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E-Torsion Free Acts Over Monoids

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Abstract : In this paper we introduce E-torsion freeness of acts over monoids, and will give a characterization of monoids by this property of their (cyclic, monocyclic, Rees factor) acts.

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1 Introduction

Throughout this paper S will denote a monoid. We refer the reader to [1] and [2] for basic definitions and terminology relating to semigroups and acts over monoids and to [3], [4], [5] and [6] for definitions and results on flatness which are used here. A monoid S is called *left(right) collapsible* if for any $s, s' \in S$ there exists $z \in S$ such that $zs = zs'$ ($sz = s'z$). A submonoid P of S is called *weakly left collapsible* if for any $s, s' \in P$, $z \in S$, $sz = s'z$ implies the existence of $u \in P$ such that $us = us'$. It is obvious that every left collapsible submonoid is weakly left collapsible, but the converse is not true. A monoid S is called *right (left) reversible*, if for any $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$ ($su = s'v$). A submonoid P of S is called *weakly right reversible*, if for any $s, s' \in P$, $z \in S$, $sz = s'z$ implies the existence of $u, v \in P$ such that $us = vs'$. A right ideal K_S of a monoid S is called *left stabilizing*, if for any $k \in K_S$, there exists $l \in K_S$ such that $lk = k$. K_S is called *left annihilating*, if for any $t \in S$, $x, y \in S \setminus K_S$, $xt, yt \in K_S$

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implies that $xt = yt$. K_S is called *strongly left annihilating*, if for all $s, t \in S \setminus K_S$ and for all homomorphisms $f :_S (St \cup Ss) \to_S S$, $f(s), f(t) \in K_S$ implies that $f(s) = f(t)$. K_S is called *completely left annihilating*, if for all $x, y, z, t, t' \in S$,

$$[(xt \neq yt') \wedge (tz = t'z)] \Rightarrow [(xt \notin K_S) \vee (yt' \notin K_S) \vee (x \in K_S) \vee (y \in K_S)]$$

K_S is called P_E*-left annihilating*, if for all $x, y, t, t' \in S$,

$$(xt \neq yt') \Rightarrow [(x \in K_S) \vee (y \in K_S) \vee (xt \notin K_S) \vee (yt' \notin K_S) \vee$$

$$(\exists u, v \in S, e, f \in E(S), et = t, ft' = t', ut = vt'$$

$$xe \neq ue \Rightarrow xe, ue \in K_S, yf \neq vf \Rightarrow yf, vf \in K_S)]$$

K_S is called E*-left annihilating*, if for all $x, y, t \in S$,

$$(xt \neq yt) \Rightarrow [(x \in K_S) \vee (y \in K_S) \vee (xt \notin K_S) \vee (yt \notin K_S) \vee$$

$$(\exists u, v \in S, e, f \in E(S), et = t = ft, ut = vt,$$

$$xe \neq ue \Rightarrow xe, ue \in K_S, yf \neq vf \Rightarrow yf, vf \in K_S)]$$

A right S-act A satisfies Condition (P), if for all $a, a' \in A$, $s, s' \in S$, $as = a's'$ implies that there exist $b \in A$, $u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vs'$. A monoid S is called *right PCP*, if all principal right ideals of S satisfy Condition (P). A right S-act A satisfies Condition (P'), if for all $a, a' \in A$, $s, s', z \in S$, $as = a's'$, $sz = s'z$ imply that there exist $b \in A$, $u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vs'$. A right S-act A satisfies Condition (P_E), if for all $a, a' \in A$, $s, s' \in S$, $as = a's'$ implies that there exist $b \in A$, $u, v, e^2 = e, f^2 = f \in S$ such that $ae = bue$, $a'f = bvf$, $es = s$, $fs' = s'$ and $us = vs'$. It is obvious that Condition (P) implies Condition (P_E), but not the converse. A satisfies Condition (E), if for all $a \in A$, $s, s' \in S$, $as = as'$ implies that there exist $b \in A$, $u \in S$ such that $a = bu$ and $us = us'$. A satisfies Condition (E'), if for all $a \in A$, $s, s', z \in S$, $as = as'$ and $sz = s'z$ implies that there exist $b \in A$, $u \in S$ such that $a = bu$ and $us = us'$. A satisfies Condition (EP), if for all $a \in A$, $s, s' \in S$, $as = as'$ implies that there exist $b \in A$, $u, v \in S$ such that $a = bu = bv$ and $us = vs'$. A satisfies Condition $(E'P)$, if for all $a \in A$, $s, s', z \in S$, $as = as'$ and $sz = s'z$ imply that there exist $b \in A$, $u, v \in S$ such that $a = bu = bv$ and $us = vs'$. It is obvious that Condition $(E) \Rightarrow$ Condition $(EP) \Rightarrow$ Condition $(E'P)$ and Condition $(E) \Rightarrow$ Condition $(E') \Rightarrow$ Condition $(E'P)$. In [7] and [8] we gave a characterization of monoids by Conditions (EP) and $(E'P)$ of their acts. A right S-act A satisfies Condition (PWP), if for all $a, a' \in A$, $s \in S$, $as = a's$ implies that there exist $b \in A$ and $u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vs$. A right S-act A satisfies Condition (PWP_E), if for all $a, a' \in A$, $s \in S$, $as = a's$ implies that there exist $b \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = bue$, $a'f = bvf$, $es = fs = s$ and $us = vs$. In [9] we gave a characterization of monoids by Conditions (PWP_E) of their acts. A right S-act A satisfies Condition (W), if $as = a't$, for $a, a' \in A_S$, $s, t \in S$, implies that there exist $b \in A_S$ and $u \in Ss \cap St$, such that $as = a't = bu$.

A_S is called *regular*, if all cyclic subacts of A are projective. A_S is called *faithful*, if for $s, t \in S$ the equality $as = at$ for all $a \in A$ implies $s = t$. A_S is called *strongly faithful*, if for $s, t \in S$ the equality $as = at$ for some $a \in A$ implies $s = t$. A_S is called *P-regular*, if all cyclic subacts of A satisfy Condition (P). In [10] we gave a characterization of monoids by P-regularity of their acts. A is called *strongly (P)-cyclic* if for any $a \in A$ there exists $z \in S$ such that $ker\lambda_a = ker\lambda_z$ and zS satisfies Condition (P). In [11] we gave a characterization of monoids S by strong (P)-cyclic of right S-acts. A_S is called *locally cyclic*, if every finitely generated subact of A is contained within a cyclic subact of A. An act A_S is called to be connected, if for all $a, a' \in A$ there exist elements $s_1, t_1, \ldots, s_n, t_n \in S$ and $a_2, \ldots, a_n \in A$ such that

$$\begin{aligned} as_1 &= a_2t_1 \\ a_2s_2 &= a_3t_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ a_ns_n &= a't_n. \end{aligned}$$

For torsionless of acts we refer the reader to [12].

2 General Properties

An element $s \in S$ acts injectively on A_S if $as = bs$, for $a, b \in A_S$, implies $a = b$. If every $s \in S$ acts injectively on A_S, then we say that S acts injectively on A_S.

