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Existence Results for Fractional Quasilinear Integrodifferential Equations with Impulsive Conditions

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Abstract : In this article we prove the existence of solutions of fractional impulsive quasilinear integrodifferential equations considered in a Banach space. Further, Cauchy problems with nonlocal initial conditions are discussed for the mentioned fractional integrodifferential equations. At the end, one example is presented.

Keywords : existence of a solution; fractional differential equation; impulsive condition; abstract spaces; fixed point theorems.
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1 Introduction

The use of fractional differential equations has emerged as a new branch of applied mathematics, which has been used for constructing many mathematical models in science and engineering. In fact fractional differential equations are considered as models alternative to nonlinear differential equations [1] and other kinds of equations [2–4]. The theory of fractional differential and integrodifferential equations has been extensively studied by many authors [5–14]. In [15,16] the authors proved the existence of solutions of abstract fractional differential equations can be expressed as fractional differential or integrodifferential equations in some Banach spaces [17].

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Byszewski [18] initiated the study of nonlocal Cauchy problems for abstract evolution differential equations. Subsequently several authors discussed the problem for different kinds of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces [17–19]. Balachandran et al [20–28] established the existence of solutions of quasilinear integrodifferential equations with local and nonlocal conditions. In these papers the quasilinear operator is unbounded. Recently N'Guerekata [29] and Balachandran and Park [30] investigated the existence of solutions of fractional abstract differential equations with nonlocal condition. Benchohra and Seba [31] studied the existence problem for impulsive fractional differential equations in Banach spaces. Balachandran and Kiruthika [32] discussed the nonlocal Cauchy problem with an impulsive condition for semilinear fractional differential equations, whereas Chang and Nieto [33] studied the same problem for neutral integrodifferential equations via fractional operators. Belmekki et al [34] studied the existence of periodic solutions of nonlinear fractional differential equations. Cuevas and Cesar de Souza [19] discussed ω-periodic solutions of fractional integrodifferential equations. In this paper we study the existence of solutions of fractional quasilinear integrodifferential equations in Banach spaces by using the fractional calculus and the Banach fixed point theorem.

2 Preliminaries

We need some basic definitions and properties of fractional calculus which are used in this paper. Let X be Banach space and $\mathcal{R}_+ = [0,\infty)$. Suppoose $f \in L_1(\mathcal{R}_+)$. Let $C(J;X)$ be the Banach space of continuous functions $x(t)$ with $x(t)$ with $x(t) \in X$ for $t \in J = [0,a]$ and $\|x\|_{C(J;X)} = \max_{t\in J}\|x(t)\|$. Let $B(X)$ denotes the Banach space of bounded linear operators from X into X with the norm $\|A\|_{B(X)} = \sup\{\|A(y)\| : \|y\| = 1\}$.

Also consider the Banach space $PC(J;X) = \{u : J \to X;\ u \in PC((t_i,\ t_{i+1}];X)$, $i = 0,1,2,\cdots,m$ and there exist $u(t_{i^-})$ and $u(t_{i^+})$ $i = 1,2,3,\cdots,m$ with $u(t_{i^-}) = u(t_i)\}$ with the norm

$$\|u\|_{PC} = \sup_{t\in J}\|u(t)\|.\ \ \mathrm{Set} J' := [0,a]\{t_1,\cdots,t_m\}.$$

Definition 2.1. The Riemann-Lioville fractional integral operator of order $\alpha > 0$ of function $f \in L_1(\mathcal{R}_+)$ is defined as

$$I^{\alpha}_{0+} = \frac{1}{\Gamma(\alpha)}\int_d^t (t-s)^{\alpha-1} f(s)ds.$$

where $k \in \mathcal{R}_+$ and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative order $\alpha > 0$ $n-1 < \alpha < n,\ n \in N$, is defined as

$$^{R-L}D^{\alpha}_{0+} = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^n \int_k^t (t-s)^{n-\alpha-1} f(s)ds$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

The Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann-Liouville sense require initial conditions in some point different to $x_0 = k$. To over come this issue Caputo [35] defined the fractional derivative in the following way.

