Thai Journal of Mathematics
Volume 14 (2016) Number 1 : 83–92
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

(α, β)-Normal Composition Operators

Anuradha Gupta[†] **and Pooja Sharma**[‡,1]

[†]Delhi College of Arts and Commerce, University of Delhi,
Netaji Nagar, New Delhi 110 023, India
e-mail : dishna2@yahoo.in

[‡]Department of Mathematics, University of Delhi,
Delhi 110 007, India
e-mail : pooja.20.sh@gmail.com

Abstract : The composition operators of (α, β)-normal operators and their adjoints have been characterized on $L^2(m)$.

Keywords : (α, β)-normal operators; composition operators; normal operators; adjoint of an operator.
2010 Mathematics Subject Classification : 47B20; 47B38; 47B33; 47B15; 47A05.

1 Introduction and Statement of Results

Let H be an infinite dimensional complex separable Hilbert space and $\mathcal{B}(H)$ be the algebra of all bounded linear operators defined on H. An operator $T \in \mathcal{B}(H)$ is called normal if $TT^* = T^*T$, hyponormal if $T^*T \geq TT^*$ which is equivalent to the condition $\|T^*x\| \leq \|Tx\|$, for all $x \in H$. For real numbers α and β with $0 \leq \alpha \leq 1 \leq \beta$, an operator T acting on a Hilbert space H is called (α, β)-normal if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$, which is equivalent to the condition $\alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\|$, for all x in H [1,2]. For $\alpha = 1 = \beta$, T is a normal operator. For $\alpha = 1$, we observe from the left inequality that T^* is hyponormal and for $\beta = 1$, from the right inequality we obtain that T is hyponormal. Takagi and K. Yokouchi [3] initiated the study of multiplication and composition operators between L^p-spaces. The study of non-normal classes of composition operators initiated by R.K. Singh [4]

[1]Corresponding author.

in 1974 and later this was studied by many authors [5]- [13]. In this paper, we obtain a necessary and sufficient condition for an operator and its adjonits to be (α,β)-normal composition operator.

Let (X,Σ,m) be a sigma-finite measure space. The space $L^2(X,\Sigma,m)\equiv L^2(m)$ is defined as:

$$L^2(m)=\left\{f:X\to\mathbb{C}:f \text{ is a measurable function and } \int_X |f|^2dm<\infty\right\}$$

with $\|f\|_2=\left(\int_X |f|^2dm\right)^{\frac{1}{2}}$.

Radon Nikodym Theorem. *If (X,Σ,m) is a σ-finite measure space and m' is a σ-finite measure on Σ such that m' is absolutely continuous with respect to m, then there exists a finite-valued non-negative measurable function h on X such that for each $A\in\Sigma$, $m'(A)=\int_A hdm$. Also, h is unique in the sense that if $m'(A)=\int_A gdm$ for each $A\in\Sigma$, then $h=g$ a.e.(m).*

A mapping $T:X\to X$ is said to be measurable if $T^{-1}(A)\in\Sigma$ whenever $A\in\Sigma$. A measurable transformation $T:X\to X$ is called non-singular if the pre-image of every null set under T is a null set. Such a transformation induces a well defined composition operator

$$\begin{aligned}&C_T:L^2(m)\to L^2(m) \text{ as}\\ &C_Tf=f\circ T \quad \text{for each } f\in L^2(m), \text{ if}\end{aligned}$$

(i) the measure $m\circ T^{-1}$ is absolutely continuous with respect to m, and
(ii) the Radon-Nikodym derivative $h=\frac{d(mT^{-1})}{dm}$ is essentially bounded.
Every essentially bounded complex-valued measurable function θ induces a bounded operator M_θ on $L^2(m)$ which is defined by $M_\theta f=\theta f$ for every $f\in L^2(m)$.

Let $E\subseteq X$, then the characteristic function of E, written as χ_E, is the function on X defined by

$$\chi_E(x)=1 \text{ for } x\in E \text{ and } \chi_E(x)=0 \text{ for } x\in(X-E)$$

2 (α,β)-Normal Composition Operators

In this section we obtain a necessary and sufficient condition for an operator to be (α,β)-normal composition operator.

The following lemma due to Harrington and Whitley [7, Lemma 1] is instrumental in the subsequent results.

Lemma 2.1. *Let P denote the projection of $L^2(m)$ on $\overline{R(C_T)}$.*

(a) $C_T^* C_T f = hf$ *and* $C_T C_T^* f = (h \circ T)Pf$ *for all* f *in* $L^2(m)$.

