Thai Journal of Mathematics
Volume 14 (2016) Number 1 : 69–83
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On Composite $l(m, n)$-Kummer's Matrix Functions of Two Complex Variables

R.A. Rashwan[†], M.S. Metwally[‡], M.T. Mohamed[♭] and A. Shehata[♯,1]

[†]Department of Mathematics, Faculty of Science, Assiut University
Assiut 71516, Egypt
e-mail : rr_rashwan54@yahoo.com

[‡]Department of Mathematics, Faculty of Science (Suez)
Suez Canal University, Egypt
e-mail : met641958@yahoo.com

[♭]Department of Mathematics, Faculty of Science(New Valley)
Assiut University, New Valley, EL-Kharga 72111, Egypt
e-mail : tawfeek200944@yahoo.com

[♯]Department of Mathematics, Faculty of Science, Assiut University
Assiut 71516, Egypt
Department of Mathematics, College of Science and Arts, Unaizah
Qassim University, Qassim, Kingdom of Saudi Arabia
e-mail : drshehata2006@yahoo.com

Abstract : The principal object of this work is to define and study of a new kummer's matrix function, namely, composite $l(m, n)$-Kummer's matrix function of two complex variables. The radius of regularity and matrix recurrence relations on this function are obtained. The effect of differential operator $\alpha(\mathbb{D})$ on this function is investigated and a solution of a certain partial differential equation is established.

Keywords : $l(m, n)$-Kummer's matrix function; matrix recurrence relations; matrix differential equation; differential operator; composition function.
2010 Mathematics Subject Classification : 33C60; 33C65; 15A60; 65F35.

---

[1]Corresponding author.

1 Introduction and Preliminaries

Special matrix functions appear in connection with statistics, mathematical physics, theoretical physics, group representation theory and orthogonal matrix polynomials are closely related [1]-[4]. In [5], the hypergeometric matrix function has been introduced as a matrix power series and an integral representation. Moreover, Jódar and Cortés introduced and studied the hypergeometric matrix function and the hypergeometric matrix differential equation in [6, 7]. In [8]-[29] and [30]-[33], extension to the matrix function framework of the classical families of p-Kummers matrix function, and q-Appell matrix function and Humbert matrix function have been proposed. The fourth author has earlier studied the p and q-Horn's H_2, $pl(m,n)$-Kummer matrix functions of two complex variables under differential operators [15, 19]. The reason of interest for this family of hypergeometric matrix function is due to their intrinsic mathematical importance.

Throughout this paper D_0 will denote the complex plane. If A is a matrix in $\mathbb{C}^{N\times N}$ its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. The two-norm of A is denoted by $||A||_2$, which is defined by [1]

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where a vector x in $\mathbb{C}^N$, $||x||_2 = (x^T x)^{\frac{1}{2}}$ is Euclidean norm of x, x^T denotes the conjugate transpose and $\Re(z)$ is the real part of the complex number z.

In [34], if $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$ such that where $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, the image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Furthermore, if

$$A + nI \quad \text{is an invertible matrix for all integers } n \geq 0 \tag{1.1}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, then $\Gamma(A)$ is an invertible matrix, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula see [6]

$$(A)_n = A(A+I)...(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); \quad n \geq 1; \quad (A)_0 = I. \tag{1.2}$$

Jódar and Cortés have proved :(see [2])

$$\Gamma(A) = \lim_{n\to\infty} (n-1)![(A)_n]^{-1} n^A, \tag{1.3}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $\Re(z) > 0$, for every eigenvalue $z \in \sigma(A)$.

For any arbitrary matrices A and B in $\mathbb{C}^{N\times N}$, then (see [1])

$$\begin{aligned} \|AB\| &\leq \|A\|\|B\|, \\ \|(A)_n\| &\leq (\|A\|)_n, \\ \|(A)_n(B)_n\| &\leq (\|A\|)_n(\|B\|)_n. \end{aligned} \tag{1.4}$$

For the purpose of this work, we denote the following relations

$$((A)+(l(m,n)-1)I)! \cong \sqrt{2\pi(A+(l(m,n)-1)I)}\left(\frac{(A+(l(m,n)-1)I)}{e}\right)^{(A+(l(m,n)-1)I)} \tag{1.5}$$

and

$$\sigma_{m,n} = \begin{cases} (\frac{m+n}{m})^{\frac{m}{2}}(\frac{m+n}{n})^{\frac{n}{2}}, & m,n\neq 0; \\ 1, & m,n=0 \end{cases} \tag{1.6}$$

where $l(m,n)=\frac{1}{2}(m+n+1)(m+n)+n$ in [35].