Definition 2.3. The Caputo fractional derivative order $\alpha > 0$ $n-1 < \alpha < n$, is defined as

$$^C D^{\alpha}_{0+} = \frac{1}{\Gamma(n-\alpha)} \int_k^t (t-s)^{n-\alpha-1} f^n(s) ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$. If $0 < \alpha < 1$, then

$$^C D^{\alpha}_{0+} = \frac{1}{\Gamma(n-\alpha)} \int_k^t \frac{f'(s)}{(t-s)^{\alpha}} ds,$$

where $f'(s) = Df(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X.

Now we shall state some properties of the operators I^{α}_{0+} and $^C D^{\alpha}_{0+}$.

Properties 2.4. For $\alpha, \beta > 0$ and f as a suitable function (for instance [34] and [36]) we have

(i) $I^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\alpha+\beta}_{0+} f(t)$;

(ii) $I^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\beta}_{0+} I^{\alpha}_{0+} f(t)$;

(iii) $I^{\alpha}_{0+}(f(t) + g(t)) = I^{\alpha}_{0+} f(t) + I^{\alpha}_{0+} g(t)$;

(iv) $I^{\alpha}_{0+}\ {}^C D^{\alpha}_{0+} f(t) = f(t) - f(0),\ 0 < \alpha < 1$;

(v) $^C D^{\alpha}_{0+} I^{\alpha}_{0+} f(t) = f(t)$;

(vi) $^C D^{\alpha}_{0+} f(t) = I^{1-\alpha}_{0+} Df(t) = I^{1-\alpha}_{0+} f'(t),\ 0 < \alpha < 1$;

(vii) $^C D^{\alpha}_{0+}\ {}^C D^{\beta}_{0+} f(t) \neq\ {}^C D^{\alpha+\beta}_{0+} f(t)$;

(viii) $^C D^{\alpha}_{0+}\ {}^C D^{\beta}_{0+} f(t) \neq\ {}^C D^{\beta}_{0+}\ {}^C D^{\alpha}_{0+} f(t)$.

We observe from the above that both the Riemann-Liouville and the Caputo fractional operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. For basic facts about fractional integral and fractional derivative one can refer the books [36–39]. For our convenience, let us take $^C D^{\beta}_{0+}$ with the notation $^C D^{\beta}$.

Consider the linear fractional impulsive evolution equation

$$\begin{aligned} ^C D^q u(t) &= A(t)u(t) + f(t), 0 \le t \le a, \\ \Delta u|_{t=t_i} &= I_i(u(t_i^-)), \quad i = 1, 2, 3, \ldots, m \\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where $0 < q < 1$, $A(t)$ is a bounded linear operator on a Banach space X, $u_0 \in X$ and $f : J \to X$ is continuous, $I_i : X \to X$ and $u_0 \in X$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$, $\Delta u|_{t=t_i} = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$ and $\lim_{h\to 0^-} u(t_k + h)$ represent the right and the left limits of $u(t)$ at $t = t_i$.

It is easy to prove that the equation (2.1) is equivalent to the integral equation

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)u(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds, if t \in [0, t_1], \\ &= u_0 + \frac{1}{\Gamma(q)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1}A(s)u(s)ds + \frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}A(s)u(s)ds \\ &+\frac{1}{\Gamma(q)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1}f(s)u(s)ds + \frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}f(s)u(s)ds \\ &+\sum_{i=1}^{m} I_i(u(t_i^-)), \quad \text{if} \quad t \in (t_i, t_{i+1}]. \end{aligned} \tag{2.2}$$

By a local solution of the abstract Cauchy problem (2.1), we mean an abstract function u such that the following conditions are satisfied:

(i) $u \in PC(J; X)$ and $u \in \mathcal{D}(A(t))$ on J' (here $\mathcal{D}$ for domain);

(ii) $\frac{d^q u}{dt^q}$ exists and continuous on J', where $0 < q < 1$;

(iii) u satisfies equation (2.1) on J' and satisfies the conditions $\Delta u|_{t=t_i} = I_i(u(t_k^-)$, $u(0) = u_0 \in X$ or that it is equivalent u satisfy the integral equation (2.2).

