Thai Journal of Mathematics
Volume 14 (2016) Number 1 : 115–130
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

# A Nonlinear Parabolic Problems with Lower Order Terms and Measure Data

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**Abstract :** We prove the existence of a renormalized solution to the nonlinear parabolic equation and the second member is assumed to be in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\Omega))$.

**Keywords :** Parabolic problems; Sobolev space; Renormalized Solutions.
**2010 Mathematics Subject Classification :** Primary 47A15; Secondary; 46A32; 47D20.

## 1 Introduction

In the present paper, we establish the existence of a renormalized solution for a class of a nonlinear parabolic equations of type:

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - div(a(x,t,u,\nabla u)) + div(\phi(x,t,u)) = \mu \quad in\ Q_T \\ u(x,t) = 0 \quad on\ \partial\Omega \times (0,T) \\ b(x,u)|_{t=0} = b(x,u_0(x)) \quad in\ \Omega. \end{cases} \tag{1.1}$$

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In the problem (1.1), $\Omega$ is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, $T$ is a positive real number, $Q_T = \Omega \times (0,T)$. Let $-div\Big(a(x,t,u,\nabla u)\Big)$ be a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$, let $\phi(x,t,u)$ be a Carathéodory function (see assumptions (2.6)-(2.8)), and $b:\ \Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function such that for every $x\in\Omega$, $b(x,.)$ is a strictly increasing $C^1$-function, the data $u_0$ is in $L^1(\Omega)$ such that $b(.,u_0)$ in $L^1(\Omega)$. The measure $\mu = f - div(F)$ with $f\in L^1(Q_T)$ and $F\in (L^{p'}(Q))^N$.

Under our assumptions, problem (1.1) does not admit, in general, a weak solution since the term $\phi(x,t,u)$ may not belong $(L^1_{loc}(Q))^N$. In order to overcome this difficulty, we work with the framework of of renormalized solutions (see Definition 3.1). The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions [1] for the study of the Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics we refer to ([2], [3], [4]).

The existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [2] in the case where $a(x,t,s,\xi)$ is independent of $s$, and with $\phi = 0$, by D. Blanchard, F. Murat and H. Redwane [5] with the large monotonicity on $a$, by L. Aharouch, J. Bennouna and A. Touzani [6] and by A. Benkirane and J. Bennouna [7] in the Orlicz spaces and degenerated spaces.

In the case where $b(x,u) = u$, the existence of renormalized solutions for (1.1) has been established by R.-Di Nardo [8]. For the degenerated parabolic equation with $b(x,u)=u,\ div(\phi(x,t,u)) = H(x,t,u,\nabla u)$ and $f\in L^1(Q)$, the existence of renormalized solution has been proved by Y. Akdim and al [9].

The case where $\phi(x,t,u) = 0$ and $f\in L^1(Q_T)$, the existence of renormalized solutions has been established by H. Redwane [10] in the classical Sobolev space, and where $div(\phi(x,t,u)) = H(x,t,u,\nabla u)$ by Y. Akdim and al [11] in the degenerate Sobolev space without the sign condition and the coercivity condition on the term $H(x,t,u,\nabla u)$.

It is our purpose, in this paper to generalize the result of ([11], [9], [8]) and we prove the existence of a renormalized solution of (1.1).

The plan of the paper is as follows: In Section 2 we give some preliminaries and basic assumptions. In Section 3 we give the definition of a renormalized solution of (1.1), and we establish (Theorem 3.1) the existence of such a solution.

# 2 Assumptions on data and Preliminaries

## 2.1 Preliminaries

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ $(N\geq 2)$, $T$ is a positive real number, and $Q_T = \Omega\times(0,T)$. We need the Sobolev embeddings result

**Theorem 2.1.** *(Gagliardo-Nirenberg) Let $v$ be a function in $W_0^{1,q}(\Omega)\cap L^\rho(\Omega)$ with $q\geq 1$ and $\rho\geq 1$. Then there exists a positive constant $C$, depending on $N$, $q$ and*

*$\rho$, such that*

$$\| v \|_{L^\gamma(\Omega)} \leq C \| \nabla v \|^{\theta}_{(L^q(\Omega))^N} \| v \|^{1-\theta}_{L^\rho(\Omega)}$$

*for every $\theta$ and $\gamma$ satisfying*

$$0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta\Big(\frac{1}{q} - \frac{1}{N}\Big) + \frac{1-\theta}{\rho}.$$

## 2.2 Assumptions

Throughout this paper, we assume that the following assumptions hold true:

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function such that for every } x \in \Omega, \tag{2.1}$$

$b(x, .)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function with $b(x,0) = 0$, for any $k > 0$, there exists a constant $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^p(\Omega)$ such that: for almost every $x$ in $\Omega$

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \leq B_k(x) \quad \forall\ |s| \leq k. \tag{2.2}$$

Let $a :\ Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that, for any $k > 0$, there exist $\nu_k$ and a function $h_k \in L^{p'}(Q_T)$ with

$$|a(x,t,s,\xi)| \leq \nu_k\Big(h_k(x,t) + |\xi|^{p-1}\Big) \quad \forall\ |s| \leq k, \tag{2.3}$$

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p \ \text{ with } \alpha > 0, \tag{2.4}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta)(\xi - \eta) > 0 \ \text{ with } \xi \neq \eta. \tag{2.5}$$

Let $\phi :\ Q_T \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function such that

$$|\phi(x,t,s)| \leq c(x,t)|s|^\gamma, \tag{2.6}$$

$$c(x,t) \in L^\tau(Q_T) \quad \text{with} \quad \tau = \frac{N+p}{p-1}, \tag{2.7}$$

$$\gamma = \frac{N+2}{N+p}(p-1) \tag{2.8}$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbb{R}$ and every $\xi,\ \eta \in \mathbb{R}^N$.

$$f \in L^1(Q_T) \quad \text{and} \quad F \in (L^{p'}(Q_T))^N. \tag{2.9}$$

$$u_0 \in L^1(\Omega) \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{2.10}$$

Throughout the paper, $T_k$ denotes the truncation function at height $k \geq 0$:

$$T_k(r) = max(-k, min(k,r)) \qquad \forall\ r \in \mathbb{R}.$$

# 3 Main Results

In this section, we study the existence of renormalized solutions to problem (1.1).