(b) $\overline{R(C_T)} = \{f \in L^2(m) : f$ *is* $T^{-1}(\Sigma)$*-measurable*$\}$.

Theorem 2.2. *A composition operator* C_T *on* $L^2(m)$ *is* (α, β)*-normal* $(0 \le \alpha \le 1 \le \beta)$ *iff* $\alpha^2 h \le (h \circ T)P \le \beta^2 h$ *a.e.*

Proof. By definition of (α, β)-normal operators, C_T is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$

$$\begin{aligned}
&\text{iff} && \alpha^2 C_T^* C_T \le C_T C_T^* \le \beta^2 C_T^* C_T \\
&\text{i.e.} && \alpha^2 \langle C_T^* C_T f, f\rangle \le \langle C_T C_T^* f, f\rangle \le \beta^2 \langle C_T^* C_T f, f\rangle \quad \forall\ f \in L^2(m) \\
&\text{iff} && \alpha^2 \langle M_h f, f\rangle \le \langle M_{(h\circ T)P} f, f\rangle \le \beta^2 \langle M_h f, f\rangle \quad \forall\ f \in L^2(m)
\end{aligned}$$

$$\begin{aligned}
&\text{iff} && \alpha^2 \langle M_h \chi_E, \chi_E\rangle \le \langle M_{(h\circ T)P} \chi_E, \chi_E\rangle \le \beta^2 \langle M_h \chi_E, \chi_E\rangle, \\
& && \quad \text{for every } \chi_E \text{ of } E \text{ in } \Sigma \text{ such that } m(E) < \infty \\
&\text{iff} && \int_E \alpha^2 h\, dm \le \int_E (h \circ T)P\, dm \le \int_E \beta^2 h\, dm, \\
& && \quad \text{for every } E \text{ in } \Sigma \text{ such that } m(E) < \infty \\
&\text{iff} && \alpha^2 h \le (h \circ T)P \le \beta^2 h \quad \text{a.e., for } 0 \le \alpha \le 1 \le \beta\,.
\end{aligned}$$

□

Theorem 2.3. *An operator* $T \in \mathcal{B}(H)$ *is* (α, β)*-normal* $(0 \le \alpha \le 1 \le \beta)$ *iff* $k^2(TT^*) + 2k\alpha^2(T^*T) + TT^* \ge 0$ *a.e. and* $k^2(T^*T) + 2k(TT^*) + \beta^4(T^*T) \ge 0$ *a.e., for all* $k \in \mathbb{R}$.

Proof. For all $x \in H$ and $0 \le \alpha \le 1 \le \beta$.

$$\begin{aligned}
& && k^2(TT^*) + 2k\alpha^2 T^*T + TT^* \ge 0 \text{ a.e. and} \\
& && \quad k^2(T^*T) + 2k(TT^*) + \beta^4(T^*T) \ge 0 \text{ a.e. for all } k \in \mathbb{R} \\
&\text{iff} && \langle (k^2 TT^* + 2k\alpha^2 T^*T + TT^*)x, x\rangle \ge 0 \text{ a.e. and} \\
& && \quad \langle (k^2 T^*T + 2kTT^* + \beta^4 T^*T)x, x\rangle \ge 0 \text{ a.e. for all } k \in \mathbb{R} \\
&\text{iff} && k^2\langle TT^*x, x\rangle + 2k\alpha^2\langle T^*Tx, x\rangle + \langle TT^*x, x\rangle \ge 0 \text{ a.e. and} \\
& && \quad k^2\langle T^*Tx, x\rangle + 2k\langle TT^*x, x\rangle + \beta^4\langle T^*Tx, x\rangle \ge 0 \text{ a.e. for all } k \in \mathbb{R} \\
&\text{iff} && k^2\langle T^*x, T^*x\rangle + 2k\alpha^2\langle Tx, Tx\rangle + \langle T^*x, T^*x\rangle \ge 0 \text{ a.e. and} \\
& && \quad k^2\langle Tx, Tx\rangle + 2k\langle T^*x, T^*x\rangle + \beta^4\langle Tx, Tx\rangle \ge 0 \text{ a.e. for all } k \in \mathbb{R} \\
&\text{iff} && k^2\|T^*x\|^2 + 2k\alpha^2\|Tx\|^2 + \|T^*x\|^2 \ge 0 \text{ a.e. and} \\
& && \quad k^2\|Tx\|^2 + 2k\|T^*x\|^2 + \beta^4\|Tx\|^2 \ge 0 \text{ a.e. for all } k \in \mathbb{R}
\end{aligned}$$

