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Non-Archimedean Random Stability of σ-Quadratic Functional Equation

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Abstract : The aim of this paper is to investigate the generalized Hyers - Ulam stability of the following quadratic functional equation

$$f(ax+by) = a^2g(x) + b^2h(y) + \frac{ab}{2}[f(x+y) - f(x+\sigma(y))]$$

in non-Archimedean RN-spaces, by using the fixed point method.

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1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940. D. H. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers theorem was generalized by T. Aoki [3] for additive mappings and by Rassias [4] for linear mappings. The paper of Rassias [4] has been influential in the development of what is now known as the generalized Hyers-Ulam stability or Hyers-Ulam Rassias stability of functional equations. A generalization of the Rasssias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference with a general control function in the spirit of Rassias approach.

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The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called the quadratic functional equation. A generalized Hyers-Ulam stability for the quadratic functional equation was proved by F. Skof [6] for the function $f: X \to Y$ where X is a normal space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. Czerwik [8] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). Park [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation in Banach modules over a $\mathbb{C}^*$ algebra. The stability problem of several functional equations have been extensively investigated by number mathematicians ([10–19]).

In [20], A. Najati and G. Park showed that the functional equation

$$f(ax+by)=a^2f(x)+b^2f(y)+\frac{ab}{2}[f(x+y)-f(x-y)] \tag{1.2}$$

is equivalent to the quadratic functional equation (1.1), if a, b are rational numbers such that $a^2+b^2 \neq 1$ and, they proved the stability problem of this equation.

Throughout this paper, assume that X be a vector space over a non-Archimedean field $\mathbb{K}$, (Y,μ,T) is a non-Archimedean random Banach space over $\mathbb{K}$ and suppose $\sigma(\sigma(x))=x$ and $\sigma(x+y)=\sigma(x)+\sigma(y)$, for all $x,y\in X$.

In this paper, using the fixed point method, we will prove the generalized stability of the following equation:

$$f(ax+by)=a^2g(x)+b^2h(y)+\frac{ab}{2}[f(x+y)-f(x+\sigma(y))] \tag{1.3}$$

where $a,b\in\mathbb{N}\backslash\{0,1\}$.

In the sequel, we shall adopt the usual terminologies, notions, and conventions of the theory of non-Archimedean random normed spaces (non-ARN-spaces) as in [21–23]. In this paper, the space of all probability distribution functions is denoted by Δ^+. Elements of Δ^+ are functions $F:\mathbb{R}\cup\{-\infty,\infty\}\to[0,1]$, such that F is left continuous and nondecreasing on $\mathbb{R}$ and $F(0)=0$, $F(+\infty)=1$. It's clear that the subset

$$D^+ := \{F\in\Delta^+ : l^-F(+\infty)=1\},$$

where $l^-f(x)=\lim_{t\to x^-}f(t)$, is a subset of Δ^+. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F\leq G$ if and only if $F(t)\leq G(t)$ for all $t\in\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t)=\begin{cases}1, & \text{if } t>0,\\ 0, & \text{if } t\leq 0.\end{cases}$$

2 Preliminaries

In this section, we give the definition and theorems that are important in the following.

Theorem 2.1 ([24])**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strict contractive mapping with a Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(J^{k+1}x, J^k x) < \infty$ for some $x \in X$, then the followings are true:*

1. *the sequence $\{J^n x\}$ converge to a fixed point x^* for J,*
2. *x^* is the unique fixed point of J in*
$$X^* = \left\{y \in X, d(J^k x, y) < \infty\right\},$$
3. *if $y \in X^*$, then*
$$d(y, x^*) \leq \frac{1}{1-L} d(Jy, y).$$

Definition 2.2 ([23])**.** *A mapping $T : [0,1]^2 \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:*

1. *T is commutative and associative;*
2. *T is continuous;*
3. *$T(a, 1) = a$ for all $a \in [0, 1]$;*
4. *$T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.*

Typical examples of continuous t-norms are $T_p(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a+b-1, 0)$ (the Lukasiewicz t-norm). Recall (see [25,26]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, ..., x_n)$ for $n \geq 1$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. It is known([26]) that for the Lukasiewicz t-norm the following holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

Definition 2.3. *By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function(valuation) $|.| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:*

1. *$|r| = 0$ if and only if $r = 0$;*
2. *$|rs| = |r||s|$;*
3. *$|r + s| \leq \max(|r|, |s|)$ for all $r, s \in \mathbb{K}$.*

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. The function $|.|$ is called the trivial valuation if $|r| = 1$, $\forall r \in \mathbb{K}$, $r \neq 0$, and $|0| = 0$.