Definition 2.1. An act A_S is called *E-torsion free (ETF)*, if $E(S)$ acts injectively on A_S, that is;

$$(\forall a, a' \in A_S)(\forall e \in E(S))(ae = a'e \Rightarrow a = a').$$

Proposition 2.2. *Let S be a monoid. Then:*

(1) The one-element act Θ_S is ETF.

(2) If $E(S) = \{1\}$, then all (left) right S-acts are ETF.

(3) S_S is ETF if and only if $E(S) = \{1\}$.

(4) If S is a regular monoid, then A_S is ETF if and only if S acts injectively on A_S.

(5) If A_i, $i \in I$, are right S-acts, then A_i, $i \in I$, are ETF if and only if $A_S = \coprod_{i \in I} A_i$ is ETF.

(6) If A_i, $i \in I$, are right S-acts, then A_i, $i \in I$, are ETF if and only if $A_S = \prod_{i \in I} A_i$ is ETF.

(7) If an act is ETF, then all its subacts are ETF.

(8) A_S is an ETF right S-act if and only if $ae = a$, for all $a \in A_S$ and $e \in E(S)$.

(9) If $S = T^1$, where T is a semigroup, then the right S-act T_S is ETF if and only if $E(T) = \emptyset$ or $te = t$, for all $t \in T$ and $e \in E(T)$.

(10) If S is an idempotent monoid, then the right S-act A_S is ETF if and only if A_S is a coproduct of one element acts.

(11) If S contains a left zero, then the right S-act A_S is ETF if and only if A_S is a coproduct of one element acts.

Proof. The statements (1) to (8) are clear from definition.
(9). It follows from (8).
(10). It follows from (5) and (8).
(11). Necessity. Let z be a left zero element of S. By (8), $az = a$, for all $a \in A_S$. Thus $as = (az)s = a(zs) = az = a$, for all $s \in S$ and $a \in A_S$. Hence A_S is a coproduct of one element acts.
Sufficiency. It follows from (1) and (5). □

3 Characterization by E-Torsion Freeness of Right Acts

In this section we characterize monoids by E-torsion freeness of right acts.

Theorem 3.1. *Let S be a monoid and (U) be a property of S-acts which S_S has property (U). Then the following statements are equivalent:*

(1) All right S-acts with property (U) are ETF.

(2) All finitely generated right S-acts with property (U) are ETF.

(3) All cyclic right S-acts with property (U) are ETF.

(4) $E(S) = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4). Since S_S is a cyclic right S-act, by assumption it is ETF, and so by Proposition 2.2(3), $E(S) = \{1\}$.
(4) $\Rightarrow$ (1). It follows from Proposition 2.2(2). □

Now we have the following corollary.

Corollary 3.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right S-acts are ETF.

(2) All torsion free right S-acts are ETF.

(3) All principally weakly flat right S-acts are ETF.

(4) All GP-flat right S-acts are ETF.

(5) All weakly flat right S-acts are ETF.

(6) All right S-acts satisfying Condition (W) are ETF.

(7) All flat right S-acts are ETF.

(8) All right S-acts satisfying Condition (WP) are ETF.

(9) All right S-acts satisfying Condition (PWP) are ETF.

(10) All translation kernel flat right S-acts are ETF.

(11) All principally weakly kernel flat right S-acts are ETF.

(12) All weakly kernel flat right S-acts are ETF.

(13) All right S-acts satisfying Condition (P) are ETF.

(14) All right S-acts satisfying Condition (P_E) are ETF.

(15) All right S-acts satisfying Condition (P') are ETF.

(16) All right S-acts satisfying Condition (PWP_E) are ETF.

(17) All equalizer flat right S-acts are ETF.

(18) All strongly flat right S-acts are ETF.

(19) All weakly pullback flat right S-acts are ETF.

(20) All projective right S-acts are ETF.

(21) All projective generators right S-acts are ETF.

(22) All generators right S-acts are ETF.

(23) All free right S-acts are ETF.

(24) All right S-acts satisfying Condition (E) are ETF.

(25) All right S-acts satisfying Condition (EP) are ETF.

(26) All right S-acts satisfying Condition (E') are ETF.

(27) All right S-acts satisfying Condition $(E'P)$ are ETF.

(28) All faithful right S-acts are ETF.

(29) All torsionless right S-acts are ETF.

(30) $E(S) = \{1\}$.

Notice that all statements in Corollary above, are also true for cyclic and finitely generated right S-acts.

Lemma 3.3. *Let S be a monoid. Then the following statements are equivalent:*

(1) S is left cancellative.

(2) There exists a strongly faithful right S-act.

Proof. (1) $\Rightarrow$ (2). It is obvious, because in this case S_S is a strongly faithful right S-act.
(2) $\Rightarrow$ (1). Suppose that A_S is a strongly faithful right S-act and let $us = ut$, for $u, s, t \in S$. Let $a \in A$. Then $(au)s = (au)t$, and so $s = t$. Thus S is left cancellative, as required. □

Theorem 3.4. *Let S be a monoid and suppose there exists a strongly faithful right S-act. Then the following statements are equivalent:*

(1) All strongly faithful right S-acts are ETF.

(2) All strongly faithful finitely generated right S-acts are ETF.

(3) All strongly faithful cyclic right S-acts are ETF.

(4) $E(S) = \{1\}$.

Proof. By Lemma 3.3 and Proposition 2.2(2) it is obvious. □

Theorem 3.5. *Let S be a monoid. Then the following statements are equivalent:*

(1) All divisible right S-acts are ETF.

(2) All principally weakly injective right S-acts are ETF.

(3) All fg-weakly injective right S-acts are ETF.

(4) All weakly injective right S-acts are ETF.

(5) All injective right S-acts are ETF.

(6) All cofree right S-acts are ETF.

(7) All indecomposable right S-acts are ETF.

(8) All locally cyclic right S-acts are ETF.

(9) $E(S) = \{1\}$.

Proof. Since cofreeness $\Rightarrow$ injectivity $\Rightarrow$ weak injectivity $\Rightarrow$ fg-weak injectivity $\Rightarrow$ principal weak injectivity $\Rightarrow$ divisibility, then implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.
Implications (9) $\Rightarrow$ (1) and (9) $\Rightarrow$ (7) follow from Proposition 2.2(2).
(7) $\Rightarrow$ (8). It follows from [13, Lemma 3.4].
(6) $\Rightarrow$ (9). Since every right S-act can be embedded into a cofree right S-act, thus by Proposition 2.2(7), all right S-acts are ETF, and so $E(S) = \{1\}$, by Corollary 3.2.
(8) $\Rightarrow$ (9). All cyclic right S-acts are locally cyclic. Thus $E(S) = \{1\}$, by Corollary 3.2. □

Here we give a characterization of monoids for which E-torsion freeness of their acts implies other properties.

Theorem 3.6. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF right S-acts are free.

(2) All ETF right S-acts are projective generators.

(3) All ETF right S-acts are generators.

(4) All ETF right S-acts are faithful.

