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Edge-Chromatic Numbers of Glued Graphs

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Abstract : Let G_1 and G_2 be any two graphs. Assume that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ are connected, not a single vertex and such that $H_1 \cong H_2$ with an isomorphism f. The *glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. We give upper bounds of the edge-chromatic numbers of glued graphs; one is in terms of the edge-chromatic numbers of their original graphs where we give a characterization of graphs satisfying its equality. We further obtain a better upper bound of the chromatic numbers of glued graphs when the original graphs are line graphs.

Keywords : Graph coloring; Glued graph.
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1 Introduction

Let G_1 and G_2 be any graphs, $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be connected, not a single vertex and such that $H_1 \cong H_2$ with an isomorphism f. The *glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f. If H is the copy of H_1 and H_2 in the glued graph, H is referred as its *clone*, and G_1 and G_2 are referred as its *original graphs*. The *glued graph* $G_1 \lhd\!\rhd G_2$ *at the clone* H means that there exist a subgraph H_1 of G_1, a subgraph H_2 of G_2, and an isomorphism f such that $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ and H is the copy of H_1 and H_2 in the resulting graph. Unless we define specifically, we denote $G_1 \lhd\!\rhd G_2$ as an arbitrary graph resulting from gluing G_1 and G_2.

A k*-edge-coloring* of a graph G is a labelling $f : E(G) \to S$, where $|S| = k$. The labels are *colors*; the edges of one color form a *color class*. A k-edge-coloring is *proper* if incident edges have different labels. A graph is k*-edge-colorable* if it has a proper k-edge-coloring. The *edge-chromatic number* of a loopless graph G, $\chi'(G)$, is the least k such that G is k-edge-colorable.

Let $\Delta(G)$ be the maximum degree of a graph G. Since all edges incident to a vertex with maximum degree cannot be labelled by the same color, $\chi'(G) \leq \Delta(G)$. For simple graph G, a well-known result was independently proved by Vizing [5] and Gupta [1] that

$$\chi'(G) \leq \Delta(G) + 1.$$

We referred to it as *Vizing and Gupta's upper bound.* Then we denote that G is *Class 1* if $\chi'(G) = \Delta(G)$ and G is *Class 2* if $\chi'(G) = \Delta(G) + 1$. Nevertheless, Vizing and Gupta's upper bound is not satisfied by loopless non-simple graphs, Shannon [4] proved that

$$\chi'(G) \leq \frac{3}{2}\Delta(G)$$

which we refer to as *Shannon's upper bound.* The sharpness of this bound is provided by the *fat triangles*; the loopless triangles with multiple edges similar to the graph in Figure 1.

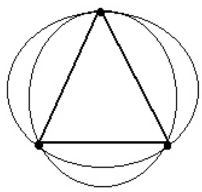


Figure 1: A fat triangle

We note few facts that the copy of both original graphs are subgraphs of their glued graphs. The graph gluing does not create an edge. Also, a glued graph of simple graphs may not be simple. Some interesting properties of glued graphs and the chromatic numbers of glued graphs are studied in [3]. Here we investigate the edge-chromatic numbers of glued graphs. In section 3, we apply our result to obtain a better upper bound of the chromatic numbers of glued graphs when original graphs are line graphs. The notation $C_n(v_1, \ldots, v_n)$ denotes a cycle of n vertices on the vertex set $\{v_1, \ldots, v_n\}$.

2 Bounds of the Edge-Chromatic Numbers of Glued Graphs

For any glued graph $G_1 \lhd\!\rhd G_2$, since G_1 and G_2 are subgraphs $G_1 \lhd\!\rhd G_2$, the edge-chromatic number of $G_1 \lhd\!\rhd G_2$ is at least $\chi'(G_1)$ and $\chi'(G_2)$. We therefore get a lower bound for any graphs G_1 and G_2 that

$$\chi'(G_1 \lhd\!\rhd G_2) \geq \max\{\chi'(G_1), \chi'(G_2)\}.$$

An upper bound of the edge-chromatic number of a glued graph is in terms of the sum of the edge-chromatic numbers of their original graphs. This result is shown next along with its sharpness.

Remark 2.1 Since the graph gluing does not identify vertices of the original graphs, non-incident edges in original graphs are still non-incident in a glued graph.