2 Composite $l(m,n)$-Kummer's Matrix Function

Let A_i and B_i are commutative matrices in $\mathbb{C}^{N\times N}$ such that B_i+nI are invertible matrices for all integers $n\geq 0$, $i=1,2$. We define the $l(m,n)$-kummer's matrix function $\Phi_2(A;B;z,w)$ of two complex variables in the form

$$\Phi_2(A_1;B_1;z_1,w_1) = \sum_{l(m_1,n_1)\geq 0} \frac{(A_1)_{l(m_1,n_1)}[(B_1)_{l(m_1,n_1)}]^{-1}}{l(m_1,n_1)!} z_1^{m_1} w_1^{n_1} \tag{2.1}$$

and

$$\Phi_2'(A_2;B_2;z_2,w_2) = \sum_{l(m_2,n_2)\geq 0} \frac{(A_2)_{l(m_2,n_2)}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_2,n_2)!} z_2^{m_2} w_2^{n_2}. \tag{2.2}$$

Suppose that the matrix functions $\Phi(A_1,A_2;B_1,B_2;z_1,w_1,z_2,w_2)$ is the composite $l(m,n)$-Kummer's matrix function of [5] the $l(m,n)$-Kummer's matrix functions (2.1), (2.2) and can be written as

$$\begin{aligned} &\Phi(A_1,A_2;B_1,B_2;z_1,z_2,w_1,w_2) = \Phi_2(A_1;B_1;z_1,w_1)\Phi_2'(A_2;B_2;z_2,w_2) \\ &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \frac{(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} \\ &\qquad (z_1^{m_1}w_1^{n_1}z_2^{m_2}w_2^{n_2}) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2) \\ &= \frac{(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} z_1^{m_1}w_1^{n_1}z_2^{m_2}w_2^{n_2} \end{aligned}$$

For simplicity, we can write the $\Phi(A_1 \pm I, A_2; B_1, B_2; z_1, z_2, w_1, w_2)$ in the form $\Phi(A_1\pm)$, $\Phi(A_1, A_2 \pm I; B_1, B_2; z_1, z_2, w_1, w_2)$ in the form $\Phi(A_2\pm)$, $\Phi(A_1, A_2; B_1 \pm I, B_2; z_1, z_2, w_1, w_2)$ in the form $\Phi(B_1\pm)$ and $\Phi(A_1, A_2; B_1, B_2 \pm I; z_1, z_2, w_1, w_2)$ in the form $\Phi(B_2\pm)$.

We begin the study of this function by calculating the radius of regularity R of such function for this purpose, we recall relation (1.3.10) of [35] and keeping in mind that $\sigma_{m_1,m_2,n_1,n_2} \geq 1$. Hence

$$
\begin{aligned}
\frac{1}{R} &= \lim_{m_1+m_2+n_1+n_2\to\infty} \sup \left(\frac{\|U_{m_1,m_2,n_1,n_2}\|}{\sigma_{m_1,m_2,n_1,n_2}} \right)^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&= \lim_{m_1+m_2+n_1+n_2\to\infty} \sup \left\| \left(\left(\frac{(A_1 + (l(m_1,n_1) - 1)I)}{e} \right)^{(A_1+(l(m_1,n_1)-1)I)} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&\quad \left\| \left(\left(\frac{(A_2 + (l(m_2,n_2) - 1)I)}{e} \right)^{(A_2+(l(m_2,n_2)-1)I)} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&\quad \left\| \left(\left(\frac{(B_1 + (l(m_1,n_1) - 1)I)}{e} \right)^{-(B_1+(l(m_1,n_1)-1)I)} \left(\frac{l(m_1,n_1)}{e} \right)^{-l(m_1,n_1)} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&\quad \left\| \left(\left(\frac{(B_2 + (l(m_2,n_2) - 1)I)}{e} \right)^{-(B_2+(l(m_2,n_2)-1)I)} \left(\frac{l(m_2,n_2)}{e} \right)^{-l(m_2,n_2)} \frac{1}{\sigma_{m_1,m_2,n_1,n_2}} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&\leq \lim_{m_1+m_2+n_1+n_2\to\infty} \sup \left\| \left(\left(I + \frac{A_1 - I}{l(m_1,n_1)} \right)^{l(m_1,n_1)I} \left(I + \frac{B_1 - I}{l(m_1,n_1)} \right)^{-l(m_1,n_1)I} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&\quad \left\| \left(\left(I + \frac{A_2 - I}{l(m_2,n_2)} \right)^{l(m_2,n_2)I} \left(I + \frac{B_2 - I}{l(m_2,n_2)} \right)^{-l(m_2,n_2)I} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} \\
&\quad \left\| \left(\left(\frac{e}{l(m_1,n_1)} \right)^{l(m_1,n_1)} \left(\frac{e}{l(m_2,n_2)} \right)^{l(m_2,n_2)} \right) \right\|^{\frac{1}{m_1+m_2+n_1+n_2}} = 0
\end{aligned}
$$