3 Quasilinear Integrodifferential Equations

Consider the fractional quasilinear integrodifferential equation of the form

$$\begin{aligned} {}^C D^q u(t) &= A(t,u)u(t) + f(t,u(t)) + \int_0^t h(t,s,u(s)ds), 0 \le t \le a, && (3.1) \\ \Delta u|_{t=t_i} &= I_i(u(t_i^-)), \quad i = 1, 2, 3, \ldots, m && (3.2) \\ u(0) &= u_0, && (3.3) \end{aligned}$$

where $A(t, u)$ is a bounded linear operator on X and $f : J \times X \to X$, $h : \Omega \times X \to X$ are continuous. The nonlinear function f of this type with integral term h occurs in mathematical problems concerned with heat flow in materials with memory and viscoelastic problems in which the integral term represents the viscosity part of the problem [16]. Here $\Omega = (t, s) : 0 \le s \le t \le a$.

It is easy to prove that the equation (3.1) is equivalent to the following integral equation

$$
\begin{aligned}
u(t) &= u_0 + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} A(s,u)u(s)ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} A(s,u)u(s)ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau \Big] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau \Big] ds \\
&\quad + \sum_{0<t_i<t} I_i(u(t_i^-)). \qquad (3.4)
\end{aligned}
$$

We need the following assumptions to prove the existence of solution of the integral equation (3.1)-(3.3).

(E_1) $A : J\times \to B(X)$ is a continuous bounded linear operator and there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$
\begin{aligned}
\|A(t,u) - A(t,v)\| &\leq M_1\|u - v\|, \text{ for all } u, v \in X, \\
M_2 &= \sup_{t\in[0,a]} \|A(t,0)\|
\end{aligned}
$$

(E_2) $f : J \times X \to X$ is continuous and there exist constants $F_L > 0$ and $F_0 > 0$ such that

$$
\begin{aligned}
\|f(t,u) - f(t,v)\|_X &\leq F_L\|u - v\|, \text{ for all } \quad u, v \in X, \\
F_0 &= \max_{t\in J} \|f(t,0)\|.
\end{aligned}
$$

(E_3) $h : \Omega \times X \to X$ is continuous and there exist constants $H_L > 0$ and $H_0 > 0$ such that

$$
\begin{aligned}
\int_0^t \|h(t,s,u) - h(t,s,v)\|ds &\leq H_L\|u - v\|, \text{ for all } u, v \in X, \\
H_0 &= \max\{\int_0^t \|h(t,s,0)\|ds : (t,s) \in [0,a]\}.
\end{aligned}
$$

(E_4) $I_i : X \to X$ is continuous and there exist constant $l_i > 0, i = 1, 2, 3, \ldots, m$ such that

$$
\begin{aligned}
\|I_i(u) - I_i(v)\| &\leq l_i\|u - v\|, \quad u, v \in X \quad \text{and} \\
l_c = \|I_i(0)\| &
\end{aligned}
$$

Let $B_r := \{u \in X : \|u\| \leq r\}$ for some $r > 0$. For brevity let us take $\eta = \frac{a^q}{\Gamma(q+1)}$.

From (E_1) we observe that

$$\|A(t,u)\| \leq \|A(t,u) - A(t,0)\| + \|A(t,0)\| \leq M_1\|u\| + \|A(t,0)\| \leq M_1 r + M_2.$$

Further assume that

(E_5) $\|u_0\| + (m+1)\eta\Big[(M_1r + M_2)r + (F_L + H_L)r + F_0 + H_0)\Big] + m(l_i r + l_c) \leq r,$

(E_6) Let $\rho = \Big[\eta(m+1)(2M_1r + M_2 + F_L + H_L) + ml_i\Big]$ be such that $0 \leq \rho < 1$.

Theorem 3.1. *If the hypotheses* $(E_1) - (E_6)$ *are satisfied, then the fractional quasilinear integrodifferential equation* (3.1)- (3.3) *has a unique solution continuous in* J.

Proof. Let $Z = C([0,a]; B_r)$. Define the mapping $\mathcal{F} : Z \to Z$ by

$$\begin{aligned}
\mathcal{F}u(t) \quad = \quad & u_0 + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} A(s,u)u(s)ds \\
& + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} A(s,u)u(s)ds \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} A(s,u)u(s)ds\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds \\
& + \frac{1}{\Gamma(q)} \int_{ti}^{t} (t-s)^{q-1} A(s,u)u(s)ds\Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\Big]ds \\
& + \sum_{0<t_i<t} I_i(u(t_i^-)).
\end{aligned} \tag{3.5}$$

and we have to show that $\mathcal{F}$ has a fixed point. This fixed point is then a solution of the equation (3.1)- (3.3). First we show that $\mathcal{F}B_r \subset B_r$.