Theorem 2.2 *For any graph G_1 and G_2,*

$$\chi'(G_1 \triangleleft\!\triangleright G_2) \leq \chi'(G_1) + \chi'(G_2).$$

Proof. Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ be a glued graph of G_1 and G_2 at arbitrary clone H. There are proper edge-colorings $f : E(G_1) \to S_1$ and $g : E(G_2) \to S_2$ of G_1 and G_2, respectively, where S_1 and S_2 are sets of colors such that $|S_1| = \chi'(G_1)$, $|S_2| = \chi'(G_2)$ and $S_1 \cap S_2 = \phi$. Define $\alpha : E(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) \to S_1 \cup S_2$ by for all $e \in E(G_1 \underset{H}{\triangleleft\!\triangleright} G_2)$,

$$\alpha(e) = \begin{cases} f(e) & \text{if } e \in E(G_1), \\ g(e) & \text{if } e \in E(G_2 \backslash H). \end{cases}$$

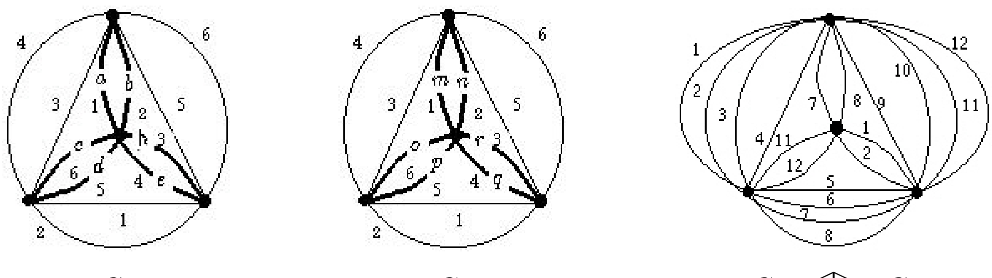
To prove that α is proper, let e_1 and e_2 be incident edges in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$.

Case 1. $e_1 \in E(G_1)$ and $e_2 \in E(G_2 \backslash H)$: Because $S_1 \cap S_2 = \phi$, we have $\alpha(e_1) \neq \alpha(e_2)$.

Case 2. e_1 and e_2 are edges in G_1: By Remark 2.1, e_1 and e_2 are incident in G_1 and hence $\alpha(e_1) = f(e_1) \neq f(e_2) = \alpha(e_2)$.

Case 3. e_1 and e_2 are edges in $G_2 \backslash H$: Similar to case 2, we have that $\alpha(e_1) = g(e_1) \neq g(e_2) = \alpha(e_2)$.

Therefore α is proper and hence $\chi'(G_1 \triangleleft\!\triangleright G_2) \leq \chi'(G_1) + \chi'(G_2)$. $\square$



G_1 G_2 $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$

Figure 2: The sharpness of Theorem 2.2

Consider graphs G_1 and G_2 with proper 6-edge-colorings in Figure 2. Note that both graphs have the maximum degree six. Thus $\chi'(G_1) = 6 = \chi'(G_2)$. We glue G_1 and G_2 with the isomorphism f defined by $f(a) = m$, $f(b) = n$, $f(c) = o$,

$f(d) = p$, $f(e) = q$ and $f(h) = r$. The glued graph $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ with a proper 12-edge-coloring is shown as in Figure 2. Since a fat triangle with the maximum degree 8 is subgraph of $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$, we have $\chi'(G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2) \geq \frac{3}{2}(8) = 12$. Hence $\chi'(G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2) = 12$. Therefore $\chi'(G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2) = \chi'(G_1) + \chi'(G_2)$, and hence the upper bound of the edge-chromatic number in Theorem 2.2 is sharp.

Now consider another upper bound of the edge-chromatic number of any glued graph. It shall be expressed in terms of the maximum degree of its original graphs and the minimum degree of its clone. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of a graph G, respectively.

Lemma 2.3 *Let G_1 and G_2 be graphs and let H be the clone of a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$. Then*

$$\Delta(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).$$

Proof. Let G_1 and G_2 be graphs and let H be the clone of a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$. For convenience, let $G = G_1 \underset{H}{\lhd\!\rhd} G_2$. Let v be a vertex with maximum degree of G. If v is not in H, then $\deg_G(v) = \max\{\Delta(G_1), \Delta(G_2)\} \leq \Delta(G_1) + \Delta(G_2) - \delta(H)$. Suppose that v is in H. So v is in both G_1 and G_2. Since each edge which is incident to v in H contributes twice in the degree sum,

$$\deg_G(v) = \deg_{G_1}(v) + \deg_{G_2}(v) - \deg_H(v).$$

Since $v \in H$, we get that $\deg_H(v) \geq \delta(H)$. Hence

$$\deg_G(v) = \deg_{G_1}(v) + \deg_{G_2}(v) - \deg_H(v) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).$$

□

Remark 2.4 Consequently from Lemma 2.3, since $\delta(H) \geq 1$, $\Delta(G_1 \lhd\!\rhd G_2) \leq \Delta(G_1) + \Delta(G_2) - 1$.