**Definition 3.1.** *A measurable function $u$ is a renormalized solution to problem (1.1), if*

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{3.1}$$

$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for any } k>0, \tag{3.2}$$

$$\lim_{n\to+\infty} \frac{1}{n}\int_{\{(x,t)\in Q_T:\ |u(x,t)|\le n\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0, \tag{3.3}$$

*and if for every function $S$ in $W^{2,\infty}(\mathbb{R})$ which is piecewise $C^1$ and such that $S'$ has a compact support*

$$\begin{aligned}\frac{\partial B_S(x,u)}{\partial t} &- div\Big(a(x,t,u,\nabla u)S'(u)\Big) + S^{''}(u)a(x,t,u,\nabla u)\nabla u \\ &+ div\Big(\phi(x,t,u)S'(u)\Big) - S''(u)\phi(x,t,u)\nabla u \\ &= fS'(u) - div(S'(u)F) + S''(u)F\nabla u \text{ in } D^{'}(Q_T),\end{aligned} \tag{3.4}$$

*and*

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad in \quad \Omega, \tag{3.5}$$

*where* $B_S(x,z) = \displaystyle\int_0^z \frac{\partial b(x,s)}{\partial s} S^{'}(s)ds.$

Equation (3.4) is formally obtained through pointwise multiplication of equation (1.1) by $S'(u)$. However while $a(x,t,u,\nabla u)$ and $\phi(x,t,u)$ does not in general make sense in (1.1). Recall that for a renormalized solution, due to (3.2), each term in (3.4) has a meaning in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$ (see e.g. [5], [2], [12], [13], [14]).
We have

$$\frac{\partial B_S(x,u)}{\partial t} \text{ belongs to } L^{p^{'}}(0,T;W^{-1,p^{'}}(\Omega)) + L^1(Q). \tag{3.6}$$

The properties of $S$, assumptions (2.2) and (3.2) imply that if $K$ is such that $supp\ S^{'} \subset [-K,K]$

$$\Big|\nabla B_S(x,u)\Big| \le \|A_K\|_{L^\infty(\Omega)}|DT_K(u)|\|S'\|_{L^\infty(\mathbb{R})} + K\|S'\|_{L^\infty(\mathbb{R})}B_K(x) \tag{3.7}$$

and

$$B_S(x,u) \text{ belongs to } L^p(0,T;W_0^{1,p}(\Omega)). \tag{3.8}$$

Then (3.6) and (3.8) imply that $B_S(x,u)$ belongs to $C^0([0,T];L^1(\Omega))$ (for a proof of this trace result see [15]), so that the initial condition (3.5) makes sense.

**Remark 3.1.** *For every $S \in W^{1,\infty}(\mathbb{R})$, nondecreasing function such that $suppS' \subset [-K,K]$, in view (2.2) we have*

$$\lambda_K|S(r)-S(r')| \leq \Big|B_S(x,r)-B_S(x,r')\Big| \leq \|A_K\|_{L^\infty(\Omega)}|S(r)-S(r')| \qquad (3.9)$$

*for almost every $x \in \Omega$ and for every $r,\ r' \in \mathbb{R}$.*

**Theorem 3.2.** *Under assumptions (2.2)-(2.10), then problem (1.1) admits a renormalized solution $u$ in the sense of Definition 3.1.*

**Step 1: Approximate problem and a priori estimates.** For each $\epsilon > 0$, we define the following approximations

$$b_\epsilon(x,r) = T_{\frac{1}{\epsilon}}(b(x,r)) + \epsilon\ r \quad \forall\ r \in \mathbb{R}, \qquad (3.10)$$

$$a_\epsilon(x,t,s,\xi) = a(x,t,T_{\frac{1}{\epsilon}}(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ s \in \mathbb{R},\ \forall\ \xi \in \mathbb{R}^N, \qquad (3.11)$$

$$\phi_\epsilon(x,t,r) = \phi(x,t,T_{\frac{1}{\epsilon}}(r)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ r \in \mathbb{R}, \qquad (3.12)$$

Let $f_\epsilon \in L^{p'}(Q_T)$ such that

$$\|f_\epsilon\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)} \text{ and } f_\epsilon \to f \text{ strongly in } L^1(Q_T). \qquad (3.13)$$

Let $u_{0\epsilon} \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\|b_\epsilon(x,u_{0\epsilon})\|_{L^1(\Omega)} \leq \|b(x,u_0)\|_{L^1(\Omega)} \text{ and } b_\epsilon(x,u_{0\epsilon}) \to b(x,u_0) \text{ strongly in } L^1(\Omega). \qquad (3.14)$$

In view of (3.10), $b_\epsilon$ is a Carathéodory function and satisfies (2.2), there exists $\lambda_\epsilon > 0$ and a function $A_\epsilon \in L^\infty(\Omega)$ and $B_\epsilon \in L^p(\Omega)$ such that:

$$\lambda_\epsilon \leq \frac{\partial b_\epsilon(x,s)}{\partial s} \leq A_\epsilon(x) \qquad \text{and} \qquad |\nabla_x(\frac{\partial b_\epsilon(x,s)}{\partial s})| \leq B_\epsilon(x) \quad a.e.\ x \in \Omega,\ \forall s \in \mathbb{R}.$$