From the assumptions we have

$$\begin{aligned}
\|\mathcal{F}u(t)\| \quad \leq \quad & \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \|A(s,u)\| \|u(s)\| ds \\
& + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \|A(s,u)\| \|u(s)\| ds \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \Big[\|f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\|\Big]ds \\
& + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \Big[\|f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau\|\Big]ds \\
& + \sum_{0<t_i<t} \|I_i(u(t_i^-))\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \|A(s,u)\| \|u(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \|A(s,u)\| \|u(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \Big[\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\| \\
&\quad + \int_0^s [\|h(s,\tau,u(\tau)) - h(s,\tau,0)\| + \|h(s,\tau,0)\|] d\tau \Big] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \Big[\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\| \\
&\quad + \int_0^s [\|h(s,\tau,u(\tau)) - h(s,\tau,0)\| + \|h(s,\tau,0)\|] d\tau \Big] ds \\
&\quad + \sum_{0<t_i<t} \|I_i(u(t_i^-)) - I_i(0) + I_i(0)\| \\
&\leq \|u_0\| + (m+1)[M_1 r + M_2] r \frac{a^q}{\Gamma(q+1)} \\
&\quad + (m+1) \Big[(F_L + H_L) r + F_0 + H_0 \Big] \frac{a^q}{\Gamma(q+1)} + m(l_i r + l_c) \\
&= \|u_0\| + (m+1)\eta \Big[(M_1 r + M_2) r + (F_L + H_L) r + F_0 + H_0 \Big] \\
&\quad + m(l_i r + l_c) \\
&\leq r.
\end{aligned}
$$

Thus, $\mathcal{F}$ maps B_r into itself. Now, for $u_1, u_2 \in Z$, we have

$$
\begin{aligned}
\|\mathcal{F}u_1(t) - \mathcal{F}u_2(t)\| &\leq \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \|A(s,u_1)u_1(s) - A(s,u_2)u_2(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \|A(s,u_1)u_1(s) - A(s,u_2)u_2(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \Big[\|f(s,u_1(s)) - f(s,u_2(s))\| \\
&\quad + \int_0^s \|h(s,\tau,u_1(\tau)) - h(s,\tau,u_2(\tau))\| d\tau \Big] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} \Big[\|f(s,u_1(s)) - f(s,u_2(s))\| \\
&\quad + \sum_{0<t_i<t} \|I_i(u_1(t_i^-)) - I_i(u_2(t_i^-))\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} [\|A(s,u_1)(u_1(s) - u_2(s)\| \\
&\quad + \|(A(s,u_1) - A(s,u_2))u_2(s)\|] ds \\
&= \rho \, \|u_1(t) - u_2(t)\|.
\end{aligned}
$$

Since $0 \le \rho < 1$, then $\mathcal{F}$ is a contraction mapping and therefore there exists a unique fixed point $u \in Z$ such that $\mathcal{F}u(t) = u(t)$. Any fixed point of $\mathcal{F}$ is the solution of (3.1). □

4 Quasilinear Nonlocal Cauchy Problem

In this section we discuss the existence of solution of fractional impulsive quasilinear integrodifferential equation (3.1)-(3.2) with nonlocal condition of the form

$$u(0) + g(u) = u_0, \tag{4.1}$$

where $g : PC(J;X) \to X$ is a given function. We assume the following conditions

(E_7) $g : PC(J;X) \to X$ is continuous and there exists a constant $G_L > 0$ such that
$$\|g(u) - g(v)\| \le G_L \|u - v\|_{PC}, \text{ for } u, v \in PC([0,a];X).$$

(E_8) $\|u_0\| + G_L r + \|g(0)\| + (m+1)\eta\Big[(M_1 r + M_2)r + (F_L + H_L)r + F_0 + H_0)\Big] + m(l_i r + l_c) \le r.$

(E_9) $\widehat{\rho} = \Big[[G_L + (m+1)\eta\Big(2M_1 r + M_2 + F_L + H_L\Big) + m l_i\Big]$ be such that $0 \le \widehat{\rho} < 1$.