(5) All ETF right S-acts are strongly faithful.

(6) $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.
$(3) \Rightarrow (4)$. It follows from [2, III, 18.1].
Since Θ_S is an ETF right S-act, and Θ_S is (strongly) faithful if and only if $S = \{1\}$, implications $(4), (5) \Rightarrow (6)$ are obvious.
$(6) \Rightarrow (1), (5)$. If $S = \{1\}$, then all right S-acts are free (strongly faithful). □

Theorem 3.7. *Let S be a monoid with no zero element. Then the following statements are equivalent:*

(1) All ETF right S-acts are torsionless.

(2) S contains a left zero.

Proof. $(1) \Rightarrow (2)$. Since the right S-act Θ_S is ETF, it follows from [12, Lemma 2.2].
$(2) \Rightarrow (1)$. It follows from Proposition 2.2(11) and [12, Proposition 2.10]. □

Notice that all statements in Theorems 3.6 and 3.7 are also true for cyclic, finitely generated and right Rees factor S-acts.

Theorem 3.8. *Let S be an idempotent monoid. Then the following statements are equivalent:*

(1) All ETF right S-acts are strongly flat.

(2) All ETF right S-acts are equalizer flat.

(3) All ETF right S-acts are weakly pullback flat.

(4) All ETF right S-acts satisfy Condition (P).

(5) All ETF right S-acts satisfy Condition (P_E).

(6) All ETF right S-acts are weakly kernel flat.

(7) All ETF right S-acts are (WP).

(8) All ETF right S-acts are flat.

(9) All ETF right S-acts are weakly flat.

(10) All ETF right S-acts satisfy Condition (W).

(11) S is right reversible.

(12) S is left collapsible.

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). It follows from [4, Page 79].
Implications (4) $\Rightarrow$ (5) $\Rightarrow$ (9), (3) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (9), (4) $\Rightarrow$ (8) $\Rightarrow$ (9) and (1) $\Rightarrow$ (2) $\Rightarrow$ (9) are obvious.
(5) $\Leftrightarrow$ (9). It follows from [14, Theorem 2.5].
(9) $\Leftrightarrow$ (10). Since S is regular, all right S-acts are principally weakly flat, and so the result follows from [2, III, 11.4].
(9) $\Rightarrow$ (11). It follows from Proposition 2.2(1), and [2, III, 11.2].
(11) $\Rightarrow$ (12). Suppose $e, f \in S$. Since S is right reversible, there exist $g, g' \in S$ such that $ge = g'f$. If $u = ge = g'f$, then $ue = (ge)e = ge^2 = ge = g'f = g'f^2 = (g'f)f = uf$. Thus S is left collapsible.
(12) $\Rightarrow$ (4). Suppose A_S is ETF and let $as = bt$, for $a, b \in A_S$ and $s, t \in S$. Since by Proposition 2.2(8), $aS = \{a\}$, for any $a \in A_S$, we have $a = b$. Since S is left collapsible, there exists $u \in S$ such that $us = ut$. But, $a = b = au$, and so A_S satisfies Condition (P), as required. □

Notice that all statements in theorem above are also true for finitely generated and cyclic right S-acts.

4 Characterization by E-Torsion Freeness of Cyclic Right Acts

In this section we characterize monoids by E-torsion freeness of their cyclic right acts.

Proposition 4.1. *Let S be a monoid and ρ be a right congruence on S. Then the following statements are equivalent:*

(1) S/ρ is ETF.

(2) $(\forall s, t \in S)(\forall e \in E(S))((se, te) \in \rho \Rightarrow (s, t) \in \rho)$.

(3) $(\forall s \in S)(\forall e \in E(S))(se, s) \in \rho$.

Proof. It is straightforward. □

Theorem 4.2. *Let ρ be a right congruence on S. If S/ρ is ETF, then $T = [1]_\rho$ is a submonoid of S with $E(S) = E(T)$. The converse is also true when ρ is a congruence or every idempotent of S is central.*

Proof. It is obvious that T is a submonoid of S and also $E(T) \subseteq E(S)$. Let $e \in E(S)$. Then $(ee, 1e) = (e, e) \in \rho$, and so $(e, 1) \in \rho$, by Proposition 4.1. Thus $e \in T$, and hence $e \in E(T)$.
Suppose $(se, te) \in \rho$, for $s, t \in S$, and $e \in E(S)$. Since $E(S) = E(T)$, we have $(e, 1) \in \rho$. If ρ is a congruence, then $(se, s), (te, t) \in \rho$, and so $(s, t) \in \rho$. If every idempotent of S is central, then $(s, se) = (s, es) \in \rho$ and $(t, te) = (t, et) \in \rho$, and so $(s, t) \in \rho$. Thus in both cases S/ρ is ETF. □

Corollary 4.3. *Let S be an idempotent monoid and ρ be a right congruence on S. Then S/ρ is ETF if and only if $S = [1]_\rho$.*

Proof. Necessity. By Theorem 4.2, we have $S = E(S) = E(T) = T = [1]_\rho$.
Sufficiency. It is obvious. □

Theorem 4.4. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All cyclic right S-acts are ETF.*

(2) *All monocyclic right S-acts are ETF.*

(3) $\rho(x, y) \subseteq \rho(xe, ye)$, *for all* $x, y \in S$, $e \in E(S)$.

(4) $\rho(x, 1) \subseteq \rho(xe, e)$, *for all* $x \in S$, $e \in E(S)$.

(5) $\rho(x, e) \subseteq \rho(xe, e)$, *for all* $x \in S$, $e \in E(S)$.

(6) $\rho(e, f) \subseteq \rho(ef, f)$, *for all* $e, f \in E(S)$.

(7) $\rho(e, 1) \subseteq \rho(ef, f)$, *for all* $e, f \in E(S)$.

(8) $\rho(e, f) \subseteq \rho(fe, f)$, *for all* $e, f \in E(S)$.

(9) $\rho(xe, y) \subseteq \rho(x, y)$, *for all* $x, y \in S$, $e \in E(S)$.

(10) $\rho(xe, 1) \subseteq \rho(x, 1)$, *for all* $x \in S$, $e \in E(S)$.

(11) $\rho(ex, 1) \subseteq \rho(x, 1)$, *for all* $x \in S$, $e \in E(S)$.

(12) $\rho(xe, f) \subseteq \rho(x, f)$, *for all* $x \in S$, $e, f \in E(S)$.

(13) $\rho(ex, f) \subseteq \rho(x, f)$, *for all* $x \in S$, $e, f \in E(S)$.

(14) $\rho(xe, f) \subseteq \rho(x, e)$, *for all* $x \in S$, $e, f \in E(S)$.

(15) $\rho(ex, f) \subseteq \rho(x, e)$, *for all* $x \in S$, $e, f \in E(S)$.

(16) $E(S) = \{1\}$.

Proof. It is straightforward. □

Notice that Theorem 4.4, is also true when inclusions from 3 to 15 be replaced by equality.