Theorem 2.5 *Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\lhd\!\rhd} G_2$ be a glued graph of G_1 and G_2 at a clone H. Then*

$$\chi'(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \frac{3}{2}(\Delta(G_1) + \Delta(G_2) - \delta(H)).$$

In particular, if $G_1 \underset{H}{\lhd\!\rhd} G_2$ is a simple graph, then

$$\chi'(G_1 \underset{H}{\lhd\!\rhd} G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1.$$

Proof. Let G_1 and G_2 be graphs and let $G_1 \underset{H}{\diamond} G_2$ be a glued graph of G_1 and G_2 at a clone H. Following from Shannon's upper bound and Lemma 2.3, we have that $\chi'(G_1 \underset{H}{\diamond} G_2) \leq \frac{3}{2}(\Delta(G_1)+\Delta(G_2)-\delta(H))$. If $G_1 \underset{H}{\diamond} G_2$ is a simple graph, by Vizing and Gupta's upper bound and Lemma 2.3, we obtain that $\chi'(G_1 \underset{H}{\diamond} G_2) \leq \Delta(G_1)+\Delta(G_2)-\delta(H)+1$.

□

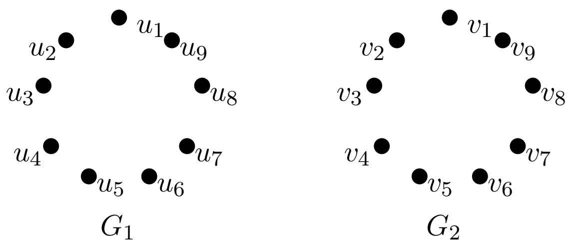


Figure 3: The sharpness of Theorem 2.5 for simple glued graphs

We now show the sharpness of Theorem 2.5. In Figure 3, consider $H_1 = C_9(u_1, u_2, \ldots, u_9)$ and $H_2 = C_9(v_1, v_2, \ldots, v_9)$. We glue G_1 and G_2 at H_1 and H_2 by isomorphism f defined by $f(u_i) = v_i$ for all $i = 1, 2, \ldots, 9$. So we have $G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2$ which is isomorphic to K_9. Note that $\chi'(K_n) = n$ when n is odd. [6] Hence $\chi'(G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2) = 9 = 6+4-2+1 = \Delta(G_1)+\Delta(G_2)-\delta(H)+1$.

For non-simple glued graphs, consider graphs G_1 and G_2 with maximum degree four in Figure 4. Gluing G_1 and G_2 at edge sets $\{a, b, c\}$ and $\{1, 2, 3\}$ with isomorphism f such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$ yields the glued graph $G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2$ as shown in Figure 4. Hence we have $\chi'(G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2) = 9 = \dfrac{3}{2}(4+4-2) = \dfrac{3}{2}(\Delta(G_1)+\Delta(G_2)-\delta(H))$.

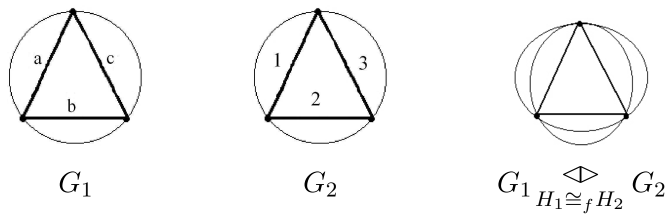


Figure 4: The sharpness of Theorem 2.5 for non-simple glued graphs

We next discuss a characterization of graphs with the edge-chromatic number is precisely the sum of the edge-chromatic number of the original graphs, that is, it satisfies the equality in Theorem 2.2.

Corollary 2.6 *If G_1, G_2 and $G_1 \lhd\rhd G_2$ are simple graphs, $\chi'(G_1 \lhd\rhd G_2) = \chi'(G_1) + \chi'(G_2)$ if and only if G_1 and G_2 are Class 1, $G_1 \lhd\rhd G_2$ is Class 2, and $\Delta(G_1 \lhd\rhd G_2) = \Delta(G_1) + \Delta(G_2) - 1$.*

Proof. Necessity. Assume $\chi'(G_1 \lhd\rhd G_2) = \chi'(G_1) + \chi'(G_2)$. By Vizing and Gupta's upper bound and Remark 2.4, we have that

$$\chi'(G_1 \lhd\rhd G_2) \leq \Delta(G_1 \lhd\rhd G_2) + 1 \leq \Delta(G_1) + \Delta(G_2) \leq \chi'(G_1) + \chi'(G_2).$$

Therefore, $\chi'(G_1 \lhd\rhd G_2) = \Delta(G_1 \lhd\rhd G_2)+1$, $\Delta(G_1 \lhd\rhd G_2) = \Delta(G_1)+\Delta(G_2)-1$, $\chi'(G_1) = \Delta(G_1)$ and $\chi'(G_2) = \Delta(G_2)$.