Definition 2.4. *Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $\|.\| : X \to \mathbb{R}$ is non-Archimedean norm (valuation) if it satisfies the following conditions:*

1. $\|x\| = 0$ *if and only if* $x = 0$*;*
2. $\|rx\| = |r|\|x\|$ *for all* $r \in \mathbb{K}$ *and* $x \in X$*;*
3. $\|x + y\| \leq \max(\|x\|, \|y\|)$ *for all* $x, y \in X$*.*
   Then, $(X, \|.\|)$ is called a non-Archimedean space. Due to the fact that

$$\|x_m - x_n\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\},$$

in which $n > m$, the sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. In a complete non-Archimedean space, every Cauchy sequence is convergent.

Definition 2.5 ([27])**.** *A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a linear space over a non-Archimedean field $\mathbb{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:*

1. $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*
2. $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ *for all* $x \in X$, $t \geq 0$ *and* $\alpha \neq 0$*;*
3. $\mu_{x+y}(\max(t, s)) \geq T(\mu_x(t), \mu_y(s))$ *for all* $x, y \in X$ *and* $t, s \geq 0$*.*

It is easy to see that if (3) holds, then (3'): $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every non-Archimedean normed linear space $(X, \|.\|)$ defines a non-Archimedean RN-space (X, μ, T_M) where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ and $x \in X$.

Definition 2.6. *Let (X, μ, T) be a non-Archimedean RN-space.*

1. *A sequence $\{x_n\}$ in X is said to be convergent to x in X if for all $t > 0$,* $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$*;*
2. *A sequence $\{x_n\}$ in X is said to be Cauchy sequence in X if for each $\varepsilon > 0$ and $t > 0$, there exist a positive integer n_0 such that for all $n \geq n_0$ and $p > 0$, we have*

$$\mu_{x_{n+p} - x_n}(t) > 1 - \varepsilon;$$

3. *A non-Archimedean RN-space (X, μ, T) is said to be complete (i.e.,(X, μ, T) is called a non-Archimedean random Banach space) if every Cauchy sequence in X is convergent to a point in X.*

Theorem 2.7 ([23]). *If (X, μ, T) is a non-Archimedean RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

3 Stability of Equation (1.3) in non-Archimedean RN-Spaces

In the rest of the paper, we take $f, g, h : X \to Y$ and we define

$$D_g^h f(x,y) = f(ax+by) - a^2 g(x) - b^2 h(y) - \frac{ab}{2}[f(x+y) - f(x+\sigma(y))]$$

where a, b in $\mathbb{N}\backslash\{0,1\}$.

Theorem 3.1 ([28, Theorem 2.1]). *A mapping $f : X \to Y$ satisfies*

$$f(ax+by) = a^2 f(x) + b^2 f(y) + \frac{ab}{2}[f(x+y) - f(x+\sigma(y))] \tag{3.1}$$

if and only if f satisfies

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \text{ and } f(x+\sigma(x)) = 0 \tag{3.2}$$

for all $x, y \in X$.

Now using fixed point approach to the non-Archimedean RN-space under arbitrary t-norm, we prove the stability of the σ- quadratic functional equation $D_g^h f(x,y) = 0$.

Theorem 3.2. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 4$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.3}$$

for all $x, y \in X$ and $t > 0$. If $f, g, h : X \to Y$ be an even mapping such that

$$\mu_{D_g^h f(x,y)}(t) \geq \varphi_{x,y}(t), \tag{3.4}$$

and $f(0) = g(0) = h(0) = 0$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (3.1) and

$$\mu_{f(x)-\frac{1}{2}f(x+\sigma(x))-Q(x)}(t) \geq \phi_{x,x}(\frac{4-\lambda}{4}t), \tag{3.5}$$

$$\mu_{g(x)-\frac{1}{2}g(x+\sigma(x))-Q(x)}(t) \geq T_M(\phi_{ax,ax}(\frac{(4-\lambda)a^2}{4}t), T_M(\varphi_{x,0}(a^2t), \varphi_{x+\sigma(x),0}(2a^2t))), \tag{3.6}$$

and

$$\mu_{h(x)-\frac{1}{2}h(x+\sigma(x))-Q(x)}(t) \geq T_M(\phi_{bx,bx}(\frac{(4-\lambda)b^2}{4}t), T_M(\varphi_{0,x}(b^2t), \varphi_{0,x+\sigma(x)}(2b^2t))), \tag{3.7}$$

for all $x \in X$ *and* $t > 0$, *where*

$$\phi_{x,y}(t) = T_M(T_M(\psi_{x,y}(t), \psi_{x+\sigma(x),y+\sigma(y)}(2t)), \psi_{x+\sigma(x),y+\sigma(y)}(4t)),$$

and

$$\psi_{x,y}(t) = T_M(\varphi_{\frac{x}{a},\frac{y}{b}}(t), \varphi_{\frac{x}{a},\frac{\sigma(y)}{b}}(t), \varphi_{\frac{x}{a},0}(\frac{t}{2}), \varphi_{0,\frac{y}{b}}(\frac{t}{2})).$$