where

$$
\sigma_{m_1,m_2,n_1,n_2} = \begin{cases} \dfrac{(m_1+m_2+n_1+n_2)^{\frac{m_1+m_2+n_1+n_2}{2}}}{m_1^{\frac{m_1}{2}} m_2^{\frac{m_2}{2}} n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}}}, & m_1, m_2, n_1, n_2 \neq 0; \\ 1, & m_1, m_2, n_1, n_2 = 0. \end{cases}
$$

Then, the radius of regularity of composite $l(m,n)$-Kummer's matrix function is an entire function.

In this connection the following contiguous functions relations follow, directly by increasing or decreasing one in original relation

$$
\begin{aligned}
&\Phi(A_1+) \\
&= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \frac{(A_1 + I)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} \\
&\qquad (z_1^{m_1} w_1^{n_1} z_2^{m_2} w_2^{n_2}) \\
&= A_1^{-1} \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} (A_1 + l(m_1,n_1)I) U_{m_1,m_2,n_1,n_2}(z_1, z_2, w_1, w_2).
\end{aligned} \tag{2.4}
$$

Similarly, we get

$$\begin{aligned}
\Phi(A_2+) &= A_2^{-1}\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(A_2+l(m_2,n_2)I)U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
\Phi(A_1-) &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(A_1-I)[(A_1+(l(m_1,n_1)-1)I)]^{-1}U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
\Phi(A_2-) &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(A_2-I)[(A_2+(l(m_2,n_2)-1)I)]^{-1}U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
\Phi(B_1+) &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}B_1[(B_1+l(m_1,n_1)I)]^{-1}U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
\Phi(B_2+) &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}B_2[(B_2+l(m_2,n_2)I)]^{-1}U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
\Phi(B_1-) &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(B_1-I)^{-1}(B_1+(l(m_1,n_1)-1)I)U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2)\\
\Phi(B_2-) &= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(B_2-I)^{-1}(B_2+(l(m_2,n_2)-1)I)U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2).
\end{aligned} \tag{2.5}$$

By the same way, we have

$$\begin{aligned}
&\Phi(A_1+;B_1+) =\\
&A_1^{-1}B_1\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(A_1+l(m_1,n_1)I)[(B_1+l(m_1,n_1)I)]^{-1}U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
&\Phi(A_1+;B_1-) =\\
&A_1^{-1}(B_1-I)^{-1}\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}(A_1+l(m_1,n_1)I)(B_1+(l(m_1,n_1)-1)I)U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),\\
&\Phi(A_1-;B_1+) =\\
&B_1(A_1-I)\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}[(A+(l(m,n)-1)I)]^{-1}[(B_1+l(m_1,n_1)I)]^{-1}U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\Phi(A_1-;B_1-) =\\
&(A_1-I)(B_1-I)^{-1}\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}[(A_1+(l(m_1,n_1)-1)I)]^{-1}(B_1+(l(m_1,n_1)-1)I)\\
&U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2).
\end{aligned} \tag{2.7}$$

For $k \geq 1$, we deduce that

$$\begin{aligned}
\Phi(A_1+kI) &= \prod_{r=1}^{k}(A_1+(r-1)I)^{-1}\\
&\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}\prod_{r=1}^{k}(A_1+(l(m_1,n_1)+(r-1))I)U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),
\end{aligned}$$

$$\Phi(A_1 - kI) = \prod_{r=1}^{k}(A_1 - rI)$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0)} \prod_{r=1}^{k}(A_1 + (l(m_1,n_1) - r)I)^{-1} U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),$$

$$\Phi(A_2 + kI) = \prod_{r=1}^{k}(A_2 + (r-1)I)^{-1}$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \prod_{r=1}^{k}(A_2 + (l(m_2,n_2) + (r-1))I) U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),$$