Let S be a monoid and $s,t \in S$. Set $F_1 = \{(x,y) \in S\times S \mid \exists e \in E(S), (xe,ye) \in \rho(s,t)\}$, $F_{i+1} = \{(x,y) \in S \times S \mid \exists e \in E(S), (xe,ye) \in \rho(F_i)\}$, for $i \in \mathbb{N}$. It can easily be seen that F_i is reflexive and symmetric, for every $i \in \mathbb{N}$. Also,

$$\rho(s,t) \subseteq F_1 \subseteq \rho(F_1) \subseteq F_2 \subseteq \rho(F_2) \subseteq \ldots \subseteq \rho(F_i) \subseteq F_{i+1} \ldots .$$

It is clear that $\rho_{ETF}(s,t) = \bigcup_{i\in\mathbb{N}} \rho(F_i)$ is a right congruence on S containing (s,t).

Theorem 4.5. *Let S be a monoid and $s,t \in S$. Then $\rho_{ETF}(s,t)$ is the smallest right congruence containing (s,t), such that $S/\rho_{ETF}(s,t)$ is ETF.*

Proof. If $(xe,ye) \in \rho_{ETF}(s,t)$, for $x,y \in S$ and $e \in E(S)$, then there exists $i \in \mathbb{N}$ such that $(xe,ye) \in \rho(F_i)$, and so $(x,y) \in F_{i+1}$. Thus $(x,y) \in \rho(F_{i+1}) \subseteq \rho_{ETF}(s,t)$, and so $S/\rho_{ETF}(s,t)$ is ETF by Proposition 4.1.
Let τ be a right congruence on S containing (s,t), such that S/τ is ETF. We show that $\rho_{ETF}(s,t) \subseteq \tau$. Since $(s,t) \in \tau$, we have $\rho(s,t) \subseteq \tau$. If $(x,y) \in F_1$, then there exists $e \in E(S)$ such that $(xe,ye) \in \rho(s,t)$, and so $(xe,ye) \in \tau$. Since S/τ is ETF, we have $(x,y) \in \tau$. Thus $F_1 \subseteq \tau$, and so $\rho(F_1) \subseteq \tau$. Suppose then that $\rho(F_i) \subseteq \tau$, $i \in \mathbb{N}$. If $(x,y) \in F_{i+1}$, then there exists $e \in E(S)$ such that $(xe,ye) \in \rho(F_i) \subseteq \tau$. Since S/τ is ETF, $(x,y) \in \tau$. Hence $F_{i+1} \subseteq \tau$, and so $\rho(F_{i+1}) \subseteq \tau$. Thus $\rho(F_i) \subseteq \tau$, for all $i \in \mathbb{N}$, and so $\rho_{ETF}(s,t) \subseteq \tau$. □

Theorem 4.6. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition (P).

(2) All ETF cyclic right S-acts satisfy Condition (P_E).

(3) For any $x,y,s,t \in S$, there exist $u,v \in S$ such that $(u,x),(v,y) \in \rho_{ETF}(xs,yt)$ and $us = vt$.

(4) For any $s,t \in S$, there exist $u,v \in S$ such that $(u,1),(v,1) \in \rho_{ETF}(s,t)$ and $us = vt$.

Proof. (1) ⇒ (2). It is obvious.
(2) ⇒ (3). The cyclic right S-act $S/\rho_{ETF}(xs,yt)$ is ETF, and so it satisfies Condition (P_E). Thus by [15, Theorem 2.5], there exist $u,v \in S$ and $e,f \in E(S)$ such that $us = vt$, $es = s$, $ft = t$, $(ue,xe),(yf,vf) \in \rho_{ETF}(xs,yt)$. Thus by Proposition 4.1, $(u,x),(v,y) \in \rho_{ETF}(xs,yt)$, as required.
(3) ⇒ (4). It is sufficient to take $x = y = 1$.
(4) ⇒ (1). Suppose τ is a right congruence on S, such that S/τ is ETF and let $(s,t) \in \tau$. Then by assumption, there exist $u,v \in S$ such that $us = vt$ and $(u,1),(v,1) \in \rho_{ETF}(s,t)$. By Theorem 4.5, $\rho_{ETF}(s,t) \subseteq \tau$, and so $(u,1),(v,1) \in \tau$. Thus S/τ satisfies Condition (P), by [2, III, 13.4]. □

Theorem 4.7. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition (P').

(2) For any $x, y, z, t, t' \in S$, the equality $tz = t'z$ implies that there exist $u, v \in S$ such that $ut = vt'$ and $(u, x), (v, y) \in \rho_{ETF}(xt, yt')$.

Proof. Using [16, Theorem 3.1] and Theorem 4.5, it is similar to that of Theorem 4.6. □

Theorem 4.8. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition (E).

(2) For any $s, t \in S$, there exists $u \in S$ such that $us = ut$, and $(u, 1) \in \rho_{ETF}(s, t)$.

Proof. Using [2, III, 14.8] and Theorem 4.5, it is similar to that of Theorem 4.6. □

Theorem 4.9. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition (E').

(2) For any $s, t, z \in S$, the equality $sz = tz$ implies that there exists $u \in S$ such that $us = ut$ and $(u, 1) \in \rho_{ETF}(s, t)$.

Proof. It follows from Theorem 4.5, and a similar argument as in the proof of Theorem 4.6. □

Theorem 4.10. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition $(E'P)$.

(2) For any $x, y, z \in S$, the equality $xz = yz$ implies that there exist $u, v \in S$ such that $ux = vy$ and $(u, 1), (v, 1) \in \rho_{ETF}(x, y)$.

(3) For any $x, t, t', z \in S$, the equality $tz = t'z$ implies that there exist $u, v \in S$ such that $ut = vt'$ and $(u, x), (v, x) \in \rho_{ETF}(xt, xt')$.

Proof. It follows from [7, Theorem 2.10], Theorem 4.5, and a similar argument as in the proof of Theorem 4.6. □

Theorem 4.11. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts are principally weakly flat.

(2) For any $u, v, s \in S$, $(u, v) \in (\rho_{ETF}(us, vs) \vee ker\rho_s)$.

Proof. (1) ⇒ (2). Suppose $u, v, s \in S$. Then the cyclic right S-act $S/\rho_{ETF}(us, vs)$ is ETF, and so it is principally weakly flat. Since $(us, vs) \in \rho_{ETF}(us, vs)$ by [2, III, 10.7], we have $(u, v) \in (\rho_{ETF}(us, vs) \vee ker\rho_s)$.
(2) ⇒ (1). Suppose τ is a right congruence on S, such that S/τ is ETF and let $(us, vs) \in \tau$. By Theorem 4.5, $\rho_{ETF}(us, vs) \subseteq \tau$. By assumption, $(u, v) \in (\rho_{ETF}(us, vs) \vee ker\rho_s)$, and so $(u, v) \in (\tau \vee ker\rho_s)$. Thus S/τ is principally weakly flat, by [2, III, 10.7]. □

Theorem 4.12. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts are weakly flat.

