same, respectively. Similarly, one can shows that for any $e^n_j \in E(t)$ there exists a uniquely $e^n_l \in E(s)$ such that the position of e^n_j and e^n_l in the coterm t and s are the same, respectively.
We define a bijection mapping $\varphi : E(t) \to E(s)$ by $\varphi(x) = y$ for all $x \in E(t)$ and $y \in E(s)$ such that x and y have the same position in t and s, respectively. Then we finishes the prove of (ii).
Conversely, Assume that σ_t and σ_s satisfy the conditions (i) and (ii). Let $r = f[r_1, \ldots, r_n] \in CT_{(n)}$ such that $r_j = \varphi^{-1}(e^n_j)$ for all $j \in J$ and $E(s) = \{e^n_{j-1} \mid j \in J$ for some $J \subseteq \{1, \ldots, n\}\}$.
Then

$$\begin{aligned} \hat{\sigma}_t(\sigma_r(f)) &= \hat{\sigma}_s(f[r_1, \ldots, r_n]) \\ &= \sigma_s(f)[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)] \\ &= (f[s_1, \ldots, s_n])[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)] \\ &= f[s_1[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)], \ldots, s_n[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)]. \end{aligned}$$

Since $r_j = \varphi^{-1}(e^n_j)$, then $\hat{\sigma}_s(r_j) = \varphi^{-1}(e^n_j)$, so $e^n_j[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)] = \varphi^{-1}(e^n_j)$ for all $j \in J$.
Therefore, $s_i[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)] = t_i$ for all $i \in \{1, \ldots, n\}$. There follows $f[t_1, \ldots, t_n] = f[s_1[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)], \ldots, s_n[\hat{\sigma}_s(r_1), \ldots, \hat{\sigma}_s(r_n)]$.
Hence, $\sigma_t(f) = \sigma_s \hat{\circ} \sigma_r$. Similarly, one can shows that $\sigma_s = \sigma_t \hat{\circ} \sigma_w$ for some $\sigma_w \in Cohyp(n)$. This implies that $\sigma_t R \sigma_s$. □

Next, we have to characterize some Green's relation L on $Cohyp(n)$.

Theorem 2.9. *If $t = f[t_1, \ldots, t_n]$ such that $t_1, \ldots, t_n \in \{e^n_{j-1} \mid j \in \{1, \ldots, n\}\}$, then $\sigma_t L \sigma_s$ if and only if*

(*i*) $E(t) = E(s)$ *and*
(*ii*) *if* $s = f[s_1, \ldots, s_n]$*, then there exist* $K \subseteq \{1, \ldots, n\}$ *such that* $\{s_k \mid k \in K\} = E(t)$.

Proof. Let $t = f[t_1, \ldots, t_n]$ and $t_1, \ldots, t_n \in \{e^n_{j-1} \mid j \in \{1, \ldots, n\}\}$. Assume that $\sigma_t L \sigma_s$. Then there are $\sigma_u, \sigma_v \in Cohyp(n)$ such that $u = f[u_1, \ldots, u_n], v = f[v_1, \ldots, v_n] \in CT_{(n)}$ and $\sigma_t = \sigma_u \hat{\circ} \sigma_s$ and $\sigma_s = \sigma_v \hat{\circ} \sigma_t$.
Therefore,

$$\begin{aligned} f[t_1, \ldots, t_n] &= \hat{\sigma}_u(\sigma_s(f)) \\ &= \hat{\sigma}_u(f[s_1, \ldots, s_n]) \\ &= \sigma_u(f)[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)] \\ &= (f[u_1, \ldots, u_n])[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)] \\ &= f[u_1[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)], \ldots, u_n[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)]]. \end{aligned}$$

This implies that $t_i = u_i[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)]$ for all $i \in \{1, \ldots, n\}$.

Since $t_1, \ldots, t_n \in \{e^n_{j-1} \mid j \in \{1, \ldots, n\}\}$, then $u_1[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)], \ldots, u_n[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)] \in \{e^n_{j-1} \mid j \in \{1, \ldots, n\}\}$. There follows from the extension of σ_u, there exist $K \subseteq \{1, \ldots, n\}$ such that $t_i = s_k$ for some $k \in K$, so $E(t) \subseteq E(s)$.
Since $\sigma_s = \sigma_v \hat{\circ} \sigma_t$, then

$$\begin{aligned} f[s_1, \ldots, s_n] &= \hat{\sigma}_v(\sigma_t(f)) \\ &= \hat{\sigma}_v(f[t_1, \ldots, t_n]) \\ &= \sigma_v(f)[\hat{\sigma}_v(t_1), \ldots, \hat{\sigma}_v(t_n)] \\ &= (f[v_1, \ldots, v_n])[\hat{\sigma}_v(t_1), \ldots, \hat{\sigma}_v(t_n)] \\ &= f[v_1[\hat{\sigma}_v(t_1), \ldots, \hat{\sigma}_v(t_n)], \ldots, v_n[\hat{\sigma}_v(t_1), \ldots, \hat{\sigma}_v(t_n)]]. \end{aligned}$$