$$\Phi(A_2 - kI) = \prod_{r=1}^{k}(A_2 - rI)$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \prod_{r=1}^{k}(A_1 + (l(m_2,n_2) - r)I)^{-1} U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),$$

$$\Phi(B_1 + kI) = \prod_{r=1}^{k}(B_1 + (r-1)I)$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \prod_{r=1}^{k}(B_1 + (l(m_1,n_1) + (r-1))I)^{-1} U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),$$

$$\Phi(B_1 - kI) = \prod_{r=1}^{k}(B_1 - rI)^{-1}$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \prod_{r=1}^{k}(B_1 + (l(m_1,n_1) - r)I) U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2),$$

$$\Phi(B_2 + kI) = \prod_{r=1}^{k}(B_2 + (r-1)I)$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \prod_{r=1}^{k}(B_2 + (l(m_2,n_2) + (r-1))I)^{-1} U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2)$$

and

$$\Phi(B_2 - kI) = \prod_{r=1}^{k}(B_2 - rI)^{-1}$$
$$\sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \prod_{r=1}^{k}(B_2 + (l(m_2,n_2) - r)I) U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2).$$

2.1 The composite $l(m,n)$-Kummer matrix function under the differential operator D

We consider the following differential operator D in the form

$$D_i = z_i \frac{\partial}{\partial z_i} + w_i \frac{\partial}{\partial w_i} = d_{i1} + d_{i2}; \quad i = 1, 2. \tag{2.8}$$

Now, acting by operator $\mathbb{D}_i = \frac{1}{2}(D_i)_2 + d_{i2}$ and $\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2$. It is clear that

$$\begin{aligned}
\mathbb{D}\,\Phi = & \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \\
& \frac{[l(m_1,n_1)+l(m_2,n_2)](A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} z_1^{m_1} z_2^{m_2} w_1^{n_1} w_2^{n_2} \\
= & \sum_{l(m_1,n_1)\geq 1,l(m_2,n_2)\geq 0} \frac{(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{(l(m_1,n_1)-1)!l(m_2,n_2)!} z_1^{m_1} z_2^{m_2} w_1^{n_1} w_2^{n_2} \\
& + \sum_{l(m_1,n_1)\geq 0,l(m_2,n_2)\geq 1} \frac{(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!(l(m_2,n_2)-1)!} z_1^{m_1} z_2^{m_2} w_1^{n_1} w_2^{n_2}
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
= & \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \\
& \frac{(A_1)_{l(m_1,n_1)+1}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)+1}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} z_1^{m_1-1} z_2^{m_2} w_1^{n_1+1} w_2^{n_2} \\
& + \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \\
& \frac{(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)+1}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)+1}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} z_1^{m_1} z_2^{m_2-1} w_1^{n_1} w_2^{n_2+1} \\
= & \frac{w_1}{z_1} \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} (A_1 + l(m_1,n_1)I)[(B_1 + l(m_1,n_1)I)]^{-1} U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2) \\
& + \frac{w_2}{z_2} \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} (A_2 + l(m_2,n_2)I)[(B_2 + l(m_2,n_2)I)]^{-1} U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2) \\
= & \frac{w_1}{z_1}\,\Phi + \frac{w_1}{z_1}(A_1 - B_1)[(B_1)]^{-1}\,\Phi(B_1+) \\
& + \frac{w_2}{z_2}\,\Phi + \frac{w_2}{z_2}(A_2 - B_2)[(B_2)]^{-1}\,\Phi(B_2+).
\end{aligned} \tag{2.10}$$

i.e., the composite $l(m,n)$-Kummer's matrix function is a solution to this matrix differential equation

$$\left(\mathbb{D} - \frac{w_1}{z_1} - \frac{w_2}{z_2}\right)\Phi - \frac{w_1}{z_1}(A_1 - B_1)[(B_1)]^{-1}\,\Phi(B_1+) - \frac{w_2}{z_2}(A_2 - B_2)[(B_2)]^{-1}\,\Phi(B_2+) = 0. \tag{2.11}$$

In addition, the relation (2.8) can be written in the form

$$\mathbb{D}\,\Phi = \frac{w_1 A_1[(B_1)]^{-1}}{z_1}\,\Phi(A_1+;B_1+) + \frac{w_2 A_2[(B_2)]^{-1}}{z_2}\,\Phi(A_2+;B_2+). \tag{2.12}$$