Moreover

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}(f(2^nx) - \frac{1}{2}f(2^nx + 2^n\sigma(x))).$$

Proof. Putting $y = 0$ in (3.4) we get

$$\mu_{f(ax)-a^2g(x)}(t) \geq \varphi_{x,0}(t) \tag{3.8}$$

for all $x \in X$ and $t > 0$.
Similarly, for all $y \in X$, we can put $x = 0$ in (3.4) to obtain

$$\mu_{f(by)-b^2h(y)}(t) \geq \varphi_{0,y}(t). \tag{3.9}$$

Also replace y by $\sigma(y)$ in (3.4)

$$\mu_{D_g^h f(x,\sigma(y))}(t) \geq \varphi_{x,\sigma(y)}(t). \tag{3.10}$$

Hence, (3.8), (3.9) and (3.10) imply

$$\mu_{D_g^h f(x,y)+D_g^h f(x,\sigma(y))-2D_g^h f(x,0)-2D_g^h f(0,y)}(t) \geq T_M(\mu_{D_g^h f(x,y)}(t), \mu_{D_g^h f(x,\sigma(y))}(t), \mu_{2D_g^h f(x,0)}(t), \mu_{2D_g^h f(0,y)}(t)),$$

i.e.,

$$\mu_{D_g^h f(x,y)+D_g^h f(x,\sigma(y))-2D_g^h f(x,0)-2D_g^h f(0,y)}(t) \geq T_M(\varphi_{x,y}(t), \varphi_{x,\sigma(y)}(t), \varphi_{x,0}(\frac{t}{2}), \varphi_{0,y}(\frac{t}{2}))$$

for all $x, y \in X$ and $t > 0$.
Then, we have

$$\mu_{f(ax+by)+f(ax+b\sigma(y))-2f(ax)-2f(by)}(t) \geq T_M(\varphi_{x,y}(t), \varphi_{x,\sigma(y)}(t), \varphi_{x,0}(\frac{t}{2}), \varphi_{0,y}(\frac{t}{2})) \tag{3.11}$$

for all $x, y \in X$ and $t > 0$.
Replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$ in (3.11) we get

$$\mu_{f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)}(t) \geq \psi_{x,y}(t), \tag{3.12}$$

where $\psi_{x,y}(t) = T_M(\varphi_{\frac{x}{a},\frac{y}{b}}(t), \varphi_{\frac{x}{a},\frac{\sigma(y)}{b}}(t), \varphi_{\frac{x}{a},0}(\frac{t}{2}), \varphi_{0,\frac{y}{b}}(\frac{t}{2}))$.
Also, we can replace x by $x + \sigma(x)$ and y by $y + \sigma(y)$ in (3.12) we get

$$\mu_{f(x+\sigma(x)+y+\sigma(y))+f(x+\sigma(x)+y+\sigma(y))-2f(x+\sigma(x))-2f(y+\sigma(y))}(t) \geq \psi_{x+\sigma(x),y+\sigma(y)}(t) \tag{3.13}$$

for all $x, y \in X$ and $t > 0$.
Now we put $F(x) = f(x) - \frac{1}{2}f(x + \sigma(x))$ and by (3.12) and (3.13) we have

$$\begin{aligned}
&\mu_{F(x+y)+F(x+\sigma(y))-2F(x)-2F(y)}(t)\\
\geq\ & T_M(\mu_{f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)}(t),\\
&\mu_{-\frac{1}{2}f(x+\sigma(x)+y+\sigma(y))-\frac{1}{2}f(x+\sigma(x)+y+\sigma(y))+f(x+\sigma(x))+f(y+\sigma(y))}(t))\\
\geq\ & T_M(\psi_{x,y}(t), \mu_{f(x+\sigma(x)+y+\sigma(y))+f(x+\sigma(x)+y+\sigma(y))-2f(x+\sigma(x))-2f(y+\sigma(y))}(2t))\\
\geq\ & T_M(\psi_{x,y}(t), \psi_{x+\sigma(x),y+\sigma(y)}(2t)),
\end{aligned}$$

that is,

$$\mu_{F(x+y)+F(x+\sigma(y))-2F(x)-2F(y)}(t) \geq T_M(\psi_{x,y}(t), \psi_{x+\sigma(x),y+\sigma(y)}(2t)). \tag{3.14}$$

