



# $(G, F)$ - Closed Set and Coupled Coincidence Point Theorems for a Generalized Compatible in Partially Metric Spaces

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**Abstract :** In this work, we prove the existence of a coupled coincidence point theorem for a pair  $\{F, G\}$  of mapping  $F, G : X \times X \rightarrow X$  with  $\varphi$ - contraction mappings in complete metric spaces without  $G$ -increasing property of  $F$  and mixed monotone property of  $G$ , using concept of  $(G, F)$ -closed set. We give some examples of a nonlinear contraction mapping, which is not applied to the existence of coupled coincidence point by  $G$  using the mixed monotone property. We also show the uniqueness of a coupled coincidence point of the given mapping. Further, we apply our results to the existence and uniqueness of a coupled coincidence point of the given mapping in partially ordered metric spaces.

**Keywords :** fixed point; coupled coincidence; invariant set; closed set; partially ordered set.

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## 1 Introduction

The existence of a fixed point for contraction type of mappings in partially ordered metric spaces has been first studied by Ran and Reurings [1] and they established some new results for contractions in partially ordered metric spaces and presented applications to matrix equations. Following this line of research, Nieto and Lopez [2, 3] and Agarwal et al. [4] presented some new results for contractions in

partially ordered metric spaces. In 1987, Guo and Lakshmikantham [5] introduced the concept of coupled fixed point. Later, Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone property for contractive operators in partially ordered metric spaces. They also gave some applications in the existence and uniqueness of the coupled fixed point theorems for mappings which satisfy the mixed monotone property. Lakshmikantham and Ćirić [7] extended the results in [6] by defining the mixed  $g$ -monotone and to study the existence and uniqueness of coupled coincidence point for such mapping which satisfy the mixed monotone property in partially ordered metric space. As a continuation of this work, many authors conducted research on the coupled fixed point theory and coupled coincidence point theory in partially ordered metric spaces and different spaces. For example see ([8] - [32]).

In the case of a  $n$ -tuple fixed point or the multidimensional fixed point theorems in several spaces. Some authors show that the results of  $n$ -tuple fixed point can be obtained from fixed point theorems and equivalent against the claims of some authors about reduction of multidimensional versions to unidimensional versions. For example, Soleimani, Shukla and Rahimi [9] show the validity both  $n$ -tuple fixed point results and fixed point theorems according to another in abstract metric spaces and Metric-like spaces. They proved that  $n$ -tuple fixed point results in abstract metric spaces and metric-like spaces can be obtained from fixed point results and conversely. Moreover, the results are true for cone  $b$ -metric spaces and  $b$ -metric-like spaces. Roldan and et al. [29] show that most of the multidimensional fixed point theorems in the context of (ordered) metric spaces are consequences of well-known fixed point theorems in the literature.

One of the interesting way to developed a coupled fixed point theory in partially ordered metric spaces is to consider the mapping  $F : X \times X \rightarrow X$  without the mixed monotone property. Recently, Sintunavarat and et al. [31, 32] proved some coupled fixed point theorems for nonlinear contractions without mixed monotone property and extended some coupled fixed point theorems of Bhaskar and Lakshmikantham [6] by using the concept of  $F$ -invariant set due to Samet and Vetro [30]. Very recently, Kutbi and et al.[12] introduced the concept of  $F$ -closed set which is weaker than the concept of  $F$ -invariant set and proved some coupled fixed point theorems without the condition of mixed monotone property.

In 2014, Hussain and et al.[11] presented the new concept of generalized compatibility of a pair  $\{F, G\}$  of mappings  $F, G : X \times X \rightarrow X$  and proved some coupled coincidence point results of such mapping without mixed  $G$ -monotone property of  $F$  which generalized some recent comparable results in the literature. They also give some examples and an application to integral equations to support the result.

In this work, we generalize and extend a coupled coincidence point theorem for a pair  $\{F, G\}$  of mapping  $F, G : X \times X \rightarrow X$  with  $\varphi$ - contraction mappings in complete metric spaces without  $G$ -increasing property of  $F$  and mixed monotone property of  $G$  , using concept of  $(G, F)$ -closed set.

## 2 Preliminaries

In this section, we give some definitions, proposition, examples and remarks which are useful for main results in this paper. Throughout this paper,  $(X, \preceq)$  denotes a partially ordered set with the partial order  $\preceq$ . By  $x \preceq y$ , we mean  $y \succeq x$ . Let  $(X, \preceq)$  is a partially ordered set, the partial order  $\preceq_2$  for the product set  $X \times X$  defined in the following way, for all  $(x, y), (u, v) \in X \times X$

$$(x, y) \preceq_2 (u, v) \Rightarrow G(x, y) \preceq G(u, v) \text{ and } G(v, u) \preceq G(y, x),$$

where  $G : X \times X \rightarrow X$  is one-one.

We say that  $(x, y)$  is comparable to  $(u, v)$  if either  $(x, y) \preceq_2 (u, v)$  or  $(u, v) \preceq_2 (x, y)$ .

Guo and Lakshmikantham [5] introduced the concept of coupled fixed point as follows:

**Definition 2.1** ([5]). *An element  $(x, y) \in X \times X$  is called a coupled fixed point of a mapping  $F : X \times X \rightarrow X$  if  $F(x, y) = x$  and  $F(y, x) = y$ .*

The concept of a mixed monotone property have been introduced by Bhaskar and Lakshmikantham in [6].

**Definition 2.2** ([6]). *Let  $(X, \preceq)$  be a partially ordered set and  $F : X \times X \rightarrow X$ . We say  $F$  has the mixed monotone property if for any  $x, y \in X$*

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \text{ implies } F(x, y_1) \succeq F(x, y_2).$$

Lakshmikantham and Ćirić in [7] introduced the concept of a mixed g-monotone mapping and a coupled coincidence point.