Consider the approximate problem:

$$\begin{cases} \dfrac{\partial b_\epsilon(x,u_\epsilon)}{\partial t} - div(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)) + div(\phi_\epsilon(x,t,u_\epsilon)) = f_\epsilon - div(F) \quad in \quad Q_T \\ u_\epsilon(x,t) = 0 \quad on \quad \partial\Omega \times (0,T) \\ b_\epsilon(x,u_\epsilon)(t=0) = b_\epsilon(x,u_{0\epsilon}) \quad in \quad \Omega. \end{cases} \qquad (3.15)$$

As a consequence, proving existence of a weak solution $u_\epsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ is an easy task (see [16]).

**Step 2:** The estimates derived in this step rely on standard techniques for problems of type (3.15). Let $\tau_1 \in (0,T)$ and $t$ fixed in $(0,\tau_1)$. Using $T_k(u_\epsilon)\chi_{(0,t)}$ as test function in (3.15), we integrate between $(0,\tau_1)$, and by the condition (2.6) we have

$$\int_\Omega B_k^\epsilon(x,u_\epsilon(t))dx + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)\,dx\,ds \qquad (3.16)$$

$$\leq \int_{Q_t} c(x,t)|u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)|\,dx\,ds + \int_{Q_t} f_\epsilon T_k(u_\epsilon)\,dx\,ds + \int_\Omega B_k^\epsilon(x,u_{0\epsilon})dx + \int_{Q_t} F\nabla T_k(u)dxds$$

where $B_k^\epsilon(x,r) = \int_0^r T_k(s)\frac{\partial b_\epsilon(x,s)}{\partial s}ds$. Due to definition of $B_k^\epsilon$ we have:

$$0 \leq \int_\Omega B_k^\epsilon(x,u_{0\epsilon})dx \leq k\int_\Omega |b_\epsilon(x,u_{0\epsilon})|dx = k||b(x,u_{0\epsilon})||_{L^1(\Omega)} \quad \forall k>0 \tag{3.17}$$

Using (3.16) and (2.4) we obtain:

$$\int_\Omega B_k^\epsilon(x,u_\epsilon(t))dx + \alpha\int_{Q_t} |\nabla T_k(u_\epsilon)|^p\,dx\,ds$$

$$\leq \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,ds\,dx + k(\|b(x,u_{0\epsilon})\|_{L^1(\Omega)} + \|f\|_{L^1(Q_T)}) + \int_{Q_t} F\nabla T_k(u)dxds. \tag{3.18}$$

Let $M = \Big(||f||_{L^1(Q_T)} + ||b(x,u_{0\epsilon})||_{L^1(\Omega)}\Big)$, remark that

$$B_k^\epsilon(x,s) = \int_0^s T_k(\sigma)\frac{\partial b_\epsilon(x,\sigma)}{\partial\sigma}d\sigma \geq \frac{\lambda_\epsilon}{2}|T_k(s)|^2$$

we deduce from (3.16) and (3.17) that

$$\frac{\lambda_\epsilon}{2}\int_\Omega |T_k(u_\epsilon)|^2\,dx + \alpha\int_{Q_t}|\nabla T_k(u_\epsilon)|^p\,dx\,ds \tag{3.19}$$

$$\leq Mk + \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds + \int_{Q_t} F\nabla T_k(u)dxds.$$

By Gagliardo-Nirenberg and Young inequalities we have:

$$\int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds \leq C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})} sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx$$

$$+\,C\;\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big(\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\Big)^{(\frac{1}{p}+\frac{N\gamma}{(N+2)p})\frac{N+2}{N+2-\gamma}}. \tag{3.20}$$

Since $\gamma = \frac{(N+2)}{N+p}(p-1)$ and by using (3.19) and (3.20), we obtain

$$\frac{\lambda_\epsilon}{2}\int_\Omega |T_k(u_\epsilon)|^2\,dx + \alpha\int_{Q_t}|\nabla T_k(u_\epsilon)|^p\,dx\,ds$$

$$\leq Mk + C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})} sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx$$

$$+C\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds + (\frac{\alpha}{p})^{-(p-1)}\|F\|_{(L^{p'}(Q))^N} + \frac{\alpha}{p}\int_{Q_t}|\nabla T_k(u_\epsilon)|^p\,dx\,ds$$

Which is equivalent to

$$\Big(\frac{\lambda_\epsilon}{2}-C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big)sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx+\frac{\alpha}{p'}\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds$$
$$-\Big(C\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big)\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\leq Mk$$

If we choose $\tau_1$ such that

$$\Big(\frac{\lambda_\epsilon}{2}-C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big)\geq 0, \tag{3.21}$$

and

$$\Big(\frac{\alpha}{p'}-C\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big)\geq 0, \tag{3.22}$$

then, let us denote by C the minimum between (3.21) and (3.22), we obtain

$$sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx+\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,dt\leq CMk \tag{3.23}$$

Then, by (3.23) and lemma 3.1, we conclude that $T_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega))$ independently of $\epsilon$ and for any $k\geq 0$, so there exists a subsequence still denoted by $u_\epsilon$ such that

$$T_k(u_\epsilon)\rightharpoonup H_k \quad \text{weakly in} \quad L^p(0,T,W_0^{1,p}(\Omega)) \tag{3.24}$$