If we replace in the first y by x in (3.14) and in the second x and y by $x+\sigma(x)$ in (3.12), we obtain

$$\mu_{F(2x)+F(x+\sigma(x))-4F(x)}(t) \geq T_M(\psi_{x,x}(t), \psi_{x+\sigma(x),x+\sigma(x)}(2t)) \tag{3.15}$$

and

$$\mu_{2f(2x+2\sigma(x))-4f(x+\sigma(x))}(t) \geq \psi_{x+\sigma(x),x+\sigma(x)}(t). \tag{3.16}$$

From (3.15) and (3.16) we have

$$\begin{aligned}
\mu_{F(2x)-4F(x)}(t) &\geq T_M(\mu_{F(2x)+F(x+\sigma(x))-4F(x)}(t), \mu_{-F(x+\sigma(x))}(t))\\
&\geq T_M(T_M(\psi_{x,x}(t), \psi_{x+\sigma(x),x+\sigma(x)}(2t)), \mu_{F(x+\sigma(x))}(t))\\
&= T_M(T_M(\psi_{x,x}(t), \psi_{x+\sigma(x),x+\sigma(x)}(2t)), \mu_{\frac{-1}{4}(2f(2x+2\sigma(x))-4f(x+\sigma(x)))}(t))\\
&= T_M(T_M(\psi_{x,x}(t), \psi_{x+\sigma(x),x+\sigma(x)}(2t)), \mu_{2f(2x+2\sigma(x))-4f(x+\sigma(x))}(4t))\\
&\geq T_M(T_M(\psi_{x,x}(t), \psi_{x+\sigma(x),x+\sigma(x)}(2t)), \psi_{x+\sigma(x),x+\sigma(x)}(4t)).
\end{aligned}$$

That is,

$$\mu_{F(2x)-4F(x)}(t) \geq \phi_{x,x}(t) \tag{3.17}$$

where $\phi_{x,y}(t) = T_M(T_M(\psi_{x,y}(t), \psi_{x+\sigma(x),y+\sigma(y)}(2t)), \psi_{x+\sigma(x),y+\sigma(y)}(4t))$.

Now, we define the set S by

$$S := \{F : X \to Y\}$$

and introduce a generalized metric on S as follows

$$d_\phi(F,G) = \inf\{\varepsilon \in \mathbb{R}_+ : \mu_{F(x)-G(x)}(\varepsilon t) \geq \phi_{x,x}(t), \forall x \in X, \forall t > 0\}. \tag{3.18}$$

Then, it is easy to verify that (S, d_ϕ) is complete (see [29]). We define an operator $J : S \to S$ by

$$JL(x) = \frac{L(2x)}{4},$$

for all $x \in X$.
Let $F, G \in S$ and $\varepsilon \in \mathbb{R}_+$ be an arbitrary constant with $d_\phi(F,G) \leq \varepsilon$, that is,

$$\mu_{F(x)-G(x)}(\varepsilon t) \geq \phi_{x,x}(t) \tag{3.19}$$

for all $x \in X$ and $t > 0$. Then

$$\begin{aligned} \mu_{JF(x)-JG(x)}(\frac{\lambda\varepsilon t}{4}) = \mu_{\frac{F(2x)}{4}-\frac{G(2x)}{4}}(\frac{\lambda\varepsilon t}{4}) &= \mu_{F(2x)-G(2x)}(\lambda\varepsilon t) \\ &\geq \phi_{2x,2x}(\lambda t) \\ &\geq \phi_{x,x}(t) \end{aligned} \tag{3.20}$$

for all $x \in X$ and $t > 0$, that is, $d_\phi(JF, JG) \leq \frac{\lambda\varepsilon}{4}$. We hence conclude that

$$d_\phi(JF, JG) \leq \frac{\lambda}{4} d_\phi(F,G)$$

for any $F, G \in S$.
As $0 < \lambda < 4$, then operator J is strictly contractive.
It follows from (3.17) that