Next, we take the operator $\mathbb{D}_1\,\mathbb{D}_2$ in consideration as follows

$$\mathbb{D}_1\mathbb{D}_2\,\Phi = \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}$$

$$\frac{l(m_1,n_1)l(m_2,n_2)(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} z_1^{m_1} z_2^{m_2} w_1^{n_1} w_1^{n_1}$$

$$= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \frac{(A_1)_{l(m_1,n_1)}(A_2)_{l(m_2,n_2)}[(B_1)_{l(m_1,n_1)}]^{-1}[(B_2)_{l(m_2,n_2)}]^{-1}}{(l(m_1,n_1)-1)!(l(m_2,n_2)-1)!} z_1^{m_1} z_2^{m_2} w_1^{n_1} w_1^{n_1}$$

$$= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}$$

$$\frac{(A_1)_{l(m_1,n_1)+1}(A_2)_{l(m_2,n_2)+1}[(B_1)_{l(m_1,n_1)+1}]^{-1}[(B_2)_{l(m_2,n_2)+1}]^{-1}}{l(m_1,n_1)!l(m_2,n_2)!} z_1^{m_1-1} z_2^{m_2-1} w_1^{n_1+1} w_1^{n_1+1}$$

$$= \frac{w_1 w_2}{z_1 z_2} \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}$$

$$\Big[(A_1 + l(m_1,n_1)I)(A_2 + l(m_2,n_2)I)[(B_1 + l(m_1,n_1)I)]^{-1}[(B_2 + l(m_2,n_2)I)]^{-1}\Big]$$

$$U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2)$$

$$= \frac{w_1 w_2}{z_1 z_2} \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \Big[I + l(m_1,n_1)(A_2 - B_2)[(B_1 + l(m_1,n_1)I)]^{-1}[(B_2 + l(m_2,n_2)I)]^{-1}$$

$$+ l(m_2,n_2)(A_1 - B_1)[(B_1 + l(m_1,n_1)I)]^{-1}[(B_2 + l(m_2,n_2)I)]^{-1}$$

$$+ (A_1A_2 - B_1B_2)[(B_1 + l(m_1,n_1)I)]^{-1}[(B_2 + l(m_2,n_2)I)]^{-1}\Big] U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2)$$

$$= \frac{w_1 w_2}{z_1 z_2} \Big[\,\Phi + (A_1 - B_1)[(B_1)]^{-1}[(B_2)]^{-1}\mathbb{D}_2\,\Phi(B_1+,B_2+)$$

$$+ (A_2 - B_2)[(B_1)]^{-1}[(B_2)]^{-1}\mathbb{D}_1\,\Phi(B_1+,B_2+)$$

$$+ (A_1A_2 - B_1B_2)[(B_1)]^{-1}[(B_2)]^{-1}\,\Phi(B_1+,B_2+)\Big]$$

i.e., the composite $l(m,n)$-Kummer's matrix function is a solution to this matrix differential equation

$$\begin{aligned}&\left(\mathbb{D}_1\mathbb{D}_2 - \frac{w_1w_2}{z_1z_2}\right)\Phi - \frac{w_1w_2}{z_1z_2}\Big[(A_1A_2 - B_1B_2) + (A_1 - B_1)\mathbb{D}_2 + (A_2 - B_2)\mathbb{D}_1\Big]\\ &[(B_1)]^{-1}[(B_2)]^{-1}\,\Phi(B_1+,B_2+) = 0.\end{aligned} \tag{2.13}$$

The $\alpha(\mathbb{D})$ differential operator has been defined by Sayyed [35] in the form

$$\alpha(\mathbb{D}) = 1 + \sum_{k=1}^{N}\mathbb{D}^k; \quad \mathbb{D}^k = \mathbb{D}\,\mathbb{D}^{k-1}. \tag{2.14}$$

From (2.1), (2.3) and (2.5), we obtain

$$(\mathbb{D}\,I + B_1 + B_2 - 2I)\,\Phi = \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0}$$

$$(B_1 + B_2 + (l(m_1,n_1) + l(m_2,n_2) - 2)I)U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2)$$

$$= \sum_{l(m_1,n_1),l(m_2,n_2)\geq 0} \Big[(B_1 + (l(m_1,n_1) - 1)I) + (B_2 + (l(m_2,n_2) - 1)I)\Big] U_{m_1,m_2,n_1,n_2}(z_1,z_2,w_1,w_2)$$