(2) For any $s, t \in S$, there exist $u, v \in S$ such that $us = vt$, $(u,1) \in (\rho_{ETF}(s,t) \vee ker\rho_s)$ and $(v,1) \in (\rho_{ETF}(s,t) \vee ker\rho_t)$.

Proof. (1) $\Rightarrow$ (2). The cyclic right S-act $S/\rho_{ETF}(s,t)$ is ETF, and so it is weakly flat. Thus by [2, III, 11.5], there exist $u, v \in S$ such that $us = vt$, $(u,1) \in (\rho_{ETF}(s,t) \vee ker\rho_s)$ and $(v,1) \in (\rho_{ETF}(s,t) \vee ker\rho_t)$.
(2) $\Rightarrow$ (1). Suppose τ is a right congruence on S, such that S/τ is ETF and let $(s,t) \in \tau$. By Theorem 4.5, $\rho_{ETF}(s,t) \subseteq \tau$ and by assumption, there exist $u, v \in S$ such that $us = vt$, $(u,1) \in (\rho_{ETF}(s,t) \vee ker\rho_s)$ and $(v,1) \in (\rho_{ETF}(s,t) \vee ker\rho_t)$. Thus $(u,1) \in (\tau \vee ker\rho_s)$ and $(v,1) \in (\tau \vee ker\rho_t)$, and so S/τ is weakly flat, by [2, III, 11.5]. □

Theorem 4.13. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition (PWP).

(2) All ETF cyclic right S-acts satisfy Condition (PWP_E).

(3) For any $x, y, t \in S$, there exist $u, v \in S$ such that $ut = vt$ and $(u,x), (v,y) \in \rho_{ETF}(xt, yt)$.

Proof. Using [9, Theorem 3.7], [5, Proposition 7] and Theorem 4.5, it is similar to that of Theorem 4.6. □

Lemma 4.14. *Let S be a left PP monoid. Then the following statements are equivalent:*

(1) For any $x, y, t \in S$, $(x,y) \in (\rho_{ETF}(xt, yt) \vee ker\rho_t)$.

(2) For any $x, y, t \in S$, there exist $u, v \in S$ such that $ut = vt$ and $(u,x), (v,y) \in \rho_{ETF}(xt, yt)$.

(3) For any $x, y, t \in S$, $(x,y) \in \rho_{ETF}(xt, yt)$.

Proof. (1) $\Leftrightarrow$ (2). It follows from [9, Theorem 2.5], Theorem 4.11 and Theorem 4.13.
(2) $\Rightarrow$ (3). Let $x, y, t \in S$. By assumption there exist $u, v \in S$ such that $ut = vt$ and $(u,x), (v,y) \in \rho_{ETF}(xt, yt)$. Sine S is left PP, there exists $e \in E(S)$ such that $ker\rho_t = ker\rho_e$. Thus $ue = ve$, and so $(xe, ye) \in \rho_{ETF}(xt, yt)$. Hence $(x,y) \in \rho_{ETF}(xt, yt)$, by Proposition 4.1 and Theorem 4.5.
(3) $\Rightarrow$ (1). It is obvious. □

Theorem 4.15. *Let S be a left PP monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts satisfy Condition (PWP).

(2) All ETF cyclic right S-acts satisfy Condition (PWP_E).

(3) All ETF cyclic right S-acts are principally weakly flat.

(4) S acts injectively on every ETF cyclic right S-acts.

(5) For any $x, y, t \in S$, $(x, y) \in \rho_{ETF}(xt, yt)$.

Proof. (1) ⇔ (2). It follows from Theorem 4.13.
(2) ⇔ (3). It follows from [9, Theorem 2.5].
(3) ⇔ (5). It follows from Theorem 4.11 and Lemma 4.14.
(4) ⇒ (5). Suppose $x, y, t \in S$. Then the cyclic right S-act $S/\rho_{ETF}(xt, yt)$ is ETF, and so S acts injectively on $S/\rho_{ETF}(xt, yt)$. $(xt, yt) \in \rho_{ETF}(xt, yt)$ implies that $[x]_{\rho_{ETF}(xt,yt)}t = [y]_{\rho_{ETF}(xt,yt)}t$, and so $[x]_{\rho_{ETF}(xt,yt)} = [y]_{\rho_{ETF}(xt,yt)}$. Thus $(x, y) \in \rho_{ETF}(xt, yt)$.
(5) ⇒ (4). Suppose τ is a right congruence on S, such that S/τ is ETF. Let $[x]_\tau t = [y]_\tau t$, $x, y, t \in S$. By Theorem 4.5 and by the assumption we have $(x, y) \in \rho_{ETF}(xt, yt) \subseteq \tau$. Thus $[x]_\tau = [y]_\tau$ as required. □

Theorem 4.16. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts are torsion free.

(2) For any $s, t \in S$, $\rho_{TF}(s, t) \subseteq \rho_{ETF}(s, t)$.

(3) For any $s, t \in S$ and $c \in S$ right cancellable, $(s, t) \in \rho_{ETF}(sc, tc)$.

Proof. (1) ⇒ (2). Suppose $s, t \in S$. Then the cyclic right S-act $S/\rho_{ETF}(s, t)$ is ETF, and so it is torsion free. Thus $\rho_{TF}(s, t) \subseteq \rho_{ETF}(s, t)$, by the proof of Lemma 3.31 of [4].
(2) ⇒ (3). Suppose $s, t \in S$ and $c \in S$ right cancellable. Then $(s, t) \in \rho_{TF}(sc, tc) \subseteq \rho_{ETF}(sc, tc)$, by [4, Lemma 3.31] and [2, III, 8.4].
(3) ⇒ (1). Suppose τ is a right congruence on S, such that S/τ is ETF and let $(sc, tc) \in \tau$, $s, t \in S$ and $c \in S$ right cancellable. Then by Theorem 4.5 and by the assumption we have $(s, t) \in \rho_{ETF}(sc, tc) \subseteq \tau$. Thus S/τ is torsion free, by [2, III, 8.4]. □

Theorem 4.17. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF cyclic right S-acts are weakly pullback flat.

(2) S satisfies the following Conditions:
(a) For any $s, t, z \in S$, the equality $sz = tz$ implies that there exists $u \in S$ such that $us = ut$ and $(u, 1) \in \rho_{ETF}(s, t)$.
(b) For any $s, t \in S$, there exist $u, v \in S$ such that $us = vt$ and $(u, 1), (v, 1) \in \rho_{ETF}(s, t)$.

Proof. It follows from [5, Theorem 21], Theorem 4.6 and Theorem 4.9. □

Theorem 4.18. *Let S be a monoid. Then the following statements are equivalent:*

(1) All torsion free cyclic right S-acts are ETF.

(2) *For any* $s, t \in S$, $\rho_{ETF}(s,t) \subseteq \rho_{TF}(s,t)$.

(3) *For any* $s, t \in S$ *and* $e \in E(S)$, $(s,t) \in \rho_{ETF}(se, te)$.

(4) $E(S) = \{1\}$.

Proof. It is similar to the proof of Theorem 4.16. □

5 Characterization by E-torsion freeness of monocyclic right acts

In this section we characterize monoids by E-torsion freeness of their monocyclic right acts.