**Definition 2.3** ([7]). *Let  $(X, \preceq)$  be a partially ordered set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . We say  $F$  has the mixed g-monotone property if for any  $x, y \in X$*

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \text{ implies } F(x, y_1) \succeq F(x, y_2).$$

**Definition 2.4** ([7]). *An element  $(x, y) \in X \times X$  is called a coupled coincidence point of a mapping  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if  $F(x, y) = gx$  and  $F(y, x) = gy$ .*

**Definition 2.5** ([7]). *Let  $X$  be a non-empty set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . We say  $F$  and  $g$  are commutative if  $gF(x, y) = F(gx, gy)$  for all  $x, y \in X$ .*

Hussain et al. [11] introduced the concept of  $G$ -increasing and  $\{F, G\}$  generalized compatible as follows.

**Definition 2.6** ([11]). *Suppose that  $F, G : X \times X \rightarrow X$  are two mapping.  $F$  is said to be  $G$ -increasing with respect to  $\preceq$  if for all  $x, y, u, v \in X$ , with  $G(x, y) \preceq G(u, v)$  we have  $F(x, y) \preceq F(u, v)$ .*

**Definition 2.7** ([11]). *An element  $(x, y) \in X \times X$  is called a coupled coincidence point of mappings  $F, G : X \times X \rightarrow X$  if  $F(x, y) = G(x, y)$  and  $F(y, x) = G(y, x)$ .*

**Definition 2.8** ([11]). *Let  $F, G : X \times X \rightarrow X$ . We say that the pair  $\{F, G\}$  is generalized compatible if*

$$\begin{cases} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) \rightarrow 0 & \text{as } n \rightarrow +\infty, \\ d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) \rightarrow 0 & \text{as } n \rightarrow +\infty, \end{cases}$$

whenever  $(x_n)$  and  $(y_n)$  are sequences in  $X$  such that

$$\begin{cases} \lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} G(x_n, y_n) = t_1, \\ \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} G(y_n, x_n) = t_2. \end{cases}$$

**Definition 2.9** ([11]). *Let  $F, G : X \times X \rightarrow X$  be two maps. We say that the pair  $\{F, G\}$  is commuting if*

$$F(G(x, y), G(y, x)) = G(F(x, y), F(y, x)) \quad \text{for all } x, y \in X.$$

Let  $\Phi$  denote the set of all functions  $\phi : [0, \infty) \rightarrow [0, \infty)$  such that :

- (i)  $\phi$  is continuous and increasing,
- (ii)  $\phi(t) = 0$  if and only if  $t = 0$ ,
- (iii)  $\phi(t + s) \leq \phi(t) + \phi(s)$ , for all  $t, s \in [0, \infty)$ .

Let  $\Psi$  be the set of all functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  such that  $\lim_{t \rightarrow r} \psi(t) > 0$  for all  $r > 0$  and  $\lim_{t \rightarrow 0^+} \psi(t) = 0$ .

Hussain and et al. [11] proved the coupled coincidence point for such mappings involving  $(\psi, \phi)$ -contractive condition as follows:

**Theorem 2.10** ([11]). *Let  $(X, \preceq)$  be a partially ordered set and  $M$  be a nonempty subset of  $X^4$  and let there exist  $d$  be a metric on  $X$  such that  $(X, d)$  is a complete metric space. Assume that  $F, G : X \times X \rightarrow X$  are two generalized compatible mappings such that  $F$  is  $G$ -increasing with respect to  $\preceq$ ,  $G$  is continuous and has the mixed monotone property. suppose that for any  $x, y \in X$ , there exists  $u, v \in X$  such that  $F(x, y) = G(u, v)$  and  $F(y, x) = G(v, u)$ . Suppose that there exists  $\phi \in \Phi$  and  $\psi \in \Psi$  such that the following holds*

$$\begin{aligned} \phi(d(F(x, y), F(u, v))) &\leq \frac{1}{2} \phi(d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))) \\ &\quad - \psi \left( \frac{d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))}{2} \right) \end{aligned}$$

for all  $x, y, u, v \in X$  with  $G(x, y) \preceq G(u, v)$  and  $G(y, x) \succeq G(v, u)$ .

Also suppose also that either

(a)  $F$  is continuous or

(b)  $X$  has the following properties: for any two sequences  $\{x_n\}$  and  $\{y_n\}$  with

(i) if a non-decreasing sequence  $\{x_n\} \rightarrow x$ , then  $x_n \preceq x$  for all  $n$ ,

(ii) if a non-increasing sequence  $(\{y_n\} \rightarrow y$ , then  $y \preceq y_n$  for all  $n$ .

If there exist  $x_0, y_0 \in X \times X$  with

$$G(x_0, y_0) \preceq F(x_0, y_0) \quad \text{and} \quad G(y_0, x_0) \succeq F(y_0, x_0).$$

Then there exist  $(x, y) \in X \times X$  such that  $G(x, y) = F(x, y)$  and  $G(y, x) = F(y, x)$ , that is  $F$  and  $G$  have a coupled coincidence point.

Kutbi and et al. [12] introduced the notion of  $F$ - closed set which extended the notion of  $F$ -invariant set as follow.

**Definition 2.11** ([12]). Let  $F : X \times X \rightarrow X$  be a mapping, and let  $M$  be a subset of  $X^4$ . We say that  $M$  is an  $F$ -closed subset of  $X^4$  if, for all  $x, y, u, v \in X$ ,

$$(x, y, u, v) \in M \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in M.$$

Now, we give the notion of  $(G, F)$ -closed set which is useful for our main results.