We turn now to prove the almost every convergence of $u_\epsilon$ and $b_\epsilon(u_\epsilon)$.
Let $k>0$ be large enough and and $B_R$ be a ball of $\Omega$, we have:

$$\begin{aligned} k\ meas\Big\{\{|u_\epsilon|>k\}\cap B_R\times[0,T]\Big\} &= \int_0^T\int_{\{|u_\epsilon|>k\}\cap B_R}|T_k(u_\epsilon)|dxdt \\ &\leq \int_0^T\int_{B_R}|T_k(u_\epsilon)|dxdt \\ &\leq \Big(\int_Q|T_k(u_\epsilon)|^p dxdt\Big)^{\frac{1}{p}}\Big(\int_0^T\int_{B_R}dxdt\Big)^{\frac{1}{p'}} \\ &\leq TC_R(CMk)^{\frac{1}{p}} \end{aligned}$$

Which implies that: $meas\Big\{\{|u_\epsilon|>k\}\cap B_R\times[0,T]\Big\}\leq \dfrac{c_1}{k^{1-\frac{1}{p}}}\quad \forall k\geq 1$, so we have

$$\lim_{k\to+\infty} meas\Big\{\{|u_\epsilon|>k\}\cap B_R\times[0,T]\Big\}=0.$$

Consider now a function non decreasing $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\leq\frac{k}{2}$ and $g_k(s)=k$ for $|s|\geq k$. Multiplying the approximate equation by $g'_k(u_\epsilon)$, we get

$$\frac{\partial B_k^\epsilon(x,u_\epsilon)}{\partial t}-div\Big(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)g'_k(u_\epsilon)\Big)+a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)g''_k(u_\epsilon)\nabla u_\epsilon+div\Big(\phi_\epsilon(x,t,u_\epsilon)g'_k(u_\epsilon)\Big)$$

$$- g_k''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon = f_\epsilon g_k'(u_\epsilon) - div(Fg_k'(u_\epsilon)) + Fg_k''(u_\epsilon)\nabla u_\epsilon \qquad \text{in } D'(Q_T) \tag{3.25}$$

where $B_g^\epsilon(x,z) = \displaystyle\int_0^z \frac{\partial b_\epsilon(x,s)}{\partial s} g_k'(s)ds.$

In view of (2.3), (3.11), (3.25) and since $T_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega))$, we deduce that $g_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega))$ and $\dfrac{\partial B_g^\epsilon(x,u_\epsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}(0,T,W^{-1,p'}(\Omega))$. Indeed, since $supp(g_k')$ and $supp(g_k'')$ are both included in [-k,k] by (3.12) it follows that for: $0 < \epsilon < \frac{1}{k}$

$$\begin{aligned} |\int_{Q_T} \phi_\epsilon(x,t,u_\epsilon)^{p'}(g_k'(u_\epsilon))^{p'}\,dx\,dt| &\le \int_{Q_T} c(x,t)^{p'}|T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'\gamma}|g_k'(u_\epsilon))|^{p'}\,dx\,dt \\ &= \int_{\{|u_\epsilon|\le k\}} c(x,t)^{p'}|T_k(u_\epsilon)|^{p'\gamma}|g_k'(u_\epsilon)|^{p'}\,dx\,dt \end{aligned}$$

Furthermore, by Hölder and Gagliardo-Niremberg inequality, it results

$$\int_{\{|u_\epsilon|\le k\}} c(x,t)^{p'}|T_k(u_\epsilon)|^{p'\gamma}|g_k'(b_\epsilon(u_\epsilon))|^{p'}\,dx\,dt$$

$$\le \|g_k'\|_{L^\infty(\mathbb{R})}||c(x,t)||^{p'}_{L^\tau(Q_T)}\Big[sup_{t\in(0,T)}(\int_\Omega |T_k(u_\epsilon)|^2)^{\frac{p}{N}} + \int_{Q_T}|\nabla T_k(u_\epsilon)|^p\,dx\,dt\Big] \le c_k.$$

where $c_k$ is a constant independently of $\epsilon$ which will vary from line to line.
In the same by (2.6) we have :

$$|\int_{Q_T} \phi_\epsilon(x,t,u_\epsilon)^{p'}(g_k''(u_\epsilon)\nabla u_\epsilon)^{p'}\,dx\,dt| \le \int_{Q_T}(g_k''(u_\epsilon))^{p'}|c(x,t)|^{p'}|T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'}|\nabla u_\epsilon|^{p'}\,dx\,dt \tag{3.26}$$

Furthermore, by Hölder and Gagliardo-Niremberg inequality,we obtain for $0 < \epsilon < \frac{1}{k}$

$$\begin{aligned} &\int_{Q_T}(g_k''(u_\epsilon))^{p'}|c(x,t)|^{p'}|T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'\gamma}|\nabla u_\epsilon|^{p'}\,dx\,dt \\ &= \int_{Q_T}(g_k''(u_\epsilon))^{p'}|c(x,t)|^{p'}|T_k(u_\epsilon)|^{p'\gamma}|\nabla T_k(u_\epsilon)|^{p'}\,dx\,dt \\ &\le \|g_k''\|_{L^\infty(\mathbb{R})}\int_{Q_T}|c(x,t)|^{p'}|T_k(u_\epsilon)|^{p'\gamma}|\nabla T_k(u_\epsilon)|^{p'}\,dx\,dt \le c_k \end{aligned}$$

We conclude by (3.25) that

$$\frac{\partial g_k(u_\epsilon)}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T,W^{-1,p'}(\Omega)). \tag{3.27}$$