Using elementary properties of real quadratic forms

$$
\begin{aligned}
& k^2TT^* + 2k\alpha^2T^*T + TT^* \geq 0 \text{ a.e. and} \\
& \quad k^2T^*T + 2kTT^* + \beta^4T^*T \geq 0 \text{ a.e. for all } k \in \mathbb{R} \\
\text{iff} \quad & 4\alpha^4\|Tx\|^4 \leq 4\|T^*x\|^4 \quad \text{and} \quad 4\|T^*x\|^4 \leq 4\beta^4\|Tx\|^4 \\
\text{iff} \quad & \alpha\|Tx\| \leq \|T^*x\| \quad \text{and} \quad \|T^*x\| \leq \beta\|Tx\| \\
& T \in \mathcal{B}(H) \text{ is } (\alpha,\beta)\text{-normal operator} \\
\text{iff} \quad & \alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\|, \quad 0 \leq \alpha \leq 1 \leq \beta
\end{aligned}
$$

□

Theorem 2.4. *A composition operator* C_T *on* $L^2(m)$ *is* (α,β)*-normal operator* $(0 \leq \alpha \leq 1 \leq \beta)$ *iff* $k^2(h \circ T)P + 2k\alpha^2h + (h \circ T)P \geq 0$ *a.e. and* $k^2h + 2k(h \circ T)P + \beta^4h \geq 0$ *a.e. for all* $k \in \mathbb{R}$.

Proof. By Theorem 2.3, C_T is (α,β)-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$

$$
\begin{aligned}
\text{iff} \quad & \langle (k^2C_TC_T^* + 2k\alpha^2C_T^*C_T + C_TC_T^*)(f), f\rangle \geq 0 \quad \text{and} \\
& \langle (k^2C_T^*C_T + 2kC_TC_T^* + \beta^4C_T^*C_T)(f), f\rangle \geq 0 \\
& \quad \text{for all } f \in L^2(m) \text{ and for all } k \in \mathbb{R} \\
\text{iff} \quad & \langle (k^2C_TC_T^* + 2k\alpha^2C_T^*C_T + C_TC_T^*)\chi_E, \chi_E\rangle \geq 0 \quad \text{and} \\
& \langle (k^2C_T^*C_T + 2kC_TC_T^* + \beta^4C_T^*C_T)\chi_E, \chi_E\rangle \geq 0 \\
& \quad \text{for every } \chi_E \text{ of } E \text{ in } \Sigma \text{ such that } m(E) < \infty \text{ and } k \in \mathbb{R} \\
\text{iff} \quad & \langle (k^2M_{(h\circ T)P} + 2k\alpha^2M_h + M_{(h\circ T)P})\chi_E, \chi_E\rangle \geq 0 \quad \text{and} \\
& \langle (k^2M_h + 2kM_{(h\circ T)P} + \beta^4M_h)\chi_E, \chi_E\rangle \geq 0 \\
& \quad \text{for every } \chi_E \text{ of } E \text{ in } \Sigma \text{ such that } m(E) < \infty \text{ and } k \in \mathbb{R} \\
\text{iff} \quad & \int (k^2M_{(h\circ T)P} + 2k\alpha^2M_h + M_{(h\circ T)P})\chi_E\, dm \geq 0 \quad \text{and} \\
& \int (k^2M_h + 2kM_{(h\circ T)P} + \beta^4M_h)\chi_E\, dm \geq 0 \\
& \quad \text{for every } \chi_E \text{ of } E \text{ in } \Sigma \text{ such that } m(E) < \infty \text{ and } k \in \mathbb{R} \\
\text{iff} \quad & \int_E (k^2(h\circ T)P + 2k\alpha^2h + (h\circ T)P)\, dm \geq 0 \quad \text{and} \\
& \int_E (k^2h + 2k(h\circ T)P + \beta^4h)\, dm \geq 0 \\
& \quad \text{for every } E \text{ in } \Sigma \text{ such that } m(E) < \infty \text{ and } k \in \mathbb{R} \\
\text{iff} \quad & k^2(h\circ T)P + 2k\alpha^2h + (h\circ T)P \geq 0 \quad \text{a.e. and} \\
& k^2h + 2k(h\circ T)P + \beta^4h \geq 0 \quad \text{a.e. for all } k \in \mathbb{R}.
\end{aligned}
$$