**Definition 2.12.** Let  $F, G : X \times X \rightarrow X$  be two mapping, and let  $M$  be a subset of  $X^4$ . We say that  $M$  is an  $(G, F)$ -closed subset of  $X^4$  if, for all  $x, y, u, v \in X$ ,

$$\begin{aligned} (G(x, y), G(y, x), G(u, v), G(v, u)) \in M \\ \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in M. \end{aligned}$$

**Definition 2.13.** Let  $(X, \leq)$  be a metric space and  $M$  be a subset of  $X^4$ . We say that  $M$  satisfies the transitive property if and only if, for all  $x, y, u, v, a, b \in X$ ,

$$\begin{aligned} (G(x, y), G(y, x), G(u, v), G(v, u)) \in M \text{ and} \\ (G(u, v), G(v, u), G(a, b), G(b, a)) \in M \\ \Rightarrow (G(x, y), G(y, x), G(a, b), G(b, a)) \in M. \end{aligned}$$

**Remark** The set  $M = X^4$  is trivially  $(G, F)$ -closed set, which satisfies the transitive property.

**Example 2.14.** Let  $(X, d)$  be a metric space endowed with a partial order  $\preceq$ . Let  $F, G : X \times X \rightarrow X$  are two generalized compatible mappings such that  $F$  is  $G$ -increasing with respect to  $\preceq$ ,  $G$  is continuous and has the mixed monotone property. Define a subset  $M \subseteq X^4$  by

$$M = \{(x, y, u, v) \in X^4 : x \preceq u, y \succeq v\}.$$

Let  $(G(x, y), G(y, x), G(u, v), G(v, u)) \in M$ , It is easy to see that, since  $F$  is  $G$ -increasing with respect to  $\preceq$ , we have  $F(x, y) \preceq F(u, v)$  and  $F(y, x) \succeq F(v, u)$ , this implies that  $(F(x, y), F(y, x), F(u, v), F(v, u)) \in M$ . Then  $M$  is  $(G, F)$ -closed subset of  $X^4$ , which satisfies the transitive property.

### 3 Main Results

Let  $\Phi$  denote the set of functions  $\varphi : [0, \infty) \rightarrow [0, \infty)$  satisfying

1.  $\varphi(t) < t$  for all  $t > 0$ ,
2.  $\lim_{r \rightarrow t^+} \varphi(r) < t$  for all  $t > 0$ .

**Theorem 3.1.** Let  $M$  be a nonempty subset of  $X^4$  and let there exist  $d$  be a metric on  $X$  such that  $(X, d)$  is a complete metric space. Assume that  $F, G : X \times X \rightarrow X$  are two generalized compatible mappings such that  $G$  is continuous and for any  $x, y \in X$ , there exists  $u, v \in X$  such that  $F(x, y) = G(u, v)$  and  $F(y, x) = G(v, u)$ . Suppose that there exists  $\varphi \in \Phi$  such that the following holds

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ & \leq \varphi(d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))) \end{aligned} \quad (3.1)$$

for all  $x, y, u, v \in X$  with  $(G(x, y), G(y, x), G(u, v), G(v, u)) \in M$ .

Also suppose also that either

- (a)  $F$  is continuous.
- (b) for any two sequences  $\{x_n\}$  and  $\{y_n\}$  with

$$\begin{aligned} & (x_n, y_n, x_{n+1}, y_{n+1}) \in M \text{ and} \\ & \{G(x_n, y_n)\} \rightarrow G(x, y), \{G(y_n, x_n)\} \rightarrow G(y, x) \text{ for all } n \geq 1 \text{ implies} \end{aligned}$$

$$(G(x_n, y_n), G(y_n, x_n), G(x, y), G(y, x)) \in M, \text{ for all } n \geq 1.$$

If there exist  $x_0, y_0 \in X \times X$  such that

$$(G(x_0, y_0), G(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \in M$$

and  $M$  is an  $(G, F)$ -closed. Then there exist  $(x, y) \in X \times X$  such that  $G(x, y) = F(x, y)$  and  $G(y, x) = F(y, x)$ , that is  $F$  and  $G$  have a coupled coincidence point.

*Proof.* Let  $x_0, y_0 \in X$  be such that  $(G(x_0, y_0), G(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \in M$ . From assumption, there exists  $(x_1, y_1) \in X \times X$  such that  $F(x_0, y_0) = G(x_1, y_1)$  and  $F(y_0, x_0) = G(y_1, x_1)$ . Again from assumption, we can choose  $x_2, y_2 \in X$  such that  $F(x_1, y_1) = G(x_2, y_2)$  and  $F(y_1, x_1) = G(y_2, x_2)$ . By repeating this argument, we can construct two sequences  $\{x_n\}_{n=1}^{\infty}$  and  $\{y_n\}_{n=1}^{\infty}$  in  $X$  such that

$$F(x_n, y_n) = G(x_{n+1}, y_{n+1}) \quad \text{and} \quad F(y_n, x_n) = G(y_{n+1}, x_{n+1}) \quad \text{for all } n \geq 1. \quad (3.2)$$

Since  $(G(x_0, y_0), G(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \in M$  and  $M$  is an  $(G, F)$ -closed, we get

$$\begin{aligned} & (G(x_0, y_0), G(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \\ &= (G(x_0, y_0), G(y_0, x_0), G(x_1, y_1), G(y_1, x_1)) \in M \\ &\Rightarrow (F(x_0, y_0), F(y_0, x_0), F(x_1, y_1), F(y_1, x_1)) \\ &= (G(x_1, y_1), G(y_1, x_1), G(x_2, y_2), G(y_2, x_2)) \in M. \end{aligned}$$

Again, using the fact that  $M$  is an  $(G, F)$ -closed, we have

$$\begin{aligned} & (G(x_1, y_1), G(y_1, x_1), G(x_2, y_2), G(y_2, x_2)) \in M \\ &\Rightarrow (F(x_2, y_2), F(y_2, x_2), F(x_3, y_3), F(y_3, x_3)) \\ &= (G(x_2, y_2), G(y_2, x_2), G(x_3, y_3), G(y_3, x_3)) \in M. \end{aligned}$$

Continuing this process, for all  $n \geq 0$  we get

$$(G(x_n, y_n), G(y_n, x_n), G(x_{n+1}, y_{n+1}), G(y_{n+1}, x_{n+1})) \in M. \quad (3.3)$$

For all  $n \geq 0$ , denote

$$\delta_n = d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})). \quad (3.4)$$

