



A Class of Strong Limit Theorems for Random Field with Geometric Distributions Indexed by a Homogeneous Tree¹

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Abstract : In this paper, we establish some strong limit theorem for the random field which obeys the geometric distributions indexed by the homogeneous tree by constructing the consistent distribution and a nonnegative martingale with pure analytical methods. As corollaries, a limit theorem for the sequence of random variables with the geometric distributions is extended.

Keywords : the homogeneous tree; random field; the geometric distribution; nonnegative martingale; strong limit theorem.

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1 Introduction

Let T be a homogeneous tree on which each vertex has $N + 1$ neighboring vertices. We first fix any vertex as the "root" and label it by 0. Let σ, τ be vertices of a tree. Write $\tau \leq \sigma$ if τ is on the unique path connecting 0 to σ , $|\sigma|$ for the number of edges on this path. For any two vertices σ, τ , denote $\sigma \wedge \tau$ the vertex farthest from 0 satisfying

$$\sigma \wedge \tau \leq \sigma, \quad \text{and} \quad \sigma \wedge \tau \leq \tau.$$

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If $\sigma \neq 0$, then we let $\bar{\sigma}$ stand for the vertex satisfying $\bar{\sigma} \leq \sigma$ and $|\bar{\sigma}| = |\sigma| - 1$ (we refer to σ as a son of $\bar{\sigma}$). It is easy to see that the root has $N + 1$ sons and all other vertices have N sons. The homogeneous tree T is also called Bethe tree $T_{B,N}$. For example, we give the following Fig 1 $T_{B,2}$.

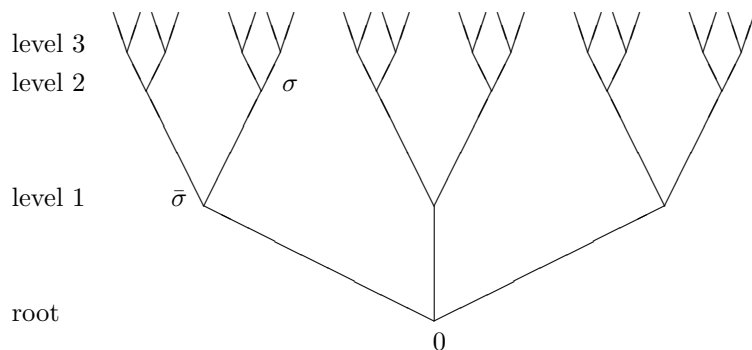


Fig 1. Bethe tree $T_{B,2}$

Two special finite tree-indexed Markov chains are introduced in Kemeny et al.(1976 [1]), Spitzer (1975 [2]), and there the finite transition matrix is assumed to be positive and reversible to its stationary distribution, and this tree-indexed Markov chains ensure that the cylinder probabilities are independent of the direction we travel along a path. In this paper, we have no such assumption.

If $|\sigma| = n$, it is said to be on the n th level on a tree T . We denote by $T^{(n)}$ the subtree of T containing the vertices from level 0 (the root) to level n , and L_n the set of all vertices on the level n . Let B be a subgraph of T . Denote $X^B = \{X_\sigma, \sigma \in B\}$, and denote by $|B|$ the number of vertices of B . Let $S(\sigma)$ be the set of all sons of vertices σ . It is easy to see that $|S(0)| = N + 1$ and $|S(\sigma)| = N$, where $\sigma \neq 0$.

Suppose that $S = \{1, 2, 3, \dots\}$ is a countable state space. Let $\Omega = S^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on T and taking values in S , and \mathcal{F} be the smallest Borel field containing all cylinder sets in Ω , μ be the probability measure on (Ω, \mathcal{F}) . Let $X = \{X_\sigma, \sigma \in T\}$ be the coordinate stochastic process defined on the measurable space (Ω, \mathcal{F}) ; that is, for any $\omega = \{\omega(t), t \in T\}$, define

$$X_t(\omega) = \omega(t), \quad t \in T^{(n)}$$

$$X^{T^{(n)}} \triangleq \{X_t, t \in T^{(n)}\}, \quad \mu(X^{T^{(n)}} = x^{T^{(n)}}) = \mu(x^{T^{(n)}}). \quad i = 1, 2. \quad (1)$$

Now we give a definition of Markov chain fields on the tree T by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see[3]).