$$\begin{aligned} \mu_{JF(x)-F(x)}(\frac{\varepsilon t}{4}) = \mu_{\frac{F(2x)}{4}-F(x)}(\frac{\varepsilon t}{4}) &= \mu_{F(2x)-4F(x)}(\varepsilon t) \\ &\geq \phi_{x,x}(t) \end{aligned} \tag{3.21}$$

for all $x \in X$ and $t > 0$, that is,

$$d_\phi(JF, F) \leq \frac{\varepsilon}{4} < \infty.$$

By Theorem 2.1, we deduce existence of a fixed point of J, that is, the existence of mapping $Q : X \to Y$ which is a fixed point of J, such that $\lim_{n\to\infty} d_\phi(J^n F, Q) = 0$.
By induction, we can easily show that

$$J^n F(x) = \frac{F(2^n x)}{2^{2n}},$$

for all $n \in \mathbb{N}$.
Also $d_\phi(F,Q) \le \frac{1}{1-L} d_\phi(JF,F)$ implies the inequality

$$d_\phi(F,Q) \le \frac{1}{1-\frac{\lambda}{4}} = \frac{4}{4-\lambda}.$$

Thus

$$\mu_{F(x)-Q(x)}(\frac{4t}{4-\lambda}) \ge \phi_{x,x}(t),$$

i.e.,

$$\mu_{F(x)-Q(x)}(t) \ge \phi_{x,x}(\frac{(4-\lambda)t}{4}). \tag{3.22}$$

Therefore

$$\begin{aligned} Q(x) = \lim_{n\to\infty} J^n F(x) &= \lim_{n\to\infty} \frac{F(2^n x)}{2^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{2^{2n}}(f(2^n x) - \frac{1}{2} f(2^n x + 2^n \sigma(x))) \end{aligned}$$

for all $x \in X$. Also Q is the unique fixed point of J on the set

$$S^* = \{G \in S : d_\phi(F,G) < \infty\}.$$

It follows from (3.14) that

$$\begin{aligned} & \mu_{\frac{F(2^n x+2^n y)+F(2^n x+2^n\sigma(y))-2F(2^n x)-2F(2^n y)}{2^{2n}}}(t) \qquad (3.23) \\ = \; & \mu_{F(2^n x+2^n y)+F(2^n x+2^n\sigma(y))-2F(2^n x)-2F(2^n y)}(2^{2n}t) \\ \ge \; & T_M(\psi_{2^n x,2^n y}(2^{2n}t), \psi_{2^n x+2^n\sigma(x),2^n y+2^n\sigma(y)}(2^{2n+1}t))) \\ \ge \; & T_M(\psi_{x,y}(\frac{2^{2n}}{\lambda^n}t), \psi_{x+\sigma(x),y+\sigma(y)}(\frac{2^{2n+1}}{\lambda^n}t)). \end{aligned}$$

As $\lim_{n\to\infty} T_M(\psi_{x,y}(\frac{2^{2n}}{\lambda^n}t), \psi_{x+\sigma(x),y+\sigma(y)}(\frac{2^{2n+1}}{\lambda^n}t)) = 1$ then

$$\lim_{n\to\infty} \frac{F(2^n x + 2^n y) + F(2^n x + 2^n\sigma(y)) - 2F(2^n x) - 2F(2^n y)}{2^{2n}} = 0.$$

Hence $Q(x+y) + Q(x+\sigma(y)) = 2Q(x) + 2Q(y)$ and $Q(2x) = 4Q(x)$ that is given $Q(x+\sigma(x)) = 0$, so Q is solution of (3.1) and

$$\mu_{f(x)-\frac{1}{2}f(x+\sigma(x))-Q(x)}(t) \ge \phi_{x,x}(\frac{4-\lambda}{4}t).$$

Now, we put $G(x) = g(x) - \frac{1}{2}g(x+\sigma(x))$ and $H(x) = h(x) - \frac{1}{2}h(x+\sigma(x))$, by (3.8) and (3.9) we have

$$\begin{aligned} \mu_{F(ax)-a^2G(x)}(t) &= \mu_{f(ax)-\frac{1}{2}f(ax+a\sigma(x))-a^2(g(x)-\frac{1}{2}g(x+\sigma(x)))}(t) \\ &\ge T_M(\mu_{f(ax)-a^2g(x)}(t), \mu_{-\frac{1}{2}(f(ax+a\sigma(x))-a^2g(x+\sigma(x)))}(t)) \\ &\ge T_M(\varphi_{x,0}(t), \varphi_{x+\sigma(x),0}(2t)), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\mu_{F(bx)-b^2H(x)}(t) &= \mu_{f(bx)-\frac{1}{2}f(bx+b\sigma(x))-b^2(h(x)-\frac{1}{2}h(x+\sigma(x)))}(t)\\ &\geq T_M(\mu_{f(ax)-b^2h(x)}(t), \mu_{-\frac{1}{2}(f(bx+b\sigma(x))-b^2h(x+\sigma(x)))}(t))\\ &\geq T_M(\varphi_{0,x}(t), \varphi_{0,x+\sigma(x)}(2t)). \end{aligned} \tag{3.25}$$