Arguing again as in [12], estimates (3.24) and (3.27) imply that, for a subsequence, still indexed by $\epsilon$,

$$u_\epsilon \to u \text{ a.e. } Q_T, \tag{3.28}$$

where $u$ is a measurable function defined on $Q_T$. Let us prove that $b(x,u)$ belongs to $L^\infty((0,T),L^1(\Omega))$. Using (3.18), (3.19), (3.20) and (3.23) we deduce that

$$\int_\Omega B_k^\epsilon(x,u_\epsilon)dx \le MkC + C_1. \tag{3.29}$$

In view of (3.28) and passing to the limit-inf in (3.29) as $\epsilon$ tends to zero, we obtain that with $B_k(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s}T_k(s)ds$. On the other hand, we have

$$\frac{1}{k}\int_\Omega B_k(x,u(\tau))dx \le C_2, \tag{3.30}$$

for almost any $\tau$ in $(0,T)$. Due to the definition of $B_k(x,s)$ and the fact that $\frac{1}{k}B_k(x,u)$ converges pointwise to $\int_0^u sg(s)\frac{\partial b(x,s)}{\partial s}\,ds = |b(x,u)|$, as $k$ tends to $+\infty$, shows that $b(x,u) \in L^\infty(0,T;L^1(\Omega))$.

**Lemma 3.3.** *The subsequence of $u_\epsilon$ defined in Step 1 satisfies*

$$\lim_{n\to+\infty}\limsup_{\epsilon\to 0}\frac{1}{n}\int_{\{|u_\epsilon|\le n\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\,dx\,dt = 0. \tag{3.31}$$

*Proof.* Using the test function $\psi_n(u_\epsilon) \equiv \frac{T_n(u_\epsilon)}{n}$ in (3.15), and by (3.12)we get

$$\int_0^T < \frac{\partial b_\epsilon(x,u_\epsilon)}{\partial t}, \psi_n(u_\epsilon) > \,dt + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon)\,dx\,dt \tag{3.32}$$

$$\le \int_{Q_T} c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\psi_n(u_\epsilon)|\,dx\,dt + \int_{Q_T} f_\epsilon\psi_n(u_\epsilon)\,dx\,dt + \int_{Q_T} F\nabla\psi_n(u_\epsilon)\,dx\,dt,$$

hence

$$\int_\Omega B_n(x,u_\epsilon)(T)dx + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon)\,dx\,dt$$

$$\le \int_{Q_T} c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\psi_n(u_\epsilon)|\,dx\,dt + \int_\Omega B_n(x,u_{0\epsilon})dx + \int_{Q_T} f_\epsilon\psi_n(u_\epsilon)\,dx\,dt + \int_{Q_T} F\nabla\psi_n(u_\epsilon)$$

where $B_n(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s}\psi_n(s)\,ds$. Since $B_n(x,u_\epsilon)(T) \ge 0$, then for every $\epsilon < \frac{1}{n}$, we have

$$\begin{aligned}\frac{1}{n}\int_{\{|u_\epsilon\}|<n} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\,dx\,dt &\le \frac{1}{n}\int_{Q_T} c(x,t)|T_n(u_\epsilon)|^\gamma|\nabla T_n(u_\epsilon)|\,dx\,dt \\ &+ \int_\Omega B_n(x,u_{0\epsilon})dx + \frac{1}{n}\int_{Q_T} f_\epsilon T_n(u_\epsilon)\,dx\,dt + \frac{1}{n}\int_{Q_T} F\nabla T_n(u_\epsilon)\,dx\,dt.\end{aligned} \tag{3.33}$$

Proceeding as in ([5], [17]), using Young inequality and Galgliardo-Niremberg inequality, we obtain for all $R < n$:

$$\frac{1}{n}\int_{\{|u_\epsilon|<n\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \, dx\, dt \tag{3.34}$$

$$\le \frac{c_1}{n}||c(x,t)\chi_{\{|u^\epsilon|\ge R\}}||_{L^r(Q_T)}\Big(sup_{t\in(0,T)}\int_\Omega |T_n(u_\epsilon)|^2dx\Big)^{\frac{1}{r}}\Big(\int_{Q_T}|T_n(u_\epsilon)|^p\Big)^{\frac{N+1}{N+p}}$$

$$+\frac{1}{n}\int_{\{|u_\epsilon|\le R\}} c(x,t)|T_R(u_\epsilon)|^\gamma|\nabla T_R(u_\epsilon)|\, dx\, dt$$

$$+\int_\Omega B_n(x,u_{0\epsilon})dx+\frac{1}{n}\int_{Q_T} f_\epsilon T_n(u_\epsilon)\, dx\, dt+\frac{\alpha}{2pn}\int_{Q_T}|\nabla T_n(u_\epsilon)|^p+\frac{2^{\frac{p'}{p}}\alpha^{\frac{-p'}{p}}}{np'}||F||^{p'}_{L^{p'}(Q)}.$$

Recalling that $u_\epsilon$ is bounded in $L^\infty(0,T;L^1(\Omega))$, we obtain

$$\frac{1}{n}\int_{\{|u_\epsilon|<n\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \, dx\, dt \tag{3.35}$$

$$\le c_2||c(x,t)\chi_{\{|u^\epsilon|\ge R\}}||_{L^r(Q_T)}+\frac{\alpha}{2pn}\int_{Q_T}|T_n(u_\epsilon)|^p\, dx\, dt$$

$$+\frac{1}{n}\int_{\{|u_\epsilon|\le R\}} c(x,t)|T_R(u_\epsilon)|^\gamma|\nabla T_R(u_\epsilon)|\, dx\, dt$$

$$+\int_\Omega B_n(x,u_{0\epsilon})dx+\frac{1}{n}\int_{Q_T} f_\epsilon T_n(u_\epsilon)\, dx\, dt+\frac{\alpha}{2np}\int_{Q_T}|\nabla T_n(u_\epsilon)|^p+\frac{2^{\frac{p'}{p}}\alpha^{\frac{-p'}{p}}}{np'}||F||^{p'}_{L^{p'}(Q)}.$$