□

Corollary 2.5. *A composition operator* C_T *on* $L^2(m)$ *with dense range is* (α,β)*-normal* $(0 \leq \alpha \leq 1 \leq \beta)$ *iff* $k^2(h \circ T) + 2k\alpha^2h + (h \circ T) \geq 0$ *a.e. and* $k^2h + 2k(h \circ T) + \beta^4h \geq 0$ *a.e. for all* $k \in \mathbb{R}$.

Corollary 2.6. *A composition operator C_T on $L^2(m)$ with dense range is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ iff $\alpha^2 h \le (h \circ T) \le \beta^2 h$ a.e.*

Corollary 2.7. *A composition operator C_T on $L^2(m)$ is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ iff for all $f \in L^2(m)$*

(a) $\|\alpha h^{\frac{1}{2}} f\| \le \|(h \circ T)^{\frac{1}{2}} P f\| \le \|\beta h^{\frac{1}{2}} f\|$.

(b) $\|\alpha h^{\frac{1}{2}} P f\| \le \|(h \circ T)^{\frac{1}{2}} P f\| \le \|\beta h^{\frac{1}{2}} P f\|$.

Theorem 2.8. *A composition operator C_T on $L^2(m)$ is (α, β)-normal, $(0 \le \alpha \le 1 \le \beta)$ iff $\alpha \dfrac{d(mT^{-2})}{dm} \le h^2 \le \beta \dfrac{d(mT^{-2})}{dm}$ a.e.*

Proof. Let a composition operator C_T on $L^2(m)$ be a (α, β)-normal operator $(0 \le \alpha \le 1 \le \beta)$.

Then by Corollary 2.7(b)

$$\|\alpha h^{\frac{1}{2}} P f\| \le \|(h \circ T) h^{\frac{1}{2}} P f\| \le \|\beta h^{\frac{1}{2}} P f\|$$

Let E be a set of finite measure in Σ. Let $A = T^{-1}(E)$. As A is $T^{-1}(\Sigma)$ measurable, therefore $P\chi_A = \chi_A$ and

$$\begin{aligned}
0 &\le \|(h \circ T)^{\frac{1}{2}} P \chi_A\|^2 - \|\alpha h^{\frac{1}{2}} P \chi_A\|^2 \\
&= \int_A (h \circ T - \alpha h) dm \\
&= \int_A (h \circ T) dm - \alpha\ d(mT^{-1})(A) \\
&= \int (h \circ T) C_T\, \chi_E dm - \alpha\ d(mT^{-1})(A) \\
&= \int (h \circ T)(\chi_E \circ T) dm - \alpha\ d(mT^{-1})(A) \\
&= \int_E \left(h^2 - \alpha \frac{dmT^{-2}}{dm} \right) dm.
\end{aligned}$$

Therefore,

$$h^2 - \alpha \frac{dmT^{-2}}{dm} \ge 0 \text{ a.e.}$$

$$\textit{or } h^2 \ge \alpha \frac{dmT^{-2}}{dm} \text{ a.e.} \tag{2.1}$$

Also,

$$\begin{aligned}
0 &\le \|\beta h^{\frac{1}{2}} P \chi_A\|^2 - \|(h \circ T)^{\frac{1}{2}} P \chi_A\|^2 \\
&= \int_E \left(\beta \frac{dmT^{-2}}{dm} - h^2 \right) dm.
\end{aligned}$$

Therefore

$$\beta\frac{dmT^{-2}}{dm}-h^2\geq 0 \text{ a.e.}$$
$$or\ \beta\frac{dmT^{-2}}{dm}\geq h^2 \text{ a.e.} \tag{2.2}$$

Combining (2.1) and (2.2)

$$\alpha\frac{dmT^{-2}}{dm}\leq h^2\leq\beta\frac{dmT^{-2}}{dm}\quad \text{a.e.}$$

Conversely, suppose that

$$\alpha\frac{d(mT^{-2})}{dm}\leq h^2\leq\beta\frac{d(mT^{-2})}{dm}\quad \text{a.e.}$$

Then, for any E in Σ such that $m(E)<\infty$, the argument above shows that the inequality of Corollary 2.7(b) holds for $f=\chi_{T^{-1}(E)}$. Suppose that f is $T^{-1}(\Sigma)$-measurable and simple. Then, we can write

$$f=\sum_j a_jA_j$$

where A_j's are disjoint sets in $T^{-1}(\Sigma)$.