$$= (B_1 - I)\,\Phi(B_1-) + (B_2 - I)\,\Phi(B_2-)$$

hence, we have

$$\mathbb{D}\,\Phi = (B_1 - I)\,\Phi(B_1-) + (B_2 - I)\,\Phi(B_2-) - (B_1 + B_2 - 2I)\Phi, \tag{2.15}$$

$$\mathbb{D}\,\Phi(B_1-) = (B_1 - 2I)\,\Phi(B_1 - 2I) + (B_2 - I)\,\Phi(B_1-, B_2-) - (B_1 + B_2 - 3I)\Phi(B_1-), \tag{2.16}$$

$$\mathbb{D}\,\Phi(B_2-) = (B_1 - I)\,\Phi(B_1-, B_2-) + (B_2 - 2I)\,\Phi(B_1, B_2 - 2I) - (B_1 + B_2 - 3I)\Phi(B_2-) \tag{2.17}$$

and

$$\begin{aligned}\mathbb{D}^2\,\Phi &= (B_1 - I)\Big[(B_1 - 2I)\,\Phi(B_1 - 2I) + (B_2 - I)\,\Phi(B_1-, B_2-) - (B_1 + B_2 - 3I)\Phi(B_1 - I)\Big]\\ &+ (B_2 - I)\Big[(B_1 - I)\,\Phi(B_1-, B_2-) + (B_2 - 2I)\,\Phi(B_1, B_2 - 2I) - (B_1 + B_2 - 3I)\Phi(B_2 - I)\Big]\\ &- (B_1 + B_2 - 2I)\Big[(B_1 - I)\,\Phi(B_1-) + (B_2 - I)\,\Phi(B_2-) - (B_1 + B_2 - 2I)\Phi\Big],\end{aligned} \tag{2.18}$$

$$\begin{aligned}\mathbb{D}^2\,\Phi &= (B_1 - I)(B_1 - 2I)\,\Phi(B_1 - 2I) + 2(B_1 - I)(B_2 - I)\,\Phi(B_1-, B_2-)\\ &+ (B_2 - I)(B_2 - 2I)\,\Phi(B_2 - 2I) - (B_1 - I)(B_1 + B_2 - 3I)\Phi(B_1-)\\ &- (B_2 - I)(B_1 + B_2 - 3I)\Phi(B_2-) - (B_1 - I)(B_1 + B_2 - 2I)\,\Phi(B_1-)\\ &- (B_2 - I)(B_1 + B_2 - 2I)\,\Phi(B_2-) + (-1)^2(B_1 + B_2 - 2I)^2\Phi,\end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathbb{D}^2\,\Phi &= (B_1 - I)(B_1 - 2I)\,\Phi(B_1 - 2I) + 2(B_1 - I)(B_2 - I)\,\Phi(B_1-, B_2-)\\ &+ (B_2 - I)(B_2 - 2I)\,\Phi(B_2 - 2I) - (B_1 - I)(2B_1 + 2B_2 - 5I)\Phi(B_1-)\\ &- (B_2 - I)(2B_1 + 2B_2 - 5I)\Phi(B_2-) + (-1)^2(B_1 + B_2 - 2I)^2\Phi,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\mathbb{D}\,\Phi(B_1 - 2I) &= (B_1 - 3I)\,\Phi(B_1 - 3I) + (B_2 - I)\,\Phi(B_1 - 2I, B_2-)\\ &\quad - (B_1 + B_2 - 4I)\Phi(B_1 - 2I),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathbb{D}\,\Phi(B_2 - 2I) &= (B_1 - I)\,\Phi(B_1-, B_2 - 2I) + (B_2 - 3I)\,\Phi(B_1, B_2 - 3I)\\ &\quad - (B_1 + B_2 - 4I)\Phi(B_2 - 2I)\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\mathbb{D}\,\Phi(B_1-, B_2-) &= (B_1 - 2I)\,\Phi(B_1 - 2I, B_2-) + (B_2 - 2I)\,\Phi(B_1-, B_2 - 2I)\\ &\quad - (B_1 + B_2 - 4I)\Phi(B_1-, B_2-).\end{aligned} \tag{2.23}$$