Definition 1.1. Let $\{p_\sigma, \sigma \in T^{(n)}\}$ be a sequence of positive real numbers, $p_\sigma \in (0, 1)$. Denote $q_\sigma = 1 - p_\sigma$, $\sigma \in T^{(n)}$. If

$$\mu_P(x^{T^{(n)}}) = \prod_{k=0}^n \prod_{\sigma \in L_k} (1 - p_\sigma)^{X_\sigma - 1} p_\sigma, \quad n \geq 0. \tag{2}$$

Then μ_P will be called a *random field* which obeys the geometric distributions (2) indexed by the homogeneous tree T .

There have been some works on limit theorems for tree-indexed stochastic processes. Benjamini and Peres have given the notion of the tree-indexed homogeneous Markov chains and studied the recurrence and ray-recurrence for them (see[4]). Berger and Ye have studied the existence of entropy rate for some stationary random fields on a homogeneous tree (see[5]). Pemantle proved a mixing property and a weak law of large numbers for a PPG-invariant and ergodic random field on a homogeneous tree (see[6]). Ye and Berger, by using Pemantle’s result and a combinatorial approach, have studied the asymptotic equipartition property (AEP) in the sense of convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree(see [7, 8]). Peng and Yang have studied a class of small deviation theorems for functionals of random field and asymptotic equipartition property (AEP) for arbitrary random field on a homogeneous trees (see [9, 10]). Recently, Yang have studied some limit theorems for countable homogeneous Markov chains indexed by a homogeneous tree and strong law of large numbers and the asymptotic equipartition property (AEP) for finite homogeneous Markov chains indexed by a homogeneous tree (see [11, 12]). Wang has also studied some Shannon-McMillan approximation theorems for arbitrary random field on the generalized Bethe tree (see[13]).

It is known to all that the geometric distribution is one of the classical probability distributions. It has comprehensive applications in all fields of the economical life. In this paper, our aim is to establish some strong limit theorems for the random field which obeys the geometric distributions indexed by the homogeneous tree by constructing the consistent distribution and a nonnegative martingale with pure analytical methods. As corollaries, some limit theorem for the sequence of random variables with the geometric distributions is extended.

2 Main Results

Theorem 2.1. Let $X = \{X_\sigma, \sigma \in T\}$ be a random field taking values in $S = \{1, 2, 3, \dots\}$ which obeys the geometric distributions (2) indexed by the homogeneous tree T . Let $\{a_n, n \geq 0\}$ be a nonnegative increasing stochastic sequence. Denote

$$\alpha = \inf \left\{ p_\sigma, \sigma \in T^{(n)}, n \geq 0 \right\} > 0, \tag{3}$$

$$D(\omega) = \left\{ \omega : \lim_{n \rightarrow \infty} a_n = \infty, \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \leq M \right\}. \tag{4}$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} (X_\sigma - \frac{1}{p_\sigma}) = 0. \quad \mu_P - a.s. \quad \omega \in D(\omega) \quad (5)$$

Proof. Consider the probability space $(\Omega, \mathcal{F}, \mu)$, let λ be an arbitrary real number, $\delta_i(j)$ be Kronecker function. We construct the following product distribution:

$$\mu_Q(x^{T^{(n)}}; \lambda) = \prod_{k=0}^n \prod_{\sigma \in L_k} (\lambda(1 - p_\sigma))^{X_\sigma - 1} (1 - \lambda(1 - p_\sigma)), \quad n \geq 0. \quad (6)$$