Lemma 5.1. *Let S be a monoid, $w, t \in S$ and $wt \neq t$. Then $\rho(wt,t) = \rho(w,1)$ if and only if t is right invertible.*

Proof. Suppose $\rho(wt,t) = \rho(w,1)$. Then by [2, III, 8.5], there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $w^m w = w^n 1 = w^n$ and $w^i w \in tS$, whenever $0 \leq i < m$, and $w^j \in tS$, whenever $0 \leq j < n$. If $n \geq 1$, then $1 = w^0 \in tS$, and so t is right invertible. If $n = 0$, then $m \geq 1$, since $w \neq 1$. Thus $w \in tS$, and so $1 = w^{m+1} \in tS$, that is, t is right invertible. The converse is obvious. □

Theorem 5.2. *Let S be a monoid and $w, e^2 = e \in S$. Then $S/\rho(we,e)$ is ETF if and only if $e = 1$ and $w^m xf = w^n yf$, for $x, y, f^2 = f \in S$, $m, n \in \mathbb{N} \cup \{0\}$, implies that $w^p x = w^q y$, for some $p, q \in \mathbb{N} \cup \{0\}$.*

Proof. Let $\rho = \rho(we,e)$.
Necessity. If $we = e$, then $S/\rho = S/\Delta_S \cong S_S$, and so we are done by Proposition 2.2(3). Thus we suppose $we \neq e$. Since $(we, 1e) \in \rho$, we have by Proposition 4.1, that $(w,1) \in \rho$. Since $(we,e) \in \rho(w,1)$, we have $\rho = \rho(w,1)$. Thus $e = 1$, by Lemma 5.1. Suppose now that $w^m xf = w^n yf$, for $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$. Then $(xf, yf) \in \rho$, by [2, III, 8.7], and so $(x,y) \in \rho$, by Proposition 4.1. Since $\rho = \rho(w,1)$, we have by [2, III, 8.7], that $w^p x = w^q y$, for some $p, q \in \mathbb{N} \cup \{0\}$.
Sufficiency. Suppose $(sf, tf) \in \rho$. Then by [2, III, 8.7], $w^m sf = w^n tf$, for some $m, n \in \mathbb{N} \cup \{0\}$. Thus by assumption, $w^p s = w^q t$, for some $p, q \in \mathbb{N} \cup \{0\}$. Again by [2, III, 8.7], $(s,t) \in \rho$, and so the result follows from Proposition 4.1. □

Theorem 5.3. *Let S be a monoid. Then all ETF right S-acts of the form $S/\rho(we,e)$ satisfy Condition (P).*

Proof. It follows from Theorem 5.2, [2, III, 13.8] and [2, III, 13.5]. □

Theorem 5.4. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All monocyclic right S-acts of the form $S/\rho(we,e)$, $w, e^2 = e \in S$, $we \neq e$, satisfying Condition (P) are ETF.*

(2) For every $1 \neq w \in S$, if there exist $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$ such that $w^m xf = w^n yf$, then there exist $p, q \in \mathbb{N} \cup \{0\}$ such that $w^p x = w^q y$.

Proof. (1) $\Rightarrow$ (2). Let $w \in S$, with $w \neq 1$. Then $S/\rho(w, 1)$ satisfies Condition (P), by [2, III, 13.8]. Thus $S/\rho(w, 1)$ is ETF, and the result follows from Theorem 5.2.
(2) $\Rightarrow$ (1). Suppose the right S-act $S/\rho(we, e)$, $we \neq e$, satisfies Condition (P). Then by [2, III, 13.8], there exists $a \in S$ such that $\rho(we, e) = \rho(a, 1)$. Since $(we, e) \in \rho(a, 1)$, by [2, III, 8.7], there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $a^m we = a^n e$. Since $we \neq e$, we have $a \neq 1$. Thus by assumption there exist $p, q \in \mathbb{N} \cup \{0\}$ such that $a^p w = a^q$. Again $(w, 1) \in \rho(a, 1) = \rho(we, e)$ by [2, III, 8.7], and so $\rho(we, e) = \rho(w, 1)$. Now the result follows from Lemma 5.1 and Theorem 5.2. □

Theorem 5.5. *Let S be a monoid. Then the following statements are equivalent:*

(1) All projective monocyclic right S-acts of the form $S/\rho(w, 1)$, $1 \neq w \in S$, are ETF.

(2) All monocyclic right S-acts of the form $S/\rho(w, 1)$, $1 \neq w \in S$, satisfying Condition (E) are ETF.

(3) If $1 \neq w$ is aperiodic, then the equality $w^n xf = w^n yf$, for $x, y, f^2 = f \in S$ and $n \in \mathbb{N}$, implies $w^n x = w^n y$.

Proof. (1) $\Leftrightarrow$ (2). It follows from [2, III, 17.14].
(2) $\Rightarrow$ (3). Suppose for $1 \neq w \in S$, there exists $n \in \mathbb{N}$ such that $w^{n+1} = w^n$. Then $S/\rho(w, 1)$ satisfies Condition (E), by [2, III, 17.14], and so it is ETF. Now the result follows from Theorem 5.2.
(3) $\Rightarrow$ (2). Suppose the right S-act $S/\rho(w, 1)$, $w \neq 1$, satisfies Condition (E). Then by [2, III, 17.14], there exists $n \in \mathbb{N}$ such that $w^{n+1} = w^n$. If $w^k xf = w^j yf$, for $x, y, f^2 = f \in S$ and $k, j \in \mathbb{N} \cup \{0\}$, then $w^n xf = w^n yf$, and so by assumption, $w^n x = w^n y$. Now the result follows from Theorem 5.2. □

Now we consider monoids over which E-torsion freeness implies projectivity and Condition (E).

Theorem 5.6. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF monocyclic right S-acts of the form $S/\rho(w, 1)$, are projective.

(2) All ETF monocyclic right S-acts of the form $S/\rho(w, 1)$, satisfy Condition (E).

(3) Every $w \in S$ is either aperiodic or there exist $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$ such that $w^m xf = w^n yf$ and $w^p x \neq w^q y$, for all $p, q \in \mathbb{N} \cup \{0\}$.

Proof. (1) $\Leftrightarrow$ (2). It follows from [2, III, 17.14].
(2) $\Rightarrow$ (3). If $w = 1$, then we are done. Suppose that $1 \neq w \in S$. If for all $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$, $w^m xf = w^n yf$ implies the existence of $p, q \in \mathbb{N} \cup \{0\}$ such that $w^p x = w^q y$, then by Theorem 5.2, $S/\rho(w, 1)$ is ETF.