Then,

$$\begin{aligned}\|\beta h^{\frac{1}{2}}Pf\|^2&=\Sigma\|\beta a_jh^{\frac{1}{2}}\chi_{A_j}\|^2\\&\geq\Sigma\|a_j(h\circ T)^{\frac{1}{2}}\chi_{A_j}\|^2\\&=\|(h\circ T)^{\frac{1}{2}}Pf\|^2\end{aligned}$$

Similarly,

$$\|\alpha h^{\frac{1}{2}}Pf\|^2\leq\|(h\circ T)\|^{\frac{1}{2}}Pf\|^2$$

As $T^{-1}(\Sigma)$-measurable simple functions are dense in $\overline{R(C_T)}$, the inequality

$$\|\alpha h^{\frac{1}{2}}Pf\|\leq\|(h\circ T)^{\frac{1}{2}}Pf\|\leq\|\beta h^{\frac{1}{2}}Pf\| \text{ holds for all } f\in L^2(m)$$

and hence, C_T is (α,β)-normal $(0\leq\alpha\leq 1\leq\beta)$. □

Example 2.9. *Let $X=\mathbb{N}$ and let m be the counting measure.*

Define $T:\mathbb{N}\to\mathbb{N}$ as

$$T(n)=2n\ \forall\ n\in\mathbb{N}$$

Then, $h(2n)=1\ \forall\ n\in\mathbb{N}$.

By Corollary 2.6, C_T is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ if

$$\begin{aligned} & \alpha^2 h \le h \circ T \le \beta^2 h, \quad a.e. \\ \text{if} \quad & \alpha^2 h(2n) \le (h \circ T)(2n) \le \beta^2 h(2n) \quad \forall\, n \in \mathbb{N} \\ \text{if} \quad & \alpha^2 \cdot 1 \le h(4n) \le \beta^2 \cdot 1 \quad \forall\, n \in \mathbb{N} \\ \text{if} \quad & \alpha^2 \le 1 \le \beta^2, \quad \text{which is true since } 0 \le \alpha \le 1 \le \beta. \end{aligned}$$

Hence, the composition operator induced by above T is (α, β)-normal operator $(0 \le \alpha \le 1 \le \beta)$.

3 Adjoint of (α, β)-Normal Composition Operators

In this section we explore the conditions under which the adjoint of a composition operator is (α, β)-normal operator.

Theorem 3.1. *An operator $C_T^* \in \mathcal{B}(L^2(m))$ is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ iff $\alpha^2 (h \circ T)P \le h \le \beta^2 (h \circ T)P$.*

Proof. By definition of (α, β)-normal operator, C_T^* is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$

$$\begin{aligned} \text{iff} \quad & \alpha^2 C_T C_T^* \le C_T^* C_T \le \beta^2 C_T C_T^* \\ \text{iff} \quad & \alpha^2 \langle C_T C_T^* f, f \rangle \le \langle C_T^* C_T f, f \rangle \le \beta^2 \langle C_T C_T^* f, f \rangle \ \forall\, f \in L^2(m) \\ \text{iff} \quad & \alpha^2 \langle M_{(h \circ T)P} f, f \rangle \le \langle M_h f, f \rangle \le \beta^2 \langle M_{(h \circ T)P} f, f \rangle \ \forall\, f \in L^2(m) \\ \text{iff} \quad & \alpha^2 \langle M_{(h \circ T)P} \chi_E, \chi_E \rangle \le \langle M_h \chi_E, \chi_E \rangle \le \beta^2 \langle M_{(h \circ T)P} \chi_E, \chi_E \rangle \ \forall\, f \in L^2(m) \\ & \quad \text{and for every } \chi_E \text{ of } E \text{ in } \Sigma \text{ such that } m(E) < \infty \\ \text{iff} \quad & \int_E \alpha^2 (h \circ T) P dm \le \int_E h dm \le \int_E \beta^2 (h \circ T) P dm \\ & \quad \text{for every } E \text{ in } \Sigma \text{ such that } m(E) < \infty \\ \text{iff} \quad & \alpha^2 (h \circ T) P \le h \le \beta^2 (h \circ T) P \text{ a.e. for } 0 \le \alpha \le 1 \le \beta \end{aligned}$$