Sufficiency. All conditions in the right hand side yield that

$$\chi'(G_1 \lhd\rhd G_2) = \Delta(G_1 \lhd\rhd G_2)+1 = (\Delta(G_1)+\Delta(G_2)-1)+1 = \chi'(G_1)+\chi'(G_2).$$

□

Determining whether a graph is Class 1 or Class 2 is generally hard [2, 6]. Gluing Class 1 graphs may get a Class 2 glued graph and vice versa. It is an open problem to determine conditions that forbid or guarantee $\Delta(G_1 \lhd\rhd G_2)$-edge-colorability.

3 The Chromatic Numbers of Glued Line Graphs

The *line graph* $L(G)$ of a connected graph G is the graph generated from G by $V(L(G)) = E(G)$ and for any two vertices $e, f \in V(L(G))$, vertex e and vertex f are adjacent in $L(G)$ if and only if edge e and edge f share a common vertex in G. If H is the line graph of G, we call G the *root graph* of H. All graphs have their line graphs, but not all graphs are line graphs. For example, there is no graph G such that $L(G) = K_{1,3}$. So the $K_{1,3}$ is not a line graph.

A *k-coloring* of a graph G is a labelling $f : V(G) \to S$, where $|S| = k$. The labels are *colors*; the vertices of one color form a *color class.* A k-coloring is *proper* if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number of graph* G, $\chi(G)$, is the least k such that G is k-colorable.

Ones may intuitively believe that $\chi(G_1 \lhd\rhd G_2) \leq \chi(G_1) + \chi(G_2)$. However, we proved in [3] and showed its sharpness that $\chi(G_1 \lhd\rhd G_2) \leq \chi(G_1)\chi(G_2)$ for any graphs G_1 and G_2. Here we shall show by using Theorem 2.2 that if G_1 and G_2 are line graphs, $G_1 \lhd\rhd G_2$ has a proper $(\chi(G_1) + \chi(G_2))$-coloring.

Remark 3.1 For any subgraph H of a graph G, $L(H) \subseteq L(G)$.

Remark 3.2 For any graph G, $\chi'(G) = \chi(L(G))$.

Lemma 3.3 *Let G_1 and G_2 be graphs.* $L(G_1) \lhd\!\rhd L(G_2) \subseteq L(G_1 \lhd\!\rhd G_2)$.

Proof. Since G_1 and G_2 are subgraphs of $G_1 \lhd\!\rhd G_2$, the line graphs $L(G_1)$ and $L(G_2)$ are subgraphs of $L(G_1 \lhd\!\rhd G_2)$. So $L(G_1) \cup L(G_2) \subseteq L(G_1 \lhd\!\rhd G_2)$. Because for each vertex and edge in $L(G_1) \lhd\!\rhd L(G_2)$ are in $L(G_1) \cup L(G_2)$ which is a subgraph of $L(G_1 \lhd\!\rhd G_2)$, so $L(G_1) \lhd\!\rhd L(G_2) \subseteq L(G_1 \lhd\!\rhd G_2)$. □

Theorem 3.4 *Let G_1 and G_2 be graphs. If G_1 and G_2 are line graphs, then* $\chi(G_1 \lhd\!\rhd G_2) \leq \chi(G_1) + \chi(G_2)$.

Proof. Let G_1 and G_2 be graphs. Assume that G_1 and G_2 are line graphs. So there are graphs G_1^* and G_2^* such that $L(G_1^*) = G_1$ and $L(G_2^*) = G_2$. By lemma 3.3, we have that $L(G_1^*) \lhd\!\rhd L(G_2^*) \subseteq L(G_1^* \lhd\!\rhd G_2^*)$. This yields $\chi(L(G_1^*) \lhd\!\rhd L(G_2^*)) \leq \chi(L(G_1^* \lhd\!\rhd G_2^*))$. Hence

$$\begin{aligned}
\chi(G_1 \lhd\!\rhd G_2) &= \chi(L(G_1^*) \lhd\!\rhd L(G_2^*)) \leq \chi(L(G_1^* \lhd\!\rhd G_2^*)) \\
&= \chi'(G_1^* \lhd\!\rhd G_2^*) \\
&\leq \chi'(G_1^*) + \chi'(G_2^*) \quad \text{(by Theorem 2.2.)} \\
&= \chi(L(G_1^*)) + \chi(L(G_2^*)) = \chi(G_1) + \chi(G_2).
\end{aligned}$$

□

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