Denote

$$U_n(\lambda, \omega) = \frac{\mu_Q(X^{T^{(n)}}; \lambda)}{\mu_P(X^{T^{(n)}})}. \quad (7)$$

By (2) and (6), we can rewrite (7) as

$$\begin{aligned} U_n(\lambda, \omega) &= \frac{1}{\mu_P(X^{T^{(n)}})} \lambda^{\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma} \prod_{k=0}^n \prod_{\sigma \in L_k} (1 - p_\sigma)^{X_\sigma - 1} p_\sigma \prod_{k=0}^n \prod_{\sigma \in L_k} \frac{1 - \lambda(1 - p_\sigma)}{\lambda p_\sigma} \\ &= \lambda^{\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma} \prod_{k=0}^n \prod_{\sigma \in L_k} \frac{1 - \lambda(1 - p_\sigma)}{\lambda p_\sigma}. \end{aligned} \quad (8)$$

We set $\mathcal{F}_n = \sigma(X^{T^{(n)}})$, denote by E_P the expectation relative to the measure μ_P . According to the independence of $\{X_\sigma, \sigma \in T\}$, we can write

$$\begin{aligned} &E_P [U_n(\lambda, \omega) | \mathcal{F}_{n-1}] \\ &= E_P \left[\lambda^{\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma} \prod_{k=0}^n \prod_{\sigma \in L_k} \frac{1 - \lambda(1 - p_\sigma)}{\lambda p_\sigma} \middle| \mathcal{F}_{n-1} \right] \\ &= U_{n-1}(\lambda, \omega) \cdot E_P \left[\lambda^{\sum_{\sigma \in L_n} X_\sigma} \prod_{\sigma \in L_n} \frac{1 - \lambda(1 - p_\sigma)}{\lambda p_\sigma} \middle| \mathcal{F}_{n-1} \right] \\ &= U_{n-1}(\lambda, \omega) \cdot E_P \left[\lambda^{\sum_{\sigma \in L_n} X_\sigma} \prod_{\sigma \in L_n} \frac{1 - \lambda(1 - p_\sigma)}{\lambda p_\sigma} \right] \\ &= U_{n-1}(\lambda, \omega) \cdot \sum_{x^{L_n} \in S} \prod_{\sigma \in L_n} \lambda^{x_\sigma} \frac{1 - \lambda(1 - p_\sigma)}{\lambda p_\sigma} (1 - p_\sigma)^{x_\sigma - 1} p_\sigma \\ &= U_{n-1}(\lambda, \omega) \cdot \prod_{\sigma \in L_n} \sum_{x_\sigma \in S} \lambda^{x_\sigma - 1} (1 - p_\sigma)^{x_\sigma - 1} (1 - \lambda(1 - p_\sigma)) \\ &= U_{n-1}(\lambda, \omega) \cdot \prod_{\sigma \in L_n} \sum_{x_\sigma \in S} (\lambda(1 - p_\sigma))^{x_\sigma - 1} (1 - \lambda(1 - p_\sigma)) \\ &= U_{n-1}(\lambda, \omega) \cdot \prod_{\sigma \in L_n} \frac{1 - \lambda(1 - p_\sigma)}{1 - \lambda(1 - p_\sigma)} = U_{n-1}(\lambda, \omega). \end{aligned} \quad (9)$$

It is easy to see that $\{U_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$ (where $\mathcal{F}_n = \sigma(X^{T(n)})$) is a nonnegative martingale. According to Doob's martingale convergence theorem, we know

$$\lim_{n \rightarrow \infty} U_n(\lambda, \omega) = U_\infty(\lambda, \omega) < \infty. \quad \mu_P - a.s. \tag{10}$$

By the first equation of (4) and (10) we have

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \log U_n(\lambda, \omega) \leq 0. \quad \mu_P - a.s. \quad \omega \in D(\omega) \tag{11}$$