□

Theorem 3.2. *An operator $C_T^* \in \mathcal{B}(L^2(m))$ is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$ iff*

$$\begin{aligned} & k^2 h + 2k\alpha^2 (h \circ T)P + h \ge 0 \ a.e. \quad and \\ & k^2 (h \circ T)P + 2kh + \beta^4 (h \circ T)P \ge 0 \ a.e. \quad for\ all \quad k \in \mathbb{R} \end{aligned}$$

Proof. By Theorem 2.3 $C_T^* \in \mathcal{B}(L^2(m))$ is (α, β)-normal $(0 \le \alpha \le 1 \le \beta)$

$$\begin{aligned} \text{iff} \quad & \langle (k^2 M_h + 2k\alpha^2 M_{(h \circ T)P} + M_h) \chi_E, \chi_E \rangle \ge 0 \text{ and} \\ & \langle (k^2 M_{(h \circ T)P} + 2k M_h + \beta^4 M_{(h \circ T)P}) \chi_E, \chi_E \rangle \ge 0 \\ & \quad \text{for every } \chi_E \text{ of } E \text{ in } \Sigma \text{ such that } m(E) < \infty \text{ and for all } \ k \in \mathbb{R} \end{aligned}$$

iff $\int (k^2 M_h + 2k\alpha^2 M_{(h\circ T)P} + M_h)\chi_E\, dm \geq 0$ and

$\int (k^2 M_{(h\circ T)P} + 2kM_h + \beta^4 M_{(h\circ T)P})\chi_E\, dm \geq 0$

for every χ_E of E in Σ such that $m(E) < \infty$ and for all $k \in \mathbb{R}$

iff $\int_E (k^2 h + 2k\alpha^2 (h\circ T)P + h) dm \geq 0$ and

$\int_E (k^2 (h\circ T)P + 2kh + \beta^4 (h\circ T)P) dm \geq 0,$

for every χ_E of E in Σ such that $m(E) < \infty$ and for all $k \in \mathbb{R}$

iff $k^2 h + 2k\alpha^2 (h\circ T)P + h \geq 0$ a.e. and

$k^2 (h\circ T)P + 2kh + \beta^4 (h\circ T)P \geq 0$ a.e. for all $k \in \mathbb{R}$ □

Corollary 3.3. *A composition operator C_T^* on $L^2(m)$ with dense range is (α,β)-normal $(0 \leq \alpha \leq 1 \leq \beta)$ iff $k^2 h + 2k\alpha^2 (h\circ T) + h \geq 0$ a.e. and $k^2(h\circ T) + 2kh + \beta^4 (h\circ T) \geq 0$ a.e. for all $k \in \mathbb{R}$.*

Corollary 3.4. *Let C_T^* on $L^2(m)$ be a composition operator with dense range. Then, C_T^* is (α,β)-normal $(0 \leq \alpha \leq 1 \leq \beta)$ iff $\alpha^2(h\circ T) \leq h \leq \beta^2 (h\circ T) \geq 0$ a.e.*

Corollary 3.5. *For an operator, the adjoint C_T^* of composition operator is (α,β)-Normal $(0 \leq \alpha \leq 1 \leq \beta)$ iff*

(a) $\sum_{\sigma(h)} \subseteq T^{-1}(\Sigma)$, *and*

(b) $\alpha^2(h\circ T) \leq h \leq \beta^2(h\circ T)$ *a.e., where* $\sum_{\sigma(h)}$ *denote the relative completion of the sigma-algebra generated by* $\{A \cap$ *support of* $h : A$ *in* $\Sigma\}$.

Proof. Suppose C_T^* is (α,β)-Normal $(0 \leq \alpha \leq 1 \leq \beta)$.

Since $\sum_{\sigma(h)} \subseteq T^{-1}(\Sigma)$, therefore $\ker C_T^* \subseteq \ker C_T$.

Therefore, (a) holds and so h is $T^{-1}(\Sigma)$-measurable.

Hence, the set $A = \{s : \alpha^2 h(T(s)) > h(s) > \beta^2 h(T(s))\}$ belongs to $T^{-1}(\Sigma)$ and so A can be written as disjoint union of sets A_n of finite measure which also belong to $T^{-1}(\Sigma)$.