Note that $T_n(u_\epsilon)$ converges to $T_n(u)$ in $L^\infty(Q_T)$ weak-$*$, and $u$ is finite almost everywhere in $Q_T$, then $\frac{1}{n}T_n(u)$ converges to zero almost everywhere in $Q_T$. Since $a$ satisfies (2.4) and in view of (3.35), we deduce that

$$\Big(\frac{p-1}{p}\Big)\frac{1}{n}\int_{\{|u_\epsilon|<n\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \, dx\, dt \tag{3.36}$$

$$\le c_2||c(x,t)\chi_{\{|u^\epsilon|\ge R\}}||_{L^r(Q_T)}+\frac{1}{n}\int_{Q_T} c(x,t)|T_R(u_\epsilon)|^\gamma|\nabla T_R(u_\epsilon)|\, dx\, dt$$

$$+\int_\Omega B_n(x,u_{0\epsilon})dx+\frac{1}{n}\int_{Q_T} f_\epsilon T_n(u_\epsilon)\, dx\, dt+\frac{2^{\frac{p'}{p}}\alpha^{\frac{-p'}{p}}}{np'}||F||^{p'}_{L^{p'}(Q_T)}.$$

In view of (2.7), (2.9), (3.13), (3.14), (3.24), (3.28), using Lebesgue's convergence theorem, and and passing to limit in (3.36) as $\epsilon$ tends to zero, then $n$ tends to $+\infty$ and then $R$ tends to $+\infty$, is an easy task and we conclude that $u_\epsilon$ satisfies lemma (3.3). □

**Step 4:** In this step we prove that the weak limit $\sigma_k$ of $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))$ can be identified with $a(x,t,T_k(u),\nabla T_k(u))$. In order to prove this result we recall the following lemma:

**Lemma 3.4.** *The subsequence of $u_\epsilon$ satisfies for any $k \geq 0$:*

$$\limsup_{\epsilon\to 0} \int_{Q_T}\int_0^t a(x,s,u_\epsilon,\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon)\,ds\,dx\,dt \leq \int_{Q_T}\int_0^t \sigma_k \nabla T_k(u)\,dx\,ds\,dt, \tag{3.37}$$

$$\lim_{\epsilon\to 0}\int_{Q_T}\int_0^t \Big(a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))-a(x,t,T_k(u_\epsilon),\nabla T_k(u))\Big)\Big(\nabla T_k(u_\epsilon)-\nabla T_k(u)\Big)=0, \tag{3.38}$$

$$\sigma_k = a(x,t,T_k(u),\nabla T_k(u))) \quad a.e.\ in\ Q_T, \tag{3.39}$$

*and as $\epsilon$ tends to 0*

$$a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \tag{3.40}$$

*weakly in $L^1(Q_T)$.*

*Proof.* We introduce a time regularization of the $T_k(u)$ for $k>0$ in order to perform the monotonicity method. This kind regularization has been introduced at the first time by R. Landes in [18]. Let $v_0^\mu$ be a sequence of function in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \leq k$ for all $\mu>0$ and $v_0^\mu$ converges to $T_k(u_0)$ a.e. in $\Omega$ and $\frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)}$ converges to 0. For $k \geq 0$ and $\mu>0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu((T_k(u))_\mu - T_k(u)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(u))_\mu(t=0) = \nu_0^\mu \text{ in } \Omega.$$

Remark that $(T_k(u))_\mu$ converges to $T_k(u)$ a.e. in $Q_T$, weakly-$*$ in $L^\infty(Q_T)$ and strongly in $L^p(0,T;W_0^p(\Omega))$ as $\mu \to +\infty$, and we have

$$||(T_k(u))_\mu||_{L^\infty(Q_T)} \leq max(||(T_k(u))||_{L^\infty(Q_T)}, ||\nu_0^\mu||_{L^\infty(\Omega)}) \leq k,\ \forall\ \mu>0,\ \forall\ k>0.$$

**Lemma 3.5.** *(see H. Redwane [19]) Let $k \geq 0$ be fixed. Let $S$ be an increasing $C^\infty(\mathbb{R})-$function such that $S(r)=r$ for $|r| \leq k$, and $suppS'$ is compact. Then*

$$\liminf_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t < \frac{\partial b_\epsilon(x,u_\epsilon)}{\partial t}, S'(u_\epsilon)(T_k(u_\epsilon)-(T_k(u))_\mu) > \geq 0.$$

*where $<.,.>$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'}(\Omega)$ and $L^\infty(\Omega)\cap W^{1,p}(\Omega)$.*

Let $S_n$ be a sequence of increasing $C^\infty$-function such that:

$S_n(r) = r$ for $|r| \leq n,\ supp(S'_n) \subset [-(n{+}1), (n{+}1)]$ and $\|S''_n\|_{L^\infty(\mathbb{R})} \leq 1$ for any $n \geq 1$.