By (8) and (11) we can write

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{a_n} \left[\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma \log \lambda - \sum_{k=0}^n \sum_{\sigma \in L_k} \log \left(\frac{\lambda p_\sigma}{1 - \lambda(1 - p_\sigma)} \right) \right] \leq 0. \\ \mu_P - a.s. \quad \omega \in D(\omega) \end{aligned} \tag{12}$$

Letting $\lambda \in \left(1, \frac{1}{1-\alpha}\right)$ and dividing both sides of (12) by $\log \lambda$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{a_n} \left[\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma - \sum_{k=0}^n \sum_{\sigma \in L_k} \log \left(\frac{\lambda p_\sigma}{1 - \lambda(1 - p_\sigma)} \right) / \log \lambda \right] \leq 0. \\ \mu_P - a.s. \quad \omega \in D(\omega) \end{aligned} \tag{13}$$

According to the property of superior limit

$$\limsup_{n \rightarrow \infty} (a_n - b_n) \leq 0 \Rightarrow \limsup_{n \rightarrow \infty} (a_n - c_n) \leq \limsup_{n \rightarrow \infty} (b_n - c_n), \tag{14}$$

By (13) and the inequality $1 - 1/x \leq \ln x \leq x - 1$ ($x > 0$), we can write

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left(X_\sigma - \frac{1}{p_\sigma} \right) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\log \left(\frac{\lambda p_\sigma}{1 - \lambda(1 - p_\sigma)} \right) / \log \lambda - \frac{1}{p_\sigma} \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\left(\frac{\lambda p_\sigma}{1 - \lambda(1 - p_\sigma)} - 1 \right) / \log \lambda - \frac{1}{p_\sigma} \right] \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\left(\frac{\lambda - 1}{1 - \lambda(1 - p_\sigma)} \right) / \log \lambda - \frac{1}{p_\sigma} \right] \\
 &\leq \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\frac{\frac{\lambda - 1}{1 - \lambda(1 - p_\sigma)}}{1 - 1/\lambda} - \frac{1}{p_\sigma} \right] \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\frac{\lambda}{1 - \lambda(1 - p_\sigma)} - \frac{1}{p_\sigma} \right] \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{\lambda - 1}{p_\sigma [1 - \lambda(1 - p_\sigma)]} \\
 &\leq \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \frac{\lambda - 1}{1 - \lambda(1 - \alpha)}. \quad \mu_P - a.s. \quad \omega \in D(\omega) \quad (15)
 \end{aligned}$$

when $1 < \lambda \leq \frac{1-\alpha/2}{1-\alpha} = 1 + \frac{\alpha}{2(1-\alpha)}$, we have $1 - \lambda(1 - \alpha) \geq \alpha/2$. Thus

$$\frac{\lambda - 1}{1 - \lambda(1 - \alpha)} \leq \frac{2(\lambda - 1)}{\alpha}. \tag{16}$$

By (4), (15) and (16), we obtain

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left(X_\sigma - \frac{1}{p_\sigma} \right) \\
 &\leq \frac{2(\lambda - 1)}{\alpha} \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \leq \frac{2(\lambda - 1)}{\alpha} M. \quad \mu_P - a.s. \quad \omega \in D(\omega) \quad (17)
 \end{aligned}$$

Choose $\lambda_i \in \left(1, \frac{1-\alpha/2}{1-\alpha}\right)$ ($i = 1, 2, \dots$) such that $\lambda_i \rightarrow 1^+$ (as $i \rightarrow \infty$), we have by (17),

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left(X_\sigma - \frac{1}{p_\sigma} \right) \leq 0. \quad \mu_P - a.s. \quad \omega \in D(\omega) \tag{18}$$