We can suppose that  $\delta_n > 0$  for all  $n \geq 0$ . If not,  $(x_n, y_n)$  will be a coupled coincidence point and the proof is finished. From (3.1), (3.2) and (3.3), we have

$$\begin{aligned} & d(G(x_{n+1}, y_{n+1}), G(x_{n+2}, y_{n+2})) + d(G(y_{n+1}, x_{n+1}), G(y_{n+2}, x_{n+2})) \\ &= d(F(x_n, y_n), F(x_{n+1}, y_{n+1})) + d(F(y_n, x_n), F(y_{n+1}, x_{n+1})) \\ &\leq \varphi(d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1}))) \\ &= \varphi(\delta_n). \end{aligned} \quad (3.5)$$

Therefore, the sequence  $\{\delta_n\}_{n=1}^{\infty}$  satisfies

$$\delta_{n+1} \leq \varphi(\delta_n), \quad \text{for all } n \geq 0. \quad (3.6)$$

Using property of  $\varphi$  it follow that the sequence  $\{\delta_n\}_{n=1}^{\infty}$  is decreasing. Therefore, there exists some  $\delta \geq 0$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta_n &= \lim_{n \rightarrow \infty} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) \\ &= \delta. \end{aligned} \quad (3.7)$$

We shall prove that  $\delta = 0$ . Assume, to the contrary, that  $\delta > 0$ . Then by letting  $n \rightarrow \infty$  in (3.6) and using the property of  $\varphi$ , we have

$$\delta = \lim_{n \rightarrow \infty} \delta_{n+1} \leq \lim_{n \rightarrow \infty} \varphi(\delta_n) = \lim_{\delta_n \rightarrow \delta^+} \varphi(\delta_n) < \delta,$$

a contradiction. Thus  $\delta = 0$  and hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \delta_n &= \lim_{n \rightarrow \infty} d(G(x_n, y_n), G(x_{n+1}, y_{n+1})) + d(G(y_n, x_n), G(y_{n+1}, x_{n+1})) \\ &= 0. \end{aligned} \quad (3.8)$$

We now prove that  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  and  $\{G(y_n, x_n)\}_{n=1}^{\infty}$  are Cauchy sequences in  $(X, d)$ . Suppose, to the contrary, that at least one of the sequences  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  or  $\{G(y_n, x_n)\}_{n=1}^{\infty}$  is not a Cauchy sequence. Then exists an  $\epsilon > 0$  for which we can find subsequences  $\{G(x_{m(k)}, y_{m(k)})\}$ ,  $\{G(x_{n(k)}, y_{n(k)})\}$  of  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  and  $\{G(y_{m(k)}, x_{m(k)})\}$ ,  $\{G(y_{n(k)}, x_{n(k)})\}$  of  $\{G(y_n, x_n)\}_{n=1}^{\infty}$ , respectively, with  $n(k) > m(k) \geq k$  such that

$$\begin{aligned} D_k &= d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)})) + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)})) \\ &> \epsilon. \end{aligned} \quad (3.9)$$

Further, corresponding to  $m(k)$ , we can choose  $n(k)$  in such a way that is the smallest integer with  $n(k) > m(k) \geq k$  and satisfying (3.9). Then

$$\begin{aligned} &d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)-1}, y_{n(k)-1})) + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)-1}, x_{n(k)-1})) \\ &\leq \epsilon. \end{aligned} \quad (3.10)$$

Using (3.9) and (3.10) and the triangle inequality, we have

$$\begin{aligned} \epsilon &< D_k \\ &\leq d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)-1}, y_{n(k)-1})) \\ &\quad + d(G(x_{n(k)-1}, y_{n(k)-1}), G(x_{n(k)}, y_{n(k)})) \\ &\quad + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)-1}, x_{n(k)-1})) \\ &\quad + d(G(y_{n(k)-1}, x_{n(k)-1}), G(y_{n(k)}, x_{n(k)})) \\ &\leq \epsilon + \delta_{n(k)-1}. \end{aligned} \quad (3.11)$$

Letting  $k \rightarrow \infty$  in (3.11) and using (3.8), we get

$$\lim_{n \rightarrow \infty} D_k = \epsilon \quad (3.12)$$



Again, for all  $k \geq 0$ , we have

$$\begin{aligned}
 D_k &= d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)})) + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)})) \\
 &\leq d(G(x_{m(k)}, y_{m(k)}), G(x_{m(k)+1}, y_{m(k)+1})) \\
 &\quad + d(G(x_{m(k)+1}, y_{m(k)+1}), G(x_{n(k)+1}, y_{n(k)+1})) \\
 &\quad + d(G(x_{n(k)+1}, y_{n(k)+1}), G(x_{n(k)}, y_{n(k)})) \\
 &\quad + d(G(y_{m(k)}, x_{m(k)}), G(y_{m(k)+1}, x_{m(k)+1})) \\
 &\quad + d(G(y_{m(k)+1}, x_{m(k)+1}), G(y_{n(k)+1}, x_{n(k)+1})) \\
 &\quad + d(G(y_{n(k)+1}, x_{n(k)+1}), G(y_{n(k)}, x_{n(k)})) \\
 &\leq \delta_{m(k)} + \delta_{n(k)} + d(G(x_{m(k)+1}, y_{m(k)+1}), G(x_{n(k)+1}, y_{n(k)+1})) \\
 &\quad + d(G(y_{m(k)+1}, x_{m(k)+1}), G(y_{n(k)+1}, x_{n(k)+1}))
 \end{aligned} \tag{3.13}$$

From (3.3) and  $n(k) > m(k)$  we have

$$(G(x_{m(k)}, y_{m(k)}), G(y_{m(k)}, x_{m(k)}), G(x_{m(k)+1}, y_{m(k)+1}), G(y_{m(k)+1}, x_{m(k)+1})) \in M$$

and

$$(G(x_{m(k)+1}, y_{m(k)+1}), G(y_{m(k)+1}, x_{m(k)+1}), G(x_{m(k)+2}, y_{m(k)+2}), G(y_{m(k)+2}, x_{m(k)+2})) \in M.$$