Thus $S/\rho(w,1)$ satisfies Condition (E), and so w is aperiodic, by [2, III, 14.9].
(3) $\Rightarrow$ (2). Suppose the right S-act $S/\rho(w,1)$, $w \in S$ is ETF. If $w = 1$, then $S/\rho(w,1) = S/\Delta_S \cong S_S$ satisfies Condition (E). Thus we suppose $w \neq 1$. By Theorem 5.2, the equality $w^m xf = w^n yf$, for all $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$, implies the existence of $p, q \in \mathbb{N} \cup \{0\}$ such that $w^p x = w^q y$. Thus w is aperiodic and the result follows from [2, III, 14.9]. □

6 Characterization by E-Torsion Freeness of Right Rees Factor Acts

In this section we characterize monoids by E-torsion freeness of their right Rees factor acts. First of all we give a characterization of monoids over which all right Rees factor acts are ETF and also monoids over which all ETF right Rees factor acts have some other properties. Then we give a characterization of monoids for which right Rees factor acts with other properties are ETF. We recall that for a right ideal K_S of S, the Rees congruence ρ_K is defined by $(a,b) \in \rho_K$ if $a, b \in K_S$ or $a = b$ and the resulting factor act is called the Rees factor act and is denoted by S/K_S.

Theorem 6.1. *Let S be a monoid and K_S be a right ideal of S. Then S/K_S is ETF if and only if $K_S = S$ or $E(S) = \{1\}$.*

Proof. Necessity. Suppose S/K_S is ETF, $K_S \neq S$ and let $e \in E(S)$. Then $[1]_{\rho_K} e = [e]_{\rho_K} e$, and so $[1]_{\rho_K} = [e]_{\rho_K}$. Thus $e = 1$, and so $E(S) = \{1\}$ as required.
Sufficiency. It follows from Proposition 2.2(1),(2). □

Theorem 6.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts are ETF.

(2) All right Rees factor S-acts of the form S/sS, $s \in S$ are ETF.

(3) All right Rees factor S-acts of the form S/sS, $s \in S$ is regular, are ETF.

(4) All right Rees factor S-acts of the form S/eS, $e \in E(S)$, are ETF.

(5) $E(S) = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (5). Let $e \in E(S)$. Then by assumption S/eS is ETF, and so by Theorem 6.1, $eS = S$ or $E(S) = \{1\}$. these imply that $E(S) = \{1\}$.
(5) $\Rightarrow$ (1). It follows from Proposition 2.2(2). □

Theorem 6.3. *Let S be a monoid and (U) be a property of acts. Then the following statements are equivalent:*

(1) All ETF right Rees factor S-acts satisfy (U).

(2) Θ_S *satisfies* (U) *and* $E(S) = \{1\}$ *implies that all right Rees factor* S*-acts satisfy* (U)*.*

Proof. (1) ⇒ (2). By Proposition 2.2(1), $\Theta_S \cong S/S_S$ satisfies (U). If $E(S) = \{1\}$, then by Theorem 6.2, and assumption all right Rees factor S-acts satisfy (U).
(2) ⇒ (1). Suppose the right Rees factor S-act S/K_S is ETF. Then either $K_S = S$ or $E(S) = \{1\}$, by Theorem 6.1. If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ satisfies Condition (U). If $E(S) = \{1\}$, then by assumption, all right Rees factor S-acts satisfy (U), and so all ETF right Rees factor S-acts satisfy (U). □

Theorem 6.4. *Let* S *be a monoid. Then the following statements are equivalent:*

(1) All ETF *right Rees factor* S*-acts are torsion free.*

(2) $E(S) \neq \{1\}$ *or else, every right cancellable element of* S *is right invertible.*

Proof. It follows from Theorem 6.3, [2, III, 8.2] and [2, IV, 6.1]. □

Theorem 6.5. *Let* S *be a monoid. Then the following statements are equivalent:*

(1) All ETF *right Rees factor* S*-acts are principally weakly flat.*

(2) All ETF *right Rees factor* S*-acts satisfy Condition* (PWP)*.*

(3) All ETF *right Rees factor* S*-acts satisfy Condition* (PWP_E)*.*

(4) $E(S) \neq \{1\}$ *or else,* S *is a group.*

Proof. (1) ⇔ (4). It follows from Theorem 6.3, [2, III, 10.2], [2, IV, 6.6], and [1, II, Exercise 11].
(2) ⇔ (4). It follows from Theorem 6.3, [5, Corollary 11] and [3, Proposition 9].
(3) ⇔ (4). It follows from Theorem 6.3, and [9, Theorem 3.1]. □

Theorem 6.6. *Let* S *be a monoid. Then the following statements are equivalent:*

(1) All ETF *right Rees factor* S*-acts are flat.*

(2) All ETF *right Rees factor* S*-acts are weakly flat.*

(3) All ETF *right Rees factor* S*-acts satisfy Condition* (WP)*.*

(4) All ETF *right Rees factor* S*-acts satisfy Condition* (P)*.*

(5) All ETF *right Rees factor* S*-acts satisfy Condition* (P_E)*.*

(6) All ETF *right Rees factor* S*-acts are* P*-regular.*

(7) S *is right reversible and* $E(S) = \{1\}$ *implies that* S *is a group.*

Proof. (1) ⇔ (2). It follows from [2, III, 12.17].
(2) ⇔ (7). It follows from Theorem 6.3, [2, III, 11.2] and [2, IV, 7.3].
(3) ⇔ (7). It follows from Theorem 6.3, [5, Corollary 18] and [3, Proposition 14].
(4) ⇔ (7). It follows from Theorem 6.3, [5, Corollary 18] and [2, IV, 9.9].
(5) ⇔ (7). It follows from Theorem 6.3 and [15, Theorem 3.1].
(6) ⇔ (7). It follows from Theorem 6.3, [10, Theorem 2.1(1)] and [10, Theorem 2.3]. □

Theorem 6.7. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF right Rees factor S-acts satisfy Condition (E).

(2) All ETF right Rees factor S-acts are pullback flat.

(3) All ETF right Rees factor S-acts are equalizer flat.

(4) All ETF right Rees factor S-acts are strongly flat.

(5) S is left collapsible and $E(S) = \{1\}$ implies that $S = \{1\}$.

Proof. Implications (1) $\Leftrightarrow$ (2), (2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4) are obvious, by [2, III, 16.7].
(4) $\Leftrightarrow$ (5). By Theorem 6.3, all ETF right Rees factor S-acts are strongly flat if and only if Θ_S is strongly flat and $E(S) = \{1\}$ implies that all right Rees factor S-acts are strongly flat. By [2, III, 14.3] and [2, IV, 11.13], all ETF right Rees factor S-acts are strongly flat if and only if S is left collapsible and $E(S) = \{1\}$ implies that $S = \{1\}$. □

Theorem 6.8. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF right Rees factor S-acts are regular.

(2) All ETF right Rees factor S-acts are strongly (P)-cyclic.

(3) All ETF right Rees factor S-acts are projective.

(4) S contains a left zero.