We use the sequence $(T_k(u))_\mu$ of approximation of $T_k(u)$, and plug the test function $S'_n(u_\epsilon)(T_k(u_\epsilon) - (T_k(u))_\mu)$ for $n > 0$ and $\mu > 0$. For fixed $k \geq 0$, let $W^\epsilon_\mu = T_k(u_\epsilon) - (T_k(u))_\mu$ we obtain upon integration over $(0,t)$ and then over $(0,T)$ :

$$\int_0^T \int_0^t < \frac{\partial b_\epsilon(x,u_\epsilon)}{\partial t}, S'_n(u_\epsilon) W^\epsilon_\mu > \, ds\, dt + \int_{Q_T}\int_0^t a_\epsilon(x,s,u_\epsilon,\nabla u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, ds\, dt\, dx$$

$$+ \int_{Q_T}\int_0^t a_\epsilon(x,s,u_\epsilon,\nabla u_\epsilon) S''_n(u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, ds\, dt\, dx \tag{3.41}$$

$$- \int_{Q_T}\int_0^t \phi_\epsilon(x,s,u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, ds\, dt\, dx$$

$$- \int_{Q_T}\int_0^t S''_n(u_\epsilon)\phi_\epsilon(x,s,u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, ds\, dt\, dx = \int_{Q_T}\int_0^t f_\epsilon S'_n(u_\epsilon) W^\epsilon_\mu \, dx\, ds\, dt$$

$$+ \int_{Q_T}\int_0^t F S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, ds\, dt\, dx + \int_{Q_T}\int_0^t F S''_n(u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, ds\, dt\, dx.$$

We pass to the limit in (3.41) as $\epsilon \to 0,\ \mu \to +\infty$ and then $n \to +\infty$ for $k$ real number fixed. We use lemma 3.5 and proceeding as in ([5], [19]), then it possible to conclude that

$$\liminf_{\mu\to+\infty} \lim_{\epsilon\to 0} \int_0^T \int_0^t < \frac{\partial b_\epsilon(x,u_\epsilon)}{\partial t}, W^\epsilon_\mu > \, ds\, dt \geq 0 \qquad \text{for any } n \geq k, \tag{3.42}$$

$$\lim_{n\to+\infty} \limsup_{\mu\to+\infty} \limsup_{\epsilon\to 0} \int_{Q_T}\int_0^t a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) S''_n(u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, ds\, dt\, dx = 0, \tag{3.43}$$

$$\lim_{\mu\to+\infty} \lim_{\epsilon\to 0} \int_{Q_T}\int_0^t f_\epsilon S'_n(u_\epsilon) W^\epsilon_\mu \quad ds\, dt\, dx = 0, \tag{3.44}$$

$$\lim_{\mu\to+\infty} \int_{Q_T}\int_0^t F S'_n(u_\epsilon) \nabla W^\epsilon_\mu \quad ds\, dt\, dx = 0, \tag{3.45}$$

$$\lim_{\mu\to+\infty} \int_{Q_T}\int_0^t F S''_n(u_\epsilon) \nabla u_\epsilon W^\epsilon_\mu \quad ds\, dt\, dx = 0. \tag{3.46}$$

Now we prove that for any $n \geq 1$:

$$\lim_{\mu\to+\infty} \lim_{\epsilon\to 0} \int_{Q_T}\int_0^t \phi_\epsilon(x,t,u_\epsilon) S'_n(u_\epsilon) \nabla W^\epsilon_\mu \, ds\, dt\, dx = 0, \tag{3.47}$$

and

$$\lim_{\mu\to+\infty} \lim_{\epsilon\to 0} \int_{Q_T}\int_0^t S''_n(u_\epsilon) \phi_\epsilon(x,t,u_\epsilon) \nabla u_\epsilon \nabla W^\epsilon_\mu \, ds\, dt\, dx = 0. \tag{3.48}$$

**Proof of (3.47)**: Let us recall the main properties of $W_\mu^\epsilon$. For fixed $\mu > 0$ : $W_\mu^\epsilon$ converges to $T_k(u) - (T_k(u))_\mu$ weakly in $L^p(0, T, W_0^{1,p}(\Omega))$ as $\epsilon \to 0$. Remark that

$$||W_\mu^\epsilon||_{L^\infty(Q_T)} \leq 2k \qquad \text{for any } \epsilon > 0,\ \mu > 0, \tag{3.49}$$

then we deduce that

$$W_\mu^\epsilon \rightharpoonup T_k(u) - (T_k(u))_\mu \quad \text{a.e in } Q_T \text{ and in } L^\infty(Q_T) \text{ weak}_*,\ \text{when } \epsilon \to 0. \tag{3.50}$$

One had $suppS' \subset [-(n+1), n+1]$ for any fixed $n \geq 1$ and $0 < \epsilon < \frac{1}{n+1}$, we have $\phi_\epsilon(x, t, u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon = \phi_\epsilon(x, t, T_{n+1}(u_\epsilon))S_n'(u_\epsilon)\nabla W_\mu^\epsilon$ a.e. in $Q_T$. On the other hand $\phi_\epsilon(x, t, T_{n+1}(u_\epsilon))S_n'(u_\epsilon) \to \phi(x, t, T_{n+1}(u))S_n'(u)$ a.e. in $Q_T$ and

$$|\phi_\epsilon(x, t, T_{n+1}(u_\epsilon))S_n'(u_\epsilon)| \leq c(x, t)(n+1)^\gamma \quad \text{for } n \geq 1.$$

By (3.50) and strongly convergence of $T_k(u_\epsilon)_\mu$ in $L^p(0, T, W_0^{1,p}(\Omega))$ we obtain (3.47).