Using  $M$  is  $G$ -transitive property, we get

$$(G(x_{m(k)}, y_{m(k)}), G(y_{m(k)}, x_{m(k)}), G(x_{m(k)+2}, y_{m(k)+2}), G(y_{m(k)+2}, x_{m(k)+2})) \in M.$$

Continue this process, we have

$$(G(x_{m(k)}, y_{m(k)}), G(y_{m(k)}, x_{m(k)}), G(x_{n(k)}, y_{n(k)}), G(y_{n(k)}, x_{n(k)})) \in M. \tag{3.14}$$

From (3.1), (3.2) and (3.14), we have

$$\begin{aligned}
 &d(G(x_{m(k)+1}, y_{m(k)+1}), G(y_{n(k)+1}, x_{n(k)+1})) \\
 &\quad + d(G(x_{m(k)+1}, y_{m(k)+1}), G(y_{n(k)+1}, x_{n(k)+1})) \\
 &= d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)})) + d(F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)})) \\
 &\leq \varphi(d(G(x_{m(k)}, y_{m(k)}), G(x_{n(k)}, y_{n(k)})) + d(G(y_{m(k)}, x_{m(k)}), G(y_{n(k)}, x_{n(k)}))) \\
 &= \varphi(D_k)
 \end{aligned} \tag{3.15}$$

which, by (3.13), yields

$$D_k \leq \delta_{m(k)} + \delta_{n(k)} + \varphi(D_k). \tag{3.16}$$

Letting  $k \rightarrow \infty$  in the above inequality and using (3.8) and (3.12) we get

$$\epsilon = \lim_{k \rightarrow \infty} D_k \leq \lim_{k \rightarrow \infty} (\delta_{m(k)} + \delta_{n(k)} + \varphi(D_k)) = \lim_{D_k \rightarrow \epsilon^+} \varphi(D_k) < \epsilon,$$

a contradiction. Hence  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  and  $\{G(y_n, x_n)\}_{n=1}^{\infty}$  are Cauchy sequences in  $(X, d)$ . Since  $(X, d)$  is complete and (3.2), there exists  $x, y \in X$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} G(x_n, y_n) &= \lim_{n \rightarrow \infty} F(x_n, y_n) = x \\ \text{and} \\ \lim_{n \rightarrow \infty} G(y_n, x_n) &= \lim_{n \rightarrow \infty} F(y_n, x_n) = y. \end{aligned} \quad (3.17)$$

Since the pair  $\{F, G\}$  satisfies the generalized compatibility, from (3.17), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) &= 0 \\ \text{and} \\ \lim_{n \rightarrow \infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) &= 0. \end{aligned} \quad (3.18)$$

Suppose that assumption (a) hold. For all  $n \geq 0$ , by the triangle inequality we have

$$\begin{aligned} d(G(x, y), F(G(x_n, y_n), G(y_n, x_n))) \\ \leq d(G(x, y), G(F(x_n, y_n), F(y_n, x_n))) \\ + d(G(F(x_n, y_n), F(y_n, x_n)), F(G(x_n, y_n), G(y_n, x_n))). \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} d(G(y, x), F(G(y_n, x_n), G(x_n, y_n))) \\ \leq d(G(y, x), G(F(y_n, x_n), F(x_n, y_n))) \\ + d(G(F(y_n, x_n), F(x_n, y_n)), F(G(y_n, x_n), G(x_n, y_n))). \end{aligned} \quad (3.20)$$

Taking the limit as  $n \rightarrow \infty$  in (3.19) and (3.20), Using (3.17), (3.18) and the fact that  $F$  and  $G$  are continuous, we have

$$G(x, y) = F(x, y) \quad \text{and} \quad G(y, x) = F(y, x). \quad (3.21)$$

Therefore  $(x, y)$  is a coupled coincidence point of  $F$  and  $G$ .

Suppose now assumption (b) holds. Since  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  is converges to  $x$  and  $\{G(y_n, x_n)\}_{n=1}^{\infty}$  is converges to  $y$ . Since the pair  $\{F, G\}$  satisfies the generalized compatibility,  $G$  is continuous and by (3.17), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} G(G(x_n, y_n), G(y_n, x_n)) &= G(x, y) \\ &= \lim_{n \rightarrow \infty} G(F(x_n, y_n), F(y_n, x_n)) \\ &= \lim_{n \rightarrow \infty} F(G(x_n, y_n), G(y_n, x_n)) \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} G(G(y_n, x_n), G(x_n, y_n)) &= G(y, x) \\ &= \lim_{n \rightarrow \infty} G(F(y_n, x_n), F(x_n, y_n)) \\ &= \lim_{n \rightarrow \infty} F(G(y_n, x_n), G(x_n, y_n)) \end{aligned} \quad (3.23)$$

From (3.3) and assumption (b) , for all  $n \geq 1$  we have

$$(G(G(x_n, y_n), G(y_n, x_n)), G(G(y_n, x_n), G(x_n, y_n)), G(x, y), G(y, x)) \in M. \quad (3.24)$$

Then, by (3.1), (3.2), (3.24), (3.22) , (3.23) and triangle inequality, we have

$$\begin{aligned} & d(G(x, y), F(x, y)) + d(G(y, x), F(y, x)) \\ & \leq d(G(x, y), F(G(x_n, y_n), G(y_n, x_n))) + d(F(G(x_n, y_n), G(y_n, x_n)), F(x, y)) \\ & \quad + d(G(y, x), F(G(y_n, x_n), G(x_n, y_n))) + d(F(G(y_n, x_n), G(x_n, y_n)), F(y, x)) \\ & \leq \varphi(d(G(G(x_n, y_n), G(y_n, x_n)), G(x, y)) + d(G(G(y_n, x_n), G(x_n, y_n)), G(y, x))) \\ & \quad + d(G(x, y), F(G(x_n, y_n), G(y_n, x_n))) + d(G(y, x), F(G(y_n, x_n), G(x_n, y_n))) \end{aligned}$$