So

$$\begin{aligned}\mathbb{D}^3\,\Phi &= (B_1 - I)(B_1 - 2I)\,\mathbb{D}\Phi(B_1 - 2I) + 2(B_1 - I)(B_2 - I)\,\mathbb{D}\Phi(B_1-, B_2-)\\ &+ (B_2 - I)(B_2 - 2I)\,\mathbb{D}\Phi(B_2 - 2I) - (B_1 - I)(2B_1 + 2B_2 - 5I)\,\mathbb{D}\Phi(B_1-)\\ &- (B_2 - I)(2B_1 + 2B_2 - 5I)\mathbb{D}\Phi(B_2-) + (-1)^2(B_1 + B_2 - 2I)^2\mathbb{D}\Phi.\end{aligned} \tag{2.24}$$

$$\mathbb{D}^3\,\Phi = (B_1-I)(B_1-2I)\Big[(B_1-3I)\,\Phi(B_1-3I)+(B_2-I)\,\Phi(B_1-2I,B_2-)$$
$$-(B_1+B_2-4I)\Phi(B_1-2I)\Big]+2(B_1-I)(B_2-I)\Big[(B_1-2I)\,\Phi(B_1-2I,B_2-)$$
$$+(B_2-2I)\,\Phi(B_1-,B_2-2I)-(B_1+B_2-4I)\Phi(B_1-,B_2-)\Big]$$
$$+(B_2-I)(B_2-2I)\Big[(B_1-I)\,\Phi(B_1-,B_2-2I)+(B_2-3I)\,\Phi(B_1,B_2-3I)$$
$$-(B_1+B_2-4I)\Phi(B_2-2I)\Big]-\Big((B_1-I)(B_1+B_2-2I)+(B_1-I)(B_1+B_2-3I)\Big)$$
$$\Big[(B_1-2I)\,\Phi(B_1-2I)+(B_2-I)\,\Phi(B_1-,B_2-)-(B_1+B_2-3I)\Phi(B_1-)\Big]$$
$$-\Big((B_1-I)(B_1+B_2-2I)+(B_1-I)(B_1+B_2-3I)\Big)$$
$$\Big[(B_1-I)\,\Phi(B_1-,B_2-)+(B_2-2I)\,\Phi(B_1,B_2-2I)-(B_1+B_2-3I)\Phi(B_2-)\Big]$$
$$+(-1)^2(B_1+B_2-2I)^2\Big[(B_1-I)\,\Phi(B_1-)+(B_2-I)\,\Phi(B_2-)-(B_1+B_2-2I)\Phi,\Big]. \tag{2.25}$$

Thus by mathematical induction we have the following general form

$$\Xi(\mathbb{D})\Phi = (1+\sum_{k=1}^{N}\mathbb{D}^k)\Phi = \Phi+\sum_{k=1}^{N}\prod_{j=1}^{k}(B_1-jI)\,\Phi(B_1-jI)$$
$$+\sum_{k=1}^{N}\prod_{j=1}^{k}(B_2-jI)\,\Phi(B_2-jI)+\sum_{k=1}^{N-1}\prod_{j=1}^{k}\prod_{i=1}^{j}\Big[(B_1-jI)(B_2-iI)\,\Phi(B_1-jI,B_2-iI)$$
$$+(B_1-iI)(B_2-jI)\,\Phi(B_1-iI,B_2-jI)\Big]-\Big[\prod_{j=1}^{k}(B_1-jI)+\prod_{j=1}^{k-1}(B_1-jI)$$
$$\sum_{k=1}^{N-1}(B_1-jI)\Big]\,\Phi(B_1-(j-1)I)-\Big[\prod_{j=1}^{k}(B_2-jI)$$
$$+\prod_{j=1}^{k-1}(B_2-jI)\sum_{k=1}^{N-1}(B_2-jI)\Big]\,\Phi(B_2-(j-1)I)$$
$$+\Big[\prod_{j=1}^{k-1}(B_1-jI)\sum_{j=1}^{k-1}(B_1-jI)+\prod_{j=1}^{k-2}(B_1-jI)\big(\sum_{j=1}^{k-2}(B_1-jI)\big)^2$$
$$+\sum_{j=1}^{k-3}(B_1-jI)(B_1-(j+1)I)+\sum_{j=1}^{k-4}(B_1-jI)(B_1-(j+1)I)...\Big]\Phi(B_1-(j-2)I)$$

$$+\Big[\prod_{j=1}^{k-1}(B_2-jI)\sum_{j=1}^{k-1}(B_2-jI)+\prod_{j=1}^{k-2}(B_2-jI)\big(\sum_{j=1}^{k-2}(B_2-jI)\big)^2$$
$$+\sum_{j=1}^{k-3}(B_2-jI)(B_2-(j+1)I)+\sum_{j=1}^{k-4}(B_2-jI)(B_2-(j+1)I)...\Big]$$
$$\Phi_2^c(B_2-(j-2)I)+...+(-1)^k(B_1+B_2-(k-1)I)^k\,\Phi.$$