It follows from (3.22), (3.23) and (3.24) that

$$\begin{aligned}\mu_{Q(ax)-a^2G(x)}(t) &\geq T_M(\mu_{Q(ax)-F(ax)}(t), \mu_{F(ax)-a^2G(x)}(t))\\ &\geq T_M(\phi_{ax,ax}(\frac{4-\lambda}{4}t), T_M(\varphi_{x,0}(t), \varphi_{x+\sigma(x),0}(2t))),\end{aligned}$$

and

$$\begin{aligned}\mu_{Q(bx)-b^2H(x)}(t) &\geq T_M(\mu_{Q(bx)-F(bx)}(t), \mu_{F(bx)-b^2H(x)}(t))\\ &\geq T_M(\phi_{bx,bx}(\frac{4-\lambda}{4}t), T_M(\varphi_{0,x}(t), \varphi_{0,x+\sigma(x)}(2t))).\end{aligned}$$

Finally, we obtain

$$\mu_{G(x)-Q(x)}(t) \geq T_M(\phi_{ax,ax}(\frac{(4-\lambda)a^2}{4}t), T_M(\varphi_{x,0}(a^2t), \varphi_{x+\sigma(x),0}(2a^2t))),$$

and

$$\mu_{H(x)-Q(x)}(t) \geq T_M(\phi_{bx,bx}(\frac{(4-\lambda)b^2}{4}t), T_M(\varphi_{0,x}(b^2t), \varphi_{0,x+\sigma(x)}(2b^2t))),$$

that is,

$$\mu_{g(x)-\frac{1}{2}g(x+\sigma(x))-Q(x)}(t) \geq T_M(\phi_{ax,ax}(\frac{(4-\lambda)a^2}{4}t), T_M(\varphi_{x,0}(a^2t), \varphi_{x+\sigma(x),0}(2a^2t))),$$

and

$$\mu_{h(x)-\frac{1}{2}h(x+\sigma(x))-Q(x)}(t) \geq T_M(\phi_{bx,bx}(\frac{(4-\lambda)b^2}{4}t), T_M(\varphi_{0,x}(b^2t), \varphi_{0,x+\sigma(x)}(2b^2t))),$$

for all $x \in X$ and $t > 0$. This completes the proof of Theorem. □

Corollary 3.3. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x, y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 8$*

$$\varphi_{x,y}(\frac{\lambda}{32}t) \geq \varphi_{2x,2y}(t),$$

for all $x, y \in X$ and $t > 0$. If $f, g, h : X \to Y$ be an even mapping such that

$$\mu_{D_g^h f(x,y)}(t) \geq \varphi_{x,y}(t), \tag{3.26}$$

and $f(0) = g(0) = h(0) = 0$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (3.1) and

$$\mu_{f(x)-\frac{1}{2}f(x+\sigma(x))-Q(x)}(t) \geq \phi_{x,x}(\frac{8-\lambda}{8}t), \tag{3.27}$$

$$\mu_{g(x)-\frac{1}{2}g(x+\sigma(x))-Q(x)}(t) \geq T_M(\phi_{ax,ax}(\frac{(8-\lambda)a^2}{8}t), T_M(\varphi_{x,0}(a^2t), \varphi_{x+\sigma(x),0}(2a^2t))),$$

and

$$\mu_{h(x)-\frac{1}{2}h(x+\sigma(x))-Q(x)}(t) \geq T_M(\phi_{bx,bx}(\frac{(8-\lambda)b^2}{8}t), T_M(\varphi_{0,x}(b^2t), \varphi_{0,x+\sigma(x)}(2b^2t))),$$

for all $x \in X$ and $t > 0$, where

$$\phi_{x,y}(t) = T_M(T_M(\psi_{x,y}(t), \psi_{x+\sigma(x),y+\sigma(y)}(2t)), \psi_{x+\sigma(x),y+\sigma(y)}(4t)),$$

and

$$\psi_{x,y}(t) = T_M(\varphi_{\frac{x}{a},\frac{y}{b}}(t), \varphi_{\frac{x}{a},\frac{\sigma(y)}{b}}(t), \varphi_{\frac{x}{a},0}(\frac{t}{2}), \varphi_{0,\frac{y}{b}}(\frac{t}{2})).$$