Letting $\lambda \in (0, 1)$, dividing both sides of (12) by $\log \lambda$, we get

$$\begin{aligned}
 &\liminf_{n \rightarrow \infty} \frac{1}{a_n} \left[\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma - \sum_{k=0}^n \sum_{\sigma \in L_k} \log \left(\frac{\lambda p_\sigma}{1 - \lambda(1 - p_\sigma)} \right) / \log \lambda \right] \geq 0, \\
 &\quad \mu_P - a.s. \quad \omega \in D(\omega) \tag{19}
 \end{aligned}$$

In virtue of the property of inferior limit,

$$\liminf_{n \rightarrow \infty} (a_n - b_n) \geq 0 \Rightarrow \liminf_{n \rightarrow \infty} (a_n - c_n) \geq \liminf_{n \rightarrow \infty} (b_n - c_n), \tag{20}$$

By (19) and inequality $1 - 1/x \leq \ln x \leq x - 1, (x > 0)$, we can write

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} (X_\sigma - \frac{1}{p_\sigma}) \\
 \geq & \liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\log \left(\frac{\lambda p_\sigma}{1 - \lambda(1 - p_\sigma)} \right) \right] / \log \lambda - \frac{1}{p_\sigma} \\
 \geq & \liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\left(\frac{\lambda - 1}{1 - \lambda(1 - p_\sigma)} \right) / \log \lambda - \frac{1}{p_\sigma} \right] \\
 \geq & \liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \left[\frac{\frac{\lambda - 1}{1 - \lambda(1 - p_\sigma)}}{1 - 1/\lambda} - \frac{1}{p_\sigma} \right] \\
 \geq & \liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{\lambda - 1}{p_\sigma [1 - \lambda(1 - p_\sigma)]} \\
 \geq & \liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{\lambda - 1}{p_\sigma [1 - \lambda(1 - \alpha)]} \\
 \geq & \frac{\lambda - 1}{\alpha} \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \geq \frac{\lambda - 1}{\alpha} M. \quad \mu_P - a.s. \quad \omega \in D(\omega) \quad (21)
 \end{aligned}$$

Select $\lambda_i \in (0, 1) (i = 1, 2, \dots)$ such that $\lambda_i \rightarrow 1^-$ (as $i \rightarrow \infty$), by (21) we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} (X_\sigma - \frac{1}{p_\sigma}) \geq 0. \quad \mu_P - a.s. \quad \omega \in D(\omega) \quad (22)$$

(18) and (22) imply that (5) is also valid. □

3 Some Corollaries for Random Field with Geometric Distributions on the Homogeneous Tree.

Corollary 3.1. *Let $X = \{X_\sigma, \sigma \in T\}$ be a random field which obeys the geometric distributions (2) indexed by the homogeneous tree T . α is defined by (3). Then*

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{\sigma \in L_k} (X_\sigma - \frac{1}{p_\sigma}) = 0. \quad \mu_P - a.s. \quad (23)$$

Proof. Letting $a_n = |T^{(n)}|, n \geq 0$ and $M = 1/\alpha$, we obtain $\lim_{n \rightarrow \infty} |T^{(n)}| = +\infty$ and

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{\alpha} = \limsup_{n \rightarrow \infty} \frac{|T^{(n)}|}{|T^{(n)}|} \frac{1}{\alpha} = \frac{1}{\alpha}.$$

It means $D(\omega) = \Omega$. (23) follows from (5) immediately. □

Corollary 3.2. Let $\{X_n, n \geq 0\}$ be a sequence of random variables which obeys the geometric distributions. Denote

$$\alpha = \inf \{p_n, n \geq 0\} > 0, \tag{24}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n (X_k - \frac{1}{p_k}) = 0. \quad P - a.s. \tag{25}$$

Proof. Letting $|S(\sigma)| \equiv 1, \sigma \in T^{(n)}$ in Theorem 1, that is, the successor of each vertex of the tree T has only one vertex, the random field which obey the geometric distributions indexed by the homogeneous tree degenerates into the sequence of random variables with the geometric distribution. At the moment, $|T^{(n)}| = n + 1, p_n = p_\sigma, n \geq 0.$ (25) follows from (5) immediately. \square