Theorem 4.1. *If the hypotheses $(E_1) - (E_4)$, $(E_7) - (E_9)$ are satisfied, then the fractional quasilinear integrodifferential equation (3.1)-(3.2) with nonlocal condition (4.1) has a unique solution continuous in J.*

Proof. We want to prove that the operator $\mathcal{P} : Z \to Z$ defined by

$$
\begin{aligned}
\mathcal{P}u(t) &= u_0 - g(u) + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} A(s,u)u(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_i}^{t} (t-s)^{q-1} A(s,u)u(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau)) d\tau \Big] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{ti}^{t} (t-s)^{q-1} \Big[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau)) d\tau \Big] ds \\
&\quad + \sum_{0<t_i<t} I_i(u(t_i^-))
\end{aligned} \tag{4.2}
$$

has a fixed point. Then this fixed point is a solution of the equation (3.1)-(3.2) and (4.1). Then from the assumption (E_8) it is easy to see that $\mathcal{P}B_r \subset B_r$. Now, for $u_1, u_2 \in Z$, we have

$$\begin{aligned}\|\mathcal{P}u_1(t) - \mathcal{P}u_2(t)\| &\leq \Big[G_L + (m+1)\eta\Big(2M_1 r + M_2 + F_L + H_L\Big) \\ &\quad + ml_i\Big]\|u_1(t) - u_2(t)\| \\ &= \widehat{\rho}\,\|u_1(t) - u_2(t)\|.\end{aligned}$$

Since $0 \leq \widehat{\rho} < 1$, the result follows by the application of contraction mapping principle. $\square$

5 Quasilinear Delay Integrodifferential Equations

In this section we discuss the existence of solution of fractional impulsive delay quasilinear integrodifferential equation (3.2) and (4.1) with the nonlocal condition of the form

$$^{C}D^q u(t) = A(t,u)u(t) + f(t,u(t),u(\alpha(t))) + \int_0^t h(t,s,u(s),u(\beta(s)ds), 0 \leq t \leq a, \quad (5.1)$$

where α, β: $J \to J$ are continuous and A, f, g, h are as for above. Assume the following additional conditions:

(E_{10}) $\|u_0\| + G_L r + \|g(0)\| + (m+1)\eta\Big[(M_1 r + M_2)r + 2r(F_L + H_L) + F_0 + H_0)\Big] + m(l_i r + l_c) \leq r.$

(E_{11}) $p^* = \Big[G_L + (m+1)\eta\Big(2M_1 r + M_2 + 2F_L + 2H_L\Big) + ml_i\Big]$ be such that $0 \leq p^* < 1$.

Theorem 5.1. *If the hypotheses* $(E_1) - (E_4)$, $(E_{10}) - (E_{11})$*are satisfied, then the fractional quasilinear integrodifferential equation* (3.2), (4.1) *and* (5.1) *has a unique solution continuous in* J.

Proof. We want to prove that the operator $\mathcal{Q} : Z \to Z$ defined by

$$\begin{aligned}\mathcal{Q}u(t) &= u_0 - g(u) + \frac{1}{\Gamma(q)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1}A(s,u)u(s)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}A(s,u)u(s)ds \\ &\quad + \frac{1}{\Gamma(q)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1}\Big[f(s,u(s),u(\alpha(s))) + \int_0^s h(s,\tau,u(\tau),u(\beta(\tau))d\tau\Big]ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_i}^{t}(t-s)^{q-1}\Big[[f(s,u(s),u(\alpha(s))) + \int_0^s h(s,\tau,u(\tau),u(\beta(\tau))d\tau\Big]ds \\ &\quad + \sum_{0<t_i<t} I_i(u(t_i^-))\end{aligned} \quad (5.2)$$

has a fixed point. This fixed point is then a solution of the equation (3.2), (4.1) and (5.1). Then from the assumption (E_{10}) it is easy to see that $\mathcal{Q}B_r \subset B_r$. Now, for $u_1, u_2 \in Z$, we have

$$\begin{aligned}\|\mathcal{Q}u_1(t) - \mathcal{Q}u_2(t)\| &\leq \Big[G_L + (m+1)\eta\Big(2M_1 r + M_2 + 2F_L + 2H_L\Big) + ml_i\Big]\|u_1(t) - u_2(t)\| \\ &= p^* \, \|u_1(t) - u_2(t)\|.\end{aligned}$$