Moreover

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}(f(2^n x) - \frac{1}{2}f(2^n x + 2^n\sigma(x))).$$

Proof. It is enough to define an operator $J : S \to S$ by

$$JL(x) = 4L(\frac{x}{2}).$$

The result will be obtained from argument as in proof of Theorem 3.2. □

Corollary 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 4$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.28}$$

for all $x, y \in X$ and $t > 0$. If $f, g, h : X \to Y$ be an even mapping such that

$$\mu_{f(ax+by)-a^2g(x)-b^2h(y)-\frac{ab}{2}(f(x+y)-f(x-y))}(t) \geq \varphi_{x,y}(t), \tag{3.29}$$

and $f(0) = g(0) = h(0) = 0$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (3.1) and

$$\mu_{f(x)-Q(x)}(t) \geq \phi_{x,x}(\frac{4-\lambda}{4}t), \tag{3.30}$$

$$\mu_{g(x)-Q(x)}(t) \geq T_M(\phi_{ax,ax}(\frac{(4-\lambda)a^2}{4}t), T_M(\varphi_{x,0}(a^2t), \varphi_{0,0}(2a^2t))),$$

and

$$\mu_{h(x)-Q(x)}(t) \geq T_M(\phi_{bx,bx}(\frac{(4-\lambda)b^2}{4}t), T_M(\varphi_{0,x}(b^2 t), \varphi_{0,0}(2b^2 t))),$$

for all $x \in X$ and $t > 0$, where

$$\phi_{x,y}(t) = T_M(T_M(\psi_{x,y}(t), \psi_{0,0}(2t)), \psi_{0,0}(4t)),$$

and

$$\psi_{x,y}(t) = T_M(\varphi_{\frac{x}{a},\frac{y}{b}}(t), \varphi_{\frac{x}{a},\frac{-y}{b}}(t), \varphi_{\frac{x}{a},0}(\frac{t}{2}), \varphi_{0,\frac{y}{b}}(\frac{t}{2})).$$

Moreover

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} f(2^n x).$$

Proof. By Theorem 3.2 and $\sigma(x) = -x$ we get the result. □

Corollary 3.5. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x, y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 8$*

$$\varphi_{x,y}(\frac{\lambda}{32}t) \geq \varphi_{2x,2y}(t),$$

for all $x, y \in X$ and $t > 0$. If $f, g, h : X \to Y$ be an even mapping such that

$$\mu_{f(ax+by)-a^2 g(x)-b^2 h(y)-\frac{ab}{2}(f(x+y)-f(x-y))}(t) \geq \varphi_{x,y}(t), \tag{3.31}$$

and $f(0) = g(0) = h(0) = 0$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (3.1) and

$$\mu_{f(x)-Q(x)}(t) \geq \phi_{x,x}(\frac{8-\lambda}{8}t), \tag{3.32}$$

$$\mu_{g(x)-Q(x)}(t) \geq T_M(\phi_{ax,ax}(\frac{(8-\lambda)a^2}{8}t), T_M(\varphi_{x,0}(a^2 t), \varphi_{0,0}(2a^2 t))),$$

and

$$\mu_{h(x)-Q(x)}(t) \geq T_M(\phi_{bx,bx}(\frac{(8-\lambda)b^2}{8}t), T_M(\varphi_{0,x}(b^2 t), \varphi_{0,0}(2b^2 t))),$$

for all $x \in X$ and $t > 0$, where

$$\phi_{x,y}(t) = T_M(T_M(\psi_{x,y}(t), \psi_{0,0}(2t)), \psi_{0,0}(4t)),$$

and

$$\psi_{x,y}(t) = T_M(\varphi_{\frac{x}{a},\frac{y}{b}}(t), \varphi_{\frac{x}{a},\frac{-y}{b}}(t), \varphi_{\frac{x}{a},0}(\frac{t}{2}), \varphi_{0,\frac{y}{b}}(\frac{t}{2})).$$

Moreover

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} f(2^n x).$$

Proof. By Corollary 3.3 and $\sigma(x) = -x$ we get the result. □

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References

[1] S.M. Ulam, Problems in Modern Mathematics, John Wiley & Sons, New York, NY, USA, 1964.

[2] D.H. Hyers, On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America 27 (4) (1941) 222-224.