Proof. (1) $\Rightarrow$ (2). It follows from [11, Thorem 2.1].
(2) $\Rightarrow$ (3). It follows from [11, Corollary 3.10].
(3) $\Rightarrow$ (4). By Proposition 2.2(1), the right Rees factor S-act $S/S_S \cong \Theta_S$ is ETF, and so it is projective. Thus by [2, III, 17.2], S contains a left zero element.
(4) $\Rightarrow$ (1). Suppose the right Rees factor S-act S/K_S is ETF. By Theorem 6.1, $K_S = S$ or $E(S) = \{1\}$. If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ is regular, by [2, III, 19.4(4)]. If $E(S) = \{1\}$, then $S = \{1\}$, and so all right Rees factor S-acts are regular. □

Notice that all statements in Theorem 6.8 are also true for (cyclic) right S-acts.

Theorem 6.9. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF right Rees factor S-acts are weakly pullback flat.

(2) All ETF right Rees factor S-acts are weakly kernel flat.

(3) S is right reversible, weakly left collapsible and $E(S) = \{1\}$ implies that S is a group.

Proof. It follows from Theorem 6.3, [10, Corollary 3.10] and [3, Theorem 20]. □

Theorem 6.10. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF right Rees factor S-acts are principally weakly kernel flat.

(2) All ETF right Rees factor S-acts are translation kernel flat.

(3) $ker\rho_z$ is connected as a left S-act, for every $z \in S$ and $E(S) = \{1\}$ implies that S is a group.

Proof. It follows from Theorem 6.3, [3, Proposition 7] and [3, Theorem 20]. □

Theorem 6.11. *Let S be a monoid. Then the following statements are equivalent:*

(1) All ETF right Rees factor S-acts satisfy Condition (P').

(2) S is weakly right reversible and $E(S) = \{1\}$ implies that S is a group.

Proof. It follows from Theorem 6.3, [16, Corollary 4.4] and [16, Thorem 4.18]. □

Now we give a characterization of monoids for which right Rees factor acts with other properties are ETF.

Theorem 6.12. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts satisfying Condition (P) are ETF.

(2) All right Rees factor S-acts satisfying Condition (E) are ETF.

(3) All pullback flat right Rees factor S-acts are ETF.

(4) All equalizer flat right Rees factor S-acts are ETF.

(5) All strongly flat right Rees factor S-acts are ETF.

(6) All projective right Rees factor S-acts are ETF.

(7) All free right Rees factor S-acts are ETF.

(8) S contains no left zero element or $S = \{1\}$.

Proof. (1) $\Rightarrow$ (2). It follows from [2, III, 14.7].
Implications (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from [2, III, 16.7].
Implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.
(7) $\Rightarrow$ (8). Let $s \in S$ be a left zero. Then the right Rees factor S-act $S/sS \cong S_S$ is free, and so it is ETF, by assumption. Thus by Proposition 2.2(3), $E(S) = \{1\}$, and so $s = 1$, that is, $S = \{1\}$.
(8) $\Rightarrow$ (1). Suppose the right Rees factor S-act S/K_S satisfies Condition (P). If $K_S = S$, then by Proposition 2.2(1) we are done. Thus we suppose that $K_S \neq S$. Then by [2, III, 13.9], $|K_S| = 1$, and so S contains a left zero element. Thus, $S = \{1\}$, and so $K_S = S$, which is a contradiction. □

Theorem 6.13. *Let S be a monoid. Then the following statements are equivalent:*

(1) All weakly flat right Rees factor S-acts are ETF.

(2) All flat right Rees factor S-acts are ETF.

(3) $E(S) = \{1\}$ or S is not right reversible.

Proof. (1) $\Leftrightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Suppose $E(S) \neq \{1\}$, S is right reversible and let $e \in E(S)\setminus\{1\}$. By [2, III, 12.17], the right Rees factor S-act $S/(eS)_S$ is flat. Thus $eS = S$, by Theorem 6.1, which is a contradiction.
(3) $\Rightarrow$ (1). It follows from [2, III, 12.17] and Proposition 2.2(2). □

Theorem 6.14. *Let S be a monoid. Then the following statements are equivalent:*

(1) All torsion free right Rees factor S-acts are ETF.

(2) All principally weakly flat right Rees factor S-acts are ETF.

(3) $E(S) = \{1\}$.

Proof. (1) $\Rightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Suppose $e \in E(S)$. The right Rees factor S-act $S/(eS)_S$ is principally weakly flat, and so $E(S) = \{1\}$ or $eS = S$, by Theorem 6.1. In each case, $E(S) = \{1\}$.
(3) $\Rightarrow$ (1). It follows from Proposition 2.2(2). □

Theorem 6.15. *Let S be a monoid. Then the following statements are equivalent:*

(1) All regular right Rees factor S-acts are ETF.

(2) $S = \{1\}$ or S contains no left zero element or there exists $z \in S$ such that $ker\lambda_z \neq ker\lambda_e$, for every $e \in E(S)$.

Proof. (1) $\Rightarrow$ (2). It follows from [2, III, 19.6], [2, III, 17.16] and Theorem 6.1.
(2) $\Rightarrow$ (1). It follows from [2, III, 19.6], [2, III, 17.16] and Proposition 2.2(2). □

Theorem 6.16. *Let S be a monoid. Then the following statements are equivalent:*

(1) All P-regular right Rees factor S-acts are ETF.

(2) All strongly (P)-cyclic right Rees factor S-acts are ETF.

(3) $S = \{1\}$ or S contains no left zero element or S is not right PCP.

Proof. Since strong (P)-cyclic implies P-regularity, (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3). It follows from [11, Theorem 3.1] and Theorem 6.1.
(3) $\Rightarrow$ (1). It follows from [10, Theorem 3.1] and Theorem 6.2. □

Theorem 6.17. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts satisfying Condition (P') are ETF.

(2) $E(S) = \{1\}$ or S has no left stabilizing and completely left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [16, Theorem 4.3] and Theorem 6.1.
(2) ⇒ (1). It follows from [16, Theorem 4.3] and Theorem 6.2. □

Theorem 6.18. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts satisfying Condition (PWP) are ETF.

(2) $E(S) = \{1\}$ or S has no left stabilizing and left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [5, Theorem 10] and Theorem 6.1.
(2) ⇒ (1). It follows from [5, Theorem 10] and Theorem 6.2. □

Theorem 6.19. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts satisfying Condition (WP) are ETF.

(2) $E(S) = \{1\}$ or S is not right reversible or S has no left stabilizing and strongly left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [5, Theorem 17] and Theorem 6.1.
(2) ⇒ (1). It follows from [5, Theorem 17] and Theorem 6.2. □

Theorem 6.20. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts satisfying Condition (P_E) are ETF.

(2) $E(S) = \{1\}$ or S is not right reversible or S has no proper P_E-left annihilating right ideal.

Proof. (1) ⇒ (2). It follows from [15, Theorem 3.5] and Theorem 6.1.
(2) ⇒ (1). It follows from [15, Theorem 3.5] and Theorem 6.2. □

Theorem 6.21. *Let S be a monoid. Then the following statements are equivalent:*

(1) All right Rees factor S-acts satisfying Condition (PWP_E) are ETF.

(2) $E(S) = \{1\}$ or S has no left stabilizing and E-left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [9, Theorem 4.2] and Theorem 6.1.
(2) ⇒ (1). It follows from [9, Theorem 4.2] and Theorem 6.2. □

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