Letting now  $n \rightarrow \infty$  in the above inequality and using property of  $\varphi$  such that  $\lim_{r \rightarrow 0^+} \varphi(r) = 0$ , we have

$$d(G(x, y), F(x, y)) + d(G(y, x), F(y, x)) = 0$$

which implies that  $G(x, y) = F(x, y)$  and  $G(y, x) = F(y, x)$ . □

Next, we give an example to validate Theorem 3.1

**Example 3.2.** Let  $X = [0, 1]$ ,  $d(x, y) = |x - y|$  and  $F, G : X \times X \rightarrow X$  be defined by

$$F(x, y) = \begin{cases} \frac{x^2 - y^2}{8} & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

and

$$G(x, y) = \begin{cases} x + y & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Clearly,  $G$  does not satisfy mixed monotone property and if  $x > y$ ,  $u = v \neq 0$  , consider

$$\begin{aligned} G(x, y) \leq G(u, v) & \Rightarrow x + y \leq u + v \\ \text{but } F(x, y) & = x^2 - y^2 = (x - y)(x + y) > 0 = F(u, v). \end{aligned}$$

Then  $F$  is not  $G$ -increasing.

Now we prove that for any  $x, y \in X$ , there exists  $u, v \in X$  such that  $F(x, y) = G(u, v)$  and  $F(y, x) = G(v, u)$ . It is easy to see the following cases.

Case 1: If  $x = y$ , then we have  $F(y, x) = F(x, y) = 0 = G(0, 0)$ .

Case 2: If  $x > y$ , then  $(x - y)x > (x - y)y$  and we have

$$F(x, y) = \frac{x^2 - y^2}{8} = \frac{(x - y)x + (x - y)y}{8} = G\left(\frac{(x - y)x}{8}, \frac{(x - y)y}{8}\right)$$

and

$$F(y, x) = 0 = G\left(\frac{(x - y)y}{8}, \frac{(x - y)x}{8}\right).$$

Case 3: If  $y > x$ , then  $(y - x)y > (y - x)x$  and we have

$$F(y, x) = \frac{y^2 - x^2}{8} = \frac{(y - x)y + (y - x)x}{8} = G\left(\frac{(y - x)y}{8}, \frac{(y - x)x}{8}\right)$$

and

$$F(x, y) = 0 = G\left(\frac{(y - x)x}{8}, \frac{(y - x)y}{8}\right).$$

Now, we prove that the pair  $\{F, G\}$  satisfies the generalized compatibility hypothesis. Let  $\{x_n\}_{n=1}^{\infty}$  and  $\{y_n\}_{n=1}^{\infty}$  be two sequences in  $X$  such that

$$t_1 = \lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} G(x_n, y_n)$$

and

$$t_2 = \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} G(y_n, x_n).$$

Then we must have  $t_1 = 0 = t_2$  and it is easy to prove that

$$\begin{cases} \lim_{n \rightarrow \infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) = 0 \\ \lim_{n \rightarrow \infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) = 0. \end{cases}$$

Now, for all  $x, y, u, v \in X$  with  $(G(x, y), G(y, x), G(u, v), G(v, u)) \in M = X^4$  and let  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  be a function defined by  $\varphi(t) = \frac{t}{8}$ , we have

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &= \left| \frac{x^2 - y^2}{8} - \frac{u^2 - v^2}{8} \right| + \left| \frac{y^2 - x^2}{8} - \frac{v^2 - u^2}{8} \right| \\ &= 2 \left| \frac{x^2 - y^2}{8} - \frac{u^2 - v^2}{8} \right| \\ &= 2 \left| \frac{(x - y)(x + y)}{8} - \frac{(u - v)(u + v)}{8} \right| \\ &\leq \frac{1}{4} |(x + y) - (u + v)| \\ &= \varphi(2|(x + y) - (u + v)|) \\ &= \varphi(|(x + y) - (u + v)| + |(y + x) - (v + u)|) \\ &= \varphi(d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))). \end{aligned}$$

Therefore condition (3.1) is satisfied. Thus all the requirements of Theorem 3.1 are satisfied and  $(0, 0)$  is a coupled coincidence point of  $F$  and  $G$ .

Next, we show the uniqueness of the coupled coincidence point and coupled fixed point of  $F$  and  $G$ .

**Theorem 3.3.** *In addition to the hypotheses of Theorem 3.1, suppose that for every  $(x, y), (z, t) \in X \times X$ , there exists  $(u, v) \in X \times X$  such that*

$$\begin{aligned} &(G(x, y), G(y, x), G(u, v), G(v, u)) \in M \text{ and} \\ &(G(z, t), G(t, z), G(u, v), G(v, u)) \in M. \end{aligned}$$

*Then F and G have a unique coupled coincidence point. Moreover, if the pair  $\{F, G\}$  is commuting, then F and G have a unique coupled fixed point, that is, there exists a unique  $(a, b) \in X^2$  such that*

$$a = G(a, b) = F(a, b) \quad \text{and} \quad b = G(b, a) = F(b, a)$$

*Proof.* From Theorem 3.1, we know that F and G have a coupled coincidence points. Suppose that  $(x, y), (z, t)$  are coupled coincidence point of F and G, that is,

$$F(x, y) = G(x, y), F(y, x) = G(y, x) \text{ and } F(z, t) = G(z, t), F(t, z) = G(t, z). \tag{3.25}$$

Now we show that  $G(x, y) = G(z, t)$  and  $G(y, x) = G(t, z)$ . By the hypothesis there exists  $(u, v) \in X \times X$  such that  $(G(x, y), G(y, x), G(u, v), G(v, u)) \in M$  and  $(G(z, t), G(t, z), G(u, v), G(v, u)) \in M$ . We put  $u_0 = u$  and  $v_0 = v$  and define two sequence  $\{G(u_n, v_n)\}_{n=1}^\infty$  and  $\{G(v_n, u_n)\}_{n=1}^\infty$  as follow,

$$F(u_n, v_n) = G(u_{n+1}, v_{n+1}) \quad \text{and} \quad F(v_n, u_n) = G(v_{n+1}, u_{n+1}) \quad \text{for all } n \geq 0.$$