Since $t_1, \ldots, t_n \in \{e^n_{j-1} \mid j \in \{1, \ldots, n\}\}$, then $\hat{\sigma}_v(t_i) = t_i$ for all $i \in \{1, \ldots, n\}$. This implies that the injection symbols which occurring in the coterms $v_i[\hat{\sigma}_v(t_1), \ldots, \hat{\sigma}_v(t_n)]$ are the subset of $\{t_i \mid i \in \{1, \ldots, n\}\}$ for all $i \in \{1, \ldots, n\}$. Therefore, $E(s) \subseteq E(t)$.
To prove (*ii*), we consider the followin equation

$$f[t_1, \ldots, t_n] = f[u_1[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)], \ldots, u_n[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)]].$$

If $t_i = e^n_{j-1}$ for some $j \in \{1, \ldots, n\}$, then $u_i[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)] = e^n_{j-1}$, so u_i are injection symbols for all $i \in \{1, \ldots, n\}$. The extension of σ_u, implies that $s_k = e^n_{j-1}$ for some $k \in \{1, \ldots, n\}$. Let $K = \{k \mid s_k = t_i \text{ for some } i \in \{1, \ldots, n\}\}$. Then we finishes the prove of (*ii*).

Conversely, assume that (*i*) and (*ii*) are true. For each $i \in \{1, \ldots, n\}$, we have that $t_i = s_k$ for some $k \in K$. Then we define $\sigma_u(f) = f[u_1, \ldots, u_n]$ such that $u_i = e^n_{k-1}$ for all $i \in \{1, \ldots, n\}$. Therefore,

$$\begin{aligned} \hat{\sigma}_u(\sigma_s(f)) &= \hat{\sigma}_u(f[s_1, \ldots, s_n]) \\ &= \sigma_u(f)[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)] \\ &= (f[u_1, \ldots, u_n])[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)] \\ &= f[u_1[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)], \ldots, u_n[\hat{\sigma}_u(s_1), \ldots, \hat{\sigma}_u(s_n)]] \\ &= f[t_1, \ldots, t_n] \\ &= \sigma_t(f). \end{aligned}$$

And we define $\sigma_v(f) = f[v_1, \ldots, v_n]$ as follow:
If $k \in K$, we let $v_k = e^n_{i-1}$ and if $r \in \{1, \ldots, n\} \setminus K$, we let $v_r \equiv s_r$ such that there is a uniquely bijection $\varphi : E(s_r) \to E(v_r)$ and satisfy that if $e^n_{j-1} \in E(s_r)$, then e^n_{j-1} and $\varphi(e^n_{j-1})$ are in the same position of the coterms s_r and v_r, respectively. Since $E(t) = E(s)$, then for any $e^n_{j-1} \in E(s_r)$ such that $e^n_{j-1} = t_i$ for some $i \in \{1, \ldots, n\}$, we let $\varphi(e^n_{j-1}) = e^n_{i-1}$. Then,
$v_k[t_1, \ldots, t_n] = e^n_{i-1}[t_1, \ldots, t_n] = t_i = s_k$ for all $k \in K$, and
$v_r[t_1, \ldots, t_n] = s_r$ for all $r \in \{1, \ldots, n\} \setminus K$.
Therefore,

$$\begin{aligned}
\hat{\sigma}_v(\sigma_t(f)) &= \hat{\sigma}_v(f[t_1, \ldots, t_n]) \\
&= \sigma_v(f)[\hat{\sigma}_v(t_1), \ldots, \hat{\sigma}_v(t_n)] \\
&= (f[v_1, \ldots, v_n])[t_1, \ldots, t_n] \\
&= f[v_1[t_1, \ldots, t_n], \ldots, v_n[t_1, \ldots, t_n]] \\
&= f[s_1, \ldots, s_n] \\
&= \sigma_s(f).
\end{aligned}$$

Hence, $\sigma_t L \sigma_s$. □

Corollary 2.10. *Let* $\sigma_s, \sigma_t \in Cohyp(n)$. *If* $E(s) = E(t)$ *and* $\exists K, J \subseteq \{1, \ldots, n\}$ *such that* $E(s) = \{s_k \mid k \in K\}$ *and* $E(t) = \{t_j \mid j \in J\}$, *then* $\sigma_t L \sigma_s$.

Acknowledgements : We would like to thank the referee for his/her helpful comments and suggestions. We also would like to thank the Faculty of Science of Mahasarakham University Thailand and the Centre of Excellence in Mathematics Thailand for the financial support.

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(Received 30 September 2014)
(Accepted 7 July 2015)