Corollary 3.3. Let $X = \{X_\sigma, \sigma \in T\}$ be a random field taking values in $S = \{1, 2, 3, \dots\}$ which obeys the geometric distributions (2) indexed by the homogeneous tree $T.$ Denote

$$\alpha = \inf \{p_\sigma, \sigma \in T^{(n)}, n \geq 0\} > 0, \tag{26}$$

Then

$$\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma / \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \right) = 1. \quad \mu_P - a.s. \tag{27}$$

Proof. Letting $a_n = \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma}, n \geq 0$ and $M = 1,$ we obtain

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} > \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{\sigma \in L_k} 1 = \lim_{n \rightarrow \infty} |T^{(n)}| = \infty.$$

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} = \limsup_{n \rightarrow \infty} \left(\sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} / \sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma} \right) = 1.$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^n \sum_{\sigma \in L_k} (X_\sigma - \frac{1}{p_\sigma}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma}} \sum_{k=0}^n \sum_{\sigma \in L_k} (X_\sigma - \frac{1}{p_\sigma}) = \lim_{n \rightarrow \infty} \frac{1}{\sum_{k=0}^n \sum_{\sigma \in L_k} \frac{1}{p_\sigma}} \sum_{k=0}^n \sum_{\sigma \in L_k} X_\sigma - 1 = 0. \end{aligned} \tag{28}$$

It means $D(\omega) = \Omega.$ (27) follows from (28) immediately. \square

Theorem 3.4. *Let $X = \{X_t, t \in T\}$ be a random field which obeys the geometric distribution (2) with the parameter p indexed by the homogeneous tree T . Then*

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} X_t \leq \log\left(\frac{ep}{1 - e(1-p)}\right). \quad \mu_P - a.s. \quad (29)$$

$$\liminf_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} X_t \geq \log\left(\frac{e-1+p}{p}\right). \quad \mu_P - a.s. \quad (30)$$

Proof. By (10) we have

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \log U_n(\lambda, \omega) \leq 0. \quad \mu_P - a.s. \quad (31)$$

By use of (8), (31) can be rewritten as

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \left[\sum_{k=0}^n \sum_{t \in L_k} X_t \log \lambda - \sum_{k=0}^n \sum_{t \in L_k} \log\left(\frac{\lambda p_t}{1 - \lambda(1-p_t)}\right) \right] \leq 0. \quad (32)$$

It implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} X_t \log \lambda \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} \log\left(\frac{\lambda p_t}{1 - \lambda(1-p_t)}\right). \quad (33)$$

Letting $\lambda = e$, $p_t \equiv p$, $t \in T$ in (33), we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} X_t \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} \log\left(\frac{ep}{1 - e(1-p)}\right) = \log\left(\frac{ep}{1 - e(1-p)}\right). \quad (34)$$

In the similar way, letting $\lambda = 1/e$, $p_t \equiv p$, $t \in T$ in (32), we have

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \left[- \sum_{k=0}^n \sum_{t \in L_k} X_t \log e - \sum_{k=0}^n \sum_{t \in L_k} \log\left(\frac{p/e}{1 - (1-p)/e}\right) \right] \leq 0. \quad (35)$$

That is

$$\liminf_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \left[\sum_{k=0}^n \sum_{t \in L_k} X_t + \sum_{k=0}^n \sum_{t \in L_k} \log\left(\frac{p}{e-1+p}\right) \right] \geq 0. \quad (36)$$

Moreover,

$$\liminf_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} X_t \geq \liminf_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^n \sum_{t \in L_k} \log\left(\frac{e-1+p}{p}\right) \geq \log\left(\frac{e-1+p}{p}\right). \quad (37)$$

Hence (29), (30) follow from (34) and (37) immediately. \square

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