Since M is (G, F)-closed and  $(G(x, y), G(y, x), G(u, v), G(v, u)) \in M$ , we have

$$\begin{aligned} &(G(x, y), G(y, x), G(u, v), G(v, u)) \\ &= (G(x, y), G(y, x), G(u_0, v_0), G(v_0, u_0)) \in M \\ &\Rightarrow (F(x, y), F(y, x), F(u_0, v_0), F(v_0, u_0)) \\ &= (G(x, y), G(y, x), G(u_1, v_1), G(v_1, u_1)) \in M. \end{aligned}$$

From  $(G(x, y), G(y, x), G(u_1, v_1), G(v_1, u_1)) \in M$ , if we use again the property of (G, F)-closed, then

$$\begin{aligned} &(G(x, y), G(y, x), G(u_1, v_1), G(v_1, u_1)) \in M \\ &\Rightarrow (F(x, y), F(y, x), F(u_1, v_1), F(v_1, u_1)) \\ &= (G(x, y), G(y, x), G(u_2, v_2), G(v_2, u_2)) \in M. \end{aligned}$$

By repeating this process, we get

$$(G(x, y), G(y, x), G(u_n, v_n), G(v_n, u_n)) \in M \quad \text{for all } n \geq 0 \tag{3.26}$$

Using (3.1), (3.25) and (3.26), we have

$$\begin{aligned} &d(G(x, y), G(u_{n+1}, v_{n+1})) + d(G(y, x), G(v_{n+1}, u_{n+1})) \\ &= d(F(x, y), F(u_n, v_n)) + d(F(y, x), F(v_n, u_n)) \\ &\leq \varphi(d(G(x, y), G(u_n, v_n)) + d(G(y, x), G(v_n, u_n))) \quad \text{for all } n. \end{aligned} \tag{3.27}$$

Using property that  $\varphi(t) < t$  and repeating this process, we get

$$\begin{aligned} & d(G(x, y), G(u_{n+1}, v_{n+1})) + d(G(y, x), G(v_{n+1}, u_{n+1})) \\ & \leq \varphi^n(d(G(x, y), G(u_1, v_1)) + d(G(y, x), G(v_1, u_1))) \quad \text{for all } n. \end{aligned} \quad (3.28)$$

From  $\varphi(t) < t$  and  $\lim_{r \rightarrow t^+} \varphi(r) < t$ , it follow that  $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$  for each  $t > 0$ . Therefore, from (3.28) we have

$$\lim_{n \rightarrow \infty} (d(G(x, y), G(u_{n+1}, v_{n+1})) + d(G(y, x), G(v_{n+1}, u_{n+1}))) = 0. \quad (3.29)$$

This implies that

$$\lim_{n \rightarrow \infty} d(G(x, y), G(u_{n+1}, v_{n+1})) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(G(y, x), G(v_{n+1}, u_{n+1})) = 0. \quad (3.30)$$

Similarly, we show that

$$\lim_{n \rightarrow \infty} d(G(z, t), G(u_{n+1}, v_{n+1})) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(G(t, z), G(v_{n+1}, u_{n+1})) = 0. \quad (3.31)$$

From (3.30) and (3.31), we have

$$G(x, y) = G(z, t) \quad \text{and} \quad G(y, x) = G(t, z). \quad (3.32)$$

Now let the pair  $\{F, G\}$  is commuting, we shall prove that  $F$  and  $G$  have a unique coupled fixed point. Since

$$F(x, y) = G(x, y) \quad \text{and} \quad F(y, x) = G(y, x), \quad (3.33)$$

and  $F$  and  $G$  commutes, we have

$$\begin{aligned} & G(G(x, y), G(y, x)) = G(F(x, y), F(y, x)) = F(G(x, y), G(y, x)) \\ & \text{and} \\ & G(G(y, x), G(x, y)) = G(F(y, x), F(x, y)) = F(G(y, x), G(x, y)). \end{aligned} \quad (3.34)$$

Denote  $G(x, y) = a$  and  $G(y, x) = b$ . Then, by (3.33) and (3.34) one get

$$G(a, b) = F(a, b) \quad \text{and} \quad G(b, a) = F(b, a). \quad (3.35)$$

Therefore,  $(a, b)$  is a coupled coincidence point of  $F$  and  $G$ . Then, by (3.32) with  $z = a$  and  $t = b$ , it follows that

$$a = G(x, y) = G(a, b) \quad \text{and} \quad b = G(y, x) = G(b, a). \quad (3.36)$$

Thus  $(a, b)$  is a coupled fixed point of  $G$ , by (3.33)  $(a, b)$  is also a coupled fixed point of  $F$ . To prove the uniqueness, assume  $(p, q)$  is another coupled fixed point of  $F$  and  $G$ . Then by (3.32) and (3.36) we have

$$p = G(p, q) = G(a, b) = a \quad \text{and} \quad q = G(q, p) = G(b, a) = b.$$

□

Next, we give some application of our results to coupled coincidence point theorems.

**Corollary 3.4.** *Let  $(X, \preceq)$  be a partially ordered set and  $M$  be a nonempty subset of  $X^4$  and let there exist  $d$  be a metric on  $X$  such that  $(X, d)$  is a complete metric space. Assume that  $F, G : X \times X \rightarrow X$  are two generalized compatible mappings such that  $F$  is  $G$ -increasing with respect to  $\preceq$ ,  $G$  is continuous and has the mixed monotone property. suppose that for any  $x, y \in X$ , there exists  $u, v \in X$  such that  $F(x, y) = G(u, v)$  and  $F(y, x) = G(v, u)$ . Suppose that there exists  $\phi \in \Phi$  and  $\psi \in \Psi$  such that the following holds*

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ & \leq \varphi (d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))) \end{aligned}$$

for all  $x, y, u, v \in X$  with  $(G(x, y) \preceq G(u, v)$  and  $G(y, x) \succeq G(v, u))$ .