Since, C_T^* is (α,β)-Normal operator

$$\begin{aligned} 0 &\leq \langle (C_T^* C_T - \alpha^2 C_T C_T^*)\chi_{A_n}, \chi_{A_n}\rangle \\ &= \langle h\chi_{A_n}, \chi_{A_n}\rangle - \langle \alpha^2 (h\circ T)P\,\chi_{A_n}, \chi_{A_n}\rangle \\ &= \int_{A_n} (h - \alpha^2 (h\circ T)) dm \leq 0. \end{aligned}$$

Hence, $m(A_n) = 0$, $\forall\ n \in \mathbb{N}$ and therefore (b) holds.

Conversely, let (a) and (b) hold.

Write $f = f_1 + f_2$, where $f_1 \in (\overline{R(C_T)})$ and $f_2 \in \overline{R(C_T)}^{\perp}$.

We have,

$$\begin{aligned}\langle (C_T^* C_T - \alpha^2 C_T C_T^*), f\rangle &= \langle hf - \alpha^2 (h \circ T) Pf, f\rangle \\ &= \langle h(f_1 + f_2) - \alpha^2 (h \circ T) P(f_1 + f_2), (f_1 + f_2)\rangle\end{aligned}$$

since, $\alpha^2 (h \circ T) f_1$ is $T^{-1}(\Sigma)$-measurable, therefore it belongs to $\overline{R(C_T)}$ and so $\langle \alpha^2 (h \circ T) P f_1, f_2\rangle = 0$.

Since, $f_2 \in \ker C_T$. Therefore, $hf_2 = C_T^* C_T f_2 = 0$ and $\langle hf_1, f_2\rangle = \langle hf_2, f_1\rangle = \langle hf_2, f_2\rangle = 0$.

So,

$$\begin{aligned}\langle (C_T^* C_T - \alpha^2 C_T C_T^*)\rangle &= \langle hf_1, f_1\rangle - \alpha^2 \langle (h \circ T) f_1, f_1\rangle \\ &= \int (h - \alpha^2 (h \circ T)) |f_1|^2 dm \\ &\geq 0\end{aligned}$$

Similarly, $\beta^2 C_T C_T^* \geq C_T^* C_T$.

Therefore, C_T^* is (α, β)-normal operator. □

Acknowledgements : The authors thank the referees for their comments and suggestions. The second author is supported by the Junior Research Fellowship of Council of Scientific and Industrial Research, India (Grant no. 09/045(1139)/ 2011-EMR-I).

References

[1] M.S. Moslehian, On (α, β)-normal operators in Hilbert spaces, image 39, Problem 34-39 (2007).

[2] S.S. Dragomir, M.S. Moslehian, Some inequalities for (α, β)-normal operators in Hilbert spaces, Facta Univ. Ser. Math. Inform. 23 (2008) 39-47.

[3] H. Takagi, K. Yokouchi, Multiplication and composition operators between two L^p-spaces, Contemporary Math. 232 (1999) 321-338.

[4] R.K. Singh, Compact and quasinormal composition operators, Proc. Amer. Math. Soc. 45 (1974) 80-82.

[5] J. T. Campbell, W. E. Hornor, Semi-normal composition operators, J. Operator Theory 29 (1993) 323-343.

[6] P. Dibrell, J. T. Campbell, Hyponormal powers of composition operators, Proc. Amer. Math. Soc. 102 (4) (1988) 914-918.

[7] D.J. Harrington and R. Whitley, Seminormal composition operators, J. Operator Theory 11 (1984) 125-135.

[8] A. Lambert, Subnormal Composition Operator, Proc. Amer. Math. Soc. 3 (3) (1978) 750-754.

[9] A. Lambert, Subnormal composition operators, Bull. London Math. Soc. 18 (1986) 395-400.

[10] R. Whitley, Normal and quasinormal composition operators, Proc. Amer. Math. Soc. 70 (1978) 114-118.

[11] D. Senthilkumar, P.M. Naik and R. Santhi, Weighted composition of k-quasi-paranormal operators, Int. J. Math. Arch. 3 (2) (2012) 739-746.

[12] Gopal Datt, On k-Quasiposinormal Weighted Composition Operators, Thai J. Math. 11 (1) (2013) 131142.

[13] Anuradha Gupta and Neha Bhatia, n-Normal and n-Quasinormal composition and weighted composition operators on $L^2(\mu)$, Mat. Vesnik 66 (4) (2014) 364370.

(Received 2 January 2014)
(Accepted 13 January 2015)