Since $0 \leq p^* < 1$, the result follows by the application of contraction mapping principle. $\square$

6 Example

Consider the following fractional integrodifferential equation

$$\begin{aligned}{}^C D^q u(t) &= \frac{1}{100} sin\; u(t)u(t) + \frac{e^{-t}u(t)}{(49+e^t)(1+u(t))} + \int_0^t e^{-\frac{1}{4}u(s)} ds, &(6.1)\\ \Delta u|_{t=\frac{1}{2}} &= \frac{|u(\frac{1}{2}-|}{9+|u(\frac{1}{2}-)|}, \quad i = 1,2,3,\ldots,m, \quad t \in J, &(6.2)\\ u(0) &= u_0. &(6.3)\end{aligned}$$

where $0 < q \leq 1$. Take $X = \mathcal{R}^+,\ t \in [0,1]$ and so $a = 1$. Set

$$\begin{aligned}A(t,u) &= \frac{1}{100} sin\; u(t) I,\\ f(t,u) &= \frac{e^{-t}|u|}{(49+e^t)(1+|u|)},\\ \int_0^t h(t,s,u(s)ds &= \int_0^t e^{-\frac{1}{4}u(s)} ds, \qquad \text{and}\\ I_i(u) &= \frac{|u|}{4+|u|}, \quad u \in X.\end{aligned}$$

Let $u, v \in C([J;X)$ and $t \in J$. Then we have

$$\begin{aligned}\|\int_0^t [h(t,s,u(s)) - h(t,s,v(s))]ds\| &= \Big|\int_0^t e^{-\frac{1}{4}u(s)}ds - \int_0^t e^{-\frac{1}{4}v(s)}ds\Big| \leq \frac{1}{4}|u-v|.\\ \|f(t,u) - f(t,v)\| &= \frac{e^{-t}}{(49+e^t)}\Big|\frac{u}{1+u} - \frac{v}{1+v}\Big|\\ &= \frac{e^{-t}|u-v|}{(49+e^t)(1+u)(1+v)}\\ &= \frac{e^{-t}}{(49+e^t)}|u-v|\\ &\leq \frac{1}{50}|u-v|\end{aligned}$$

Hence the condition $(E_1) - (E_3)$ hold with $M_1 = \frac{1}{100}, F_L = \frac{1}{50}, H_L = \frac{1}{4}$. Here $M_2 = \frac{1}{100}$. Let $u, v \in X$. Then we have by (E_4)

$$\|I_i(u) - I_i(v)\| = \left|\frac{u}{9+u} - \frac{v}{9+v}\right| = \frac{9|u-v|}{(9+u)(9+v)} \leq \frac{1}{9}|u-v|.$$

Here $l_i = \frac{1}{9}$. Choose $r = 1, m = 1$. We shall check that condition

$$\eta(m+1)(2M_1r + M_2 + F_L + H_L) + ml_i < 1$$

is satisfied. Indeed

$$\eta(m+1)(2M_1r + M_2 + F_L + H_L) + ml_i < 1 \ \Leftrightarrow \Gamma(q+1) > \frac{10}{27}, \tag{6.4}$$

which is satisfied for some $q \in (0,1]$. Further (E_4) is satisfied by a suitable choice of u_0. Then by Theorem 3.1 the problem (6.1)- (6.3)has a unique solution on $[0,1]$ for the values of q satisfying (6.4). Next we consider the following nonlocal condition

$$u(0) + \sum_{i=1}^{m} c_i u(t_i) = u_0$$

for the fractional integrodifferential equation. Here the function $g(u) = \sum_{i=1}^{m} c_i u(t_i)$ and the constant $G = \sum_{i=1}^{m} c_i$. By the similar way one can easily verify that the conditions $(E_6) - (E_8)$ by properly choosing c_i and u_0. Hence the equation (6.1) with the above nonlocal condition has a unique solution by Theorem 4.1.

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