Also suppose also that either

- (a)  $F$  is continuous or
- (b)  $X$  has the following properties: for any two sequences  $\{x_n\}$  and  $\{y_n\}$  with
  - (i) if a non-decreasing sequence  $\{x_n\} \rightarrow x$ , then  $x_n \preceq x$  for all  $n$ ,
  - (ii) if a non-increasing sequence  $(\{y_n\} \rightarrow y$ , then  $y \preceq y_n$  for all  $n$ .

If there exist  $x_0, y_0 \in X \times X$  with

$$G(x_0, y_0) \preceq F(x_0, y_0) \quad \text{and} \quad G(y_0, x_0) \succeq F(y_0, x_0).$$

Then there exist  $(x, y) \in X \times X$  such that  $G(x, y) = F(x, y)$  and  $G(y, x) = F(y, x)$ , that is  $F$  and  $G$  have a coupled coincidence point.

*Proof.* We define the subset  $M \subseteq X^4$  by

$$M = \{ (x, y, u, v) \in X^4 : x \preceq u \text{ and } y \succeq v \}.$$

From Example 2.14,  $M$  is an  $(G, F)$ -closed set which satisfies the transitive property. for all  $x, y, u, v \in X$  with  $(G(x, y) \preceq G(u, v)$  and  $G(y, x) \succeq G(v, u))$ , we have  $(G(x, y), G(y, x), G(u, v), G(v, u)) \in M$ . By (3.1), we get

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ & \leq \varphi (d(G(x, y), G(u, v)) + d(G(y, x), G(v, u))) \end{aligned}$$

Since  $x_0, y_0 \in X \times X$  with

$$G(x_0, y_0) \preceq F(x_0, y_0) \quad \text{and} \quad G(y_0, x_0) \succeq F(y_0, x_0). \tag{3.37}$$

We have

$$(G(x_0, y_0), G(y_0, x_0), F(x_0, y_0), F(y_0, x_0)) \in M.$$

For the assumption (a) holds,  $F$  is continuous. By assumption (a) of Theorem 3.1, we have  $G(x, y) = F(x, y)$  and  $G(y, x) = F(y, x)$ .

Next, for the assumption (b) holds, Since  $F$  is  $G$ -increasing with respect to  $\preceq$ , using (3.37) and (3.2), we have

$$G(x_n, y_n) \preceq G(x_{n+1}, y_{n+1}) \text{ and } G(y_n, x_n) \succeq G(y_{n+1}, x_{n+1}) \text{ for all } n.$$

Therefore  $(G(x_n, y_n), G(y_n, x_n), G(x_{n+1}, y_{n+1}), G(y_{n+1}, x_{n+1})) \in M$ .

From  $H$  is continuous and by (3.17), we have

$$\lim_{n \rightarrow \infty} G(G(x_n, y_n), G(y_n, x_n)) = G(x, y)$$

and

$$\lim_{n \rightarrow \infty} G(G(y_n, x_n), G(x_n, y_n)) = G(y, x).$$

For any two sequences  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  and  $\{G(y_n, x_n)\}_{n=1}^{\infty}$  such that  $\{G(x_n, y_n)\}_{n=1}^{\infty}$  is a non-decreasing sequence in  $X$  with  $G(x_n, y_n) \rightarrow x$  and  $\{G(y_n, x_n)\}_{n=1}^{\infty}$  is a non-increasing sequence in  $X$  with  $G(y_n, x_n) \rightarrow y$ . Using assumption (b), we have

$$G(x_n, y_n) \preceq x \quad \text{and} \quad G(y_n, x_n) \succeq y \text{ for all } n.$$

Since  $G$  has the mixed monotone property, we have

$$\begin{aligned} G(G(x_n, y_n), G(y_n, x_n)) &\preceq G(x, y) \\ G(G(y_n, x_n), G(x_n, y_n)) &\succeq G(y, x). \end{aligned}$$

Therefore, we have

$$(G(G(x_n, y_n), G(y_n, x_n)), G(G(y_n, x_n), G(x_n, y_n)), G(x, y), G(y, x)) \in M.$$

for all  $n \geq 1$ , and so assumption (b) of Theorem 3.1 holds. Now, since all the hypotheses of Theorem 3.1 hold, then  $F$  and  $G$  have a coupled coincidence point. The proof is completed.  $\square$

**Corollary 3.5.** *In addition to the hypotheses of Corollary 3.4, suppose that for every  $(x, y), (z, t) \in X \times X$ , there exists  $(u, v) \in X \times X$  which is comparable to  $(x, y)$  and  $(z, t)$ . Then  $F$  and  $G$  have a unique coupled coincidence point.*

*Proof.* We define the subset  $M \subseteq X^4$  by

$$M = \{ (x, y, u, v) \in X^4 : x \preceq u \text{ and } y \succeq v \}.$$

From Example 2.14,  $M$  is an  $(G, F)$ -closed set which satisfies the transitive property. Thus, the proof of the existence of a coupled coincidence point is straightforward by following the same lines as in the proof of Corollary 3.4.

Next, we show the uniqueness of a coupled coincidence point of  $F$  and  $G$ .

Since for all  $(x, y), (z, t) \in X \times X$ , there exists  $(u, v) \in X \times X$  such that

$$G(x, y) \preceq G(u, v), \quad G(y, x) \succeq G(v, u)$$



and

$$G(z, t) \preceq G(u, v), \quad G(t, z) \succeq G(v, u),$$

we can conclude that

$$(G(x, y), G(y, x), G(u, v), G(v, u)) \in M$$

and

$$(G(z, t), G(t, z), G(u, v), G(v, u)) \in M.$$

Therefore, since all the hypotheses of Theorem 3.3 hold,  $F$  and  $G$  have a unique coupled coincidence point. The proof is completed.  $\square$

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