[3] T. Aoki, On the stability of the linear transformation n Banach spaces, J. Math. Soc. Japan 2 (1950) 64-66.

[4] Th.M. Rassias, On the stability of linear mapping in Banach spaces, Proc. Amer.Math. Soc. 72 (1978) 297-300.

[5] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, Journal of Mathematical Analysis and Applications 184 (3) (1994) 431-436.

[6] F. Skof, Local properties and approximation of operators, Rendiconti del Seminario Matematico e Fisico di Milano 53 (1983) 113-129.

[7] P.W. Cholewa, Remarks on the stability of functional equations, Aequationes Mathematicae 27 (1)(2) (1984) 76-86.

[8] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abhandlungen aus dem Mathematischen Seminar der Universit at Hamburg 62 (1992) 59-64.

[9] C.G. Park, On the stability of the quadratic mapping in Banach modules, Journal of Mathematical Analysis and Applications 276 (1) (2002) 135-144.

[10] B. Bouikhalene, E. Elqorachi, Th.M. Rassias, On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution, Nonlinear Funct. Anal. Appl. 12 (2007) 247-262.

[11] L. Cadariu, V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, in Iteration Theory, vol. 346 of Grazer Mathematische Berichte, pp. 43-52, Karl-Franzens-Universita et Graz, Graz, Austria, 2004.

[12] A. Charifi, B. Bouikhalene, E. Elqorachi, Hyers-Ulam-Rassias stability of a generalized Pexider functional equation, Banach J. Math. Anal. 1 (2007) 176-185.

[13] G. Isac, Th.M. Rassias, Stability of additive mappings: applications to nonlinear analysis, International Journal of Mathematics and Mathematical Sciences 19 (2) (1996) 219-228.

[14] K.W. Jun, Y.H. Lee, On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality, Mathematical Inequalities & Applications 4 (1) (2001) 93-118.

[15] S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, Journal of Mathematical Analysis and Applications 222 (1)(1998) 126-137.

[16] M. Mirzavaziri, M.S. Moslehian, Fixed point approach to stability of a quadratic equation, Bulletin of the Brazilian Mathematical Society 37 (3) (2006) 361-376.

[17] C.G. Park, Th. M. Rassias, Hyers-Ulam stability of a generalized Apollonius type quadratic mapping, Journal of Mathematical Analysis and Applications 322 (1) (2006) 371-381.

[18] C. Park, Generalized Hyers-Ulam Stability of Quadratic Functional Equations: A Fixed Point Approach, Hindawi Publishing Corporation Fixed Point Theory and Applications Volume 2008.

[19] Th.M. Rassias, On the stability of functional equations in Banach spaces, Journal of Mathematical Analysis and Applications 251 (1) (2000) 264-284.

[20] A. Najati, C. Park, Fixed Points and Stability of a Generalized Quadratic Functional Equation, Hindawi Publishing Corporation Journal of Inequalities and Applications Volume 2009, Article ID 193035, 19 pages doi:10.1155/2009/193035.

[21] D. Mihet, R. Saadati, S.M. Vaezpour, The stability of the quartic functional equation in random normed spaces, Acta Appl. Math. In press.

[22] R. Saadati, S.M. Vaezpour, Y.J. Cho, note to paper " On the stability of cubic mappings and quartic mappings in random normed spaces, Jour. Inequal. Appl. vol. 2009, Article ID 214530, 6 pages, 2009.

[23] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathematics, North-Holland, New York, NY, USA, 1983.

[24] J.B. Diaz, B. Margolis, Fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bulletin of the American Mathematical Society 74 (1968) 305-309.

[25] O. Hadžic, E. Pap, Fixed Point Theory in PM Spaces, Kluwer Academic Pub-lishers, Dordrecht, 2001.

[26] O. Hadžic, E. Pap, M. Budincevic, Countable extension of triangular normsand their applications to the fixed point theory in probabilistic metric spaces, Kybernetica 38 (3) (2002) 363-381.

[27] A.N. Sherstnev, On the notion of a random normed space, Dokl. Akad. Nauk SSSR 149 (1963) 280-283 (in Russian).

[28] Iz. EL-Fassi, N. Bounader, A. Chahbi, S. Kabbaj, on the stability of σ quadratic functional equation, Jyoti Academic Press 2 (2) (2013) 61-76, ISSN 2319 - 6939.

[29] D. Mihet, V. Radu, Generalized pseudo-metric and fixed points in probabilistic metric space, carpathian Journal of Mathematics 23 (1)(2) (2007) 126-132.

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