**Proof of (3.48)**: For any fixed $n \geq 1$ and $0 < \epsilon < \frac{1}{n+1}$:

$$\phi_\epsilon(x, t, u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon W_\mu^\epsilon = \phi_\epsilon(x, t, T_{n+1}(u_\epsilon))S_n''(u_\epsilon)\nabla T_{n+1}(u_\epsilon)W_\mu^\epsilon \quad \text{a.e. in } Q_T,$$

By (3.49) and (3.50) it is possible to pass to the limit for $\epsilon \to 0$, and we obtain

$$\phi_\epsilon(x, t, T_{n+1}(u_\epsilon))S_n''(u_\epsilon)W_\mu^\epsilon \to \phi(x, t, T_{n+1}(u))S_n''(u)W_\mu \quad \text{a.e. in } Q_T.$$

Since $|\phi(x, t, T_{n+1}(u))S_n''(u)W_\mu| \leq 2k|c(x, t)|(n+1)^\gamma$ a.e. in $Q_T$ and $(T_k(u))_\mu$ converges to 0 in $L^p(0, T; W_0^{1,p}(\Omega))$, we obtain (3.48).

Recalling (3.42), (3.47), (3.48), (3.43), (3.44), (3.45) and (3.46) the proof of (3.37) is complete.

Proceeding as in [5], it can be deduced from (3.37) that (3.38), (3.39) and (3.40) hold true. □

Note that, taking the limit as $\epsilon$ tends to 0 in (3.31) and using (3.40) show that u satisfies (3.3). Now we want to prove that u satisfies the equation (3.4).

Let $S$ be a function in $W^{2,\infty}(\mathbb{R})$ such that $suppS' \subset [-k, k]$ where $k$ is a real positive number. Pointwise multiplication of the approximate equation (3.15) by $S'(u_\epsilon)$ leads to

$$\begin{aligned} \frac{\partial B_S^\epsilon(x, u_\epsilon)}{\partial t} &- div\Big(a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon)S'(u_\epsilon)\Big) + S''(u_\epsilon)a(x, t, u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon \\ &+ div\Big(\phi_\epsilon(x, t, u_\epsilon)S'(u_\epsilon)\Big) - S''(u_\epsilon)\phi_\epsilon(x, t, u_\epsilon)\nabla u_\epsilon \\ &= f_\epsilon S'(u_\epsilon) - div(FS'(u_\epsilon)) + S''(u_\epsilon)F\nabla u_\epsilon \quad \text{in } D'(Q_T), \end{aligned} \tag{3.51}$$

where $B_S^\epsilon(x, r) = \displaystyle\int_0^r \frac{\partial b^\epsilon(x, s)}{\partial s}S'(s)\, ds$. In what follows we pass to the limit as $\epsilon$ tends to $O$ in each term of (3.51). Since $u_\epsilon$ converges to $u$ a.e. in $Q_T$

implies that $B_S^\epsilon(x,u_\epsilon)$ converges to $B_S(x,u)$ a.e. in $Q_T$ and $L^\infty(Q_T)$ weak-$*$, then $\dfrac{\partial B_S^\epsilon(x,u_\epsilon)}{\partial t}$ converges to $\dfrac{\partial B_S(x,u)}{\partial t}$ in $D'(Q_T)$. We observe that the term $a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon)$ can be identified with $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))S'(u_\epsilon)$ for $\epsilon \leq \frac{1}{k}$, so using the pointwise convergence of $u_\epsilon$ to $u$ in $Q_T$, the weakly convergence of $T_k(u_\epsilon)$ to $T_k(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we get

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon) \rightharpoonup a(x,t,T_k(u_\epsilon),\nabla T_k(u))S'(u) \quad \text{in } L^{p'}(Q_T),$$

and

$$S''(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \rightharpoonup S''(u)a(x,t,T_k(u_\epsilon),\nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(Q_T).$$

Furthermore, since $\phi_\epsilon(x,t,u_\epsilon)S'(u_\epsilon) = \phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon)$ a.e. in $Q_T$. By (3.12) we obtain $|\phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon)| \leq |c(x,t)|k^\gamma$, it follows that

$$\phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon) \to \phi_\epsilon(x,t,T_k(u))S'(u) \quad \text{strongly in } L^{p'}(Q_T).$$

In a similar way, it results

$$S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon = S''(T_k(u_\epsilon))\phi_\epsilon(x,t,T_k(u_\epsilon))\nabla T_k(u_\epsilon) \quad \text{a.e. in } Q_T.$$

Using the weakly convergence of $T_k(u_\epsilon)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ it is possible to prove that

$$S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon \to S''(u)\phi(x,t,u)\nabla u \quad \text{in } L^1(Q_T),$$

and $S''(u_\epsilon)F\nabla u_\epsilon$ converges to $S''(u)F\nabla u$ in $L^1(Q_T)$. Since $|S'(u_\epsilon)| \leq C$, it follow that $FS''(u_\epsilon)$ converges to $FS''(u)$ strongly in $L^{p'}(Q_T)$.

Finally by (3.13) we deduce that $f_\epsilon S'(u_\epsilon)$ converges to $fS'(u)$ in $L^1(Q_T)$. It remains to prove that $B_S(x,u)$ satisfies the initial condition $B_S(x,u)(t=0) = B_S(x,u_0)$ in $\Omega$. To this end, firstly remark that $B_S^\epsilon(x,u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ (see (3.7)). Secondly the above considerations of the behavior of the terms of this equation show that $\frac{\partial B_S^\epsilon(x,u_\epsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, $B_S^\epsilon(u_\epsilon)(t=0) = B_S^\epsilon(x,u_{0\epsilon})$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$ (for a proof of this trace result see [15]). On the other hand, the smoothness of of $S$ implies that $B_S(x,u)(t=0) = B_S(x,u_0)$ in $\Omega$. The proof of Theorem 3.1 is complete.

**Acknowledgements :** I would like to thank the referees for his comments and suggestions on the manuscript.

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(Received 23 May 2013)
(Accepted 22 July 2014)