Remark 2.1. *As special cases, we can be written the $\alpha(\mathbb{D}_1)$ differential operator to composite $l(m,n)$-Kummer's matrix function in the form*

$$\begin{aligned}\alpha(\mathbb{D}_1)\Phi =&(1+\sum_{k=1}^{N}\mathbb{D}_1^k)\Phi = \Phi + \sum_{k=1}^{N}\Bigg(\prod_{j=1}^{k}(B_1-jI)\ \Phi(B_1-jI)\\ &-\Big[\prod_{j=1}^{k}(B_1-jI)+\prod_{j=1}^{k-1}(B_1-jI)\sum_{k=1}^{N-1}(B_1-jI)\Big]\ \Phi(B_1-(j-1)I)\\ &+\Big[\prod_{j=1}^{k-1}(B_1-jI)\sum_{j=1}^{k-1}(B-jI)+\prod_{j=1}^{k-2}(B_1-jI)(\sum_{j=1}^{k-2}(B_1-jI)]^2)\\ &+\sum_{j=1}^{k-3}(B_1-jI)(B_1-(j+1)I)+\sum_{j=1}^{k-4}(B_1-jI)(B_1-(j+1)I)...\Big]\\ &\Phi(B_1-(j-2)I)+...+(-1)^k(B_1-I)^k\ \Phi\Bigg).\end{aligned} \tag{2.26}$$

where N is a finite positive integer.

Similarly, we obtain that

$$\begin{aligned}\alpha(\mathbb{D}_2)\ \Phi =&(1+\sum_{k=1}^{N}\mathbb{D}_2^k)\ \Phi = \Phi + \sum_{k=1}^{N}\Bigg(\prod_{j=1}^{k}(B_2-jI)\ \Phi(B_2-jI)\\ &-\Big[\prod_{j=1}^{k}(B_2-jI)+\prod_{j=1}^{k-1}(B_2-jI)\sum_{k=1}^{N-1}(B_2-jI)\Big]\ \Phi(B_2-(j-1)I)\\ &+\Big[\prod_{j=1}^{k-1}(B_2-jI)\sum_{j=1}^{k-1}(B_2-jI)+\prod_{j=1}^{k-2}(B_2-jI)(\sum_{j=1}^{k-2}(B_2-jI)]^2)\\ &+\sum_{j=1}^{k-3}(B_2-jI)(B_2-(j+1)I)+\sum_{j=1}^{k-4}(B_2-jI)(B_2-(j+1)I)...\Big]\\ &\Phi(B_2-(j-2)I)+...+(-1)^k(B_2-I)^k\ \ \Phi\Bigg).\end{aligned} \tag{2.27}$$

The results of this paper are original, variant, significant and so it is interesting and capable to develop its study in the future.

Open Problem

One can use the same class of new differential and integral operators for the new hypergeometric matrix functions. Hence, new results and further applications can be obtained. Further applications will be discussed in a forthcoming paper.

Acknowledgements : The authors would like to thank the anonymous referees for their several useful comments and suggestions towards the improvement of this paper.

References

[1] G. Golub, C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MA, 1989.

[2] L. Jódar, and J. C. Cortés, Some properties of gamma and beta matrix functions, *Appl. Math. Lett.*. 11 (1998) 89-93.

[3] A.M. Mathai, *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford University Press, Oxford, 1993.

[4] A.M. Mathai, *Jacobians of Matrix Transformations and Functions of Matrix Argument*, World Scientific Publishing, New York, 1997.

[5] K. A. M. Sayyed, M. S. Metwally, and M. T. Mohamed, Certain hypergeometric matrix function, *Scientiae Mathematicae Japonicae*. 69 (2009) 315-321.

[6] L. Jódar, J. C. Cortés, On the hypergeometric matrix function, *J. Comp. Appl. Math.* 99 (1998) 205-217.

[7] L. Jódar, J. C. Cortés, Closed form general solution of the hypergeometric matrix differential equation, Math. Computer Modell. 32 (2000) 1017-1028.

[8] G. S. Khammash, A. Shehata, A study on some properties of Legendre polynomials and integral transforms, Inter. J. Open Problems Computer Sci. Math. 5 (2) (2012) 1 - 6.

[9] G.S. Khammash, A. Shehata, On Humbert matrix polynomials, Asian J. Current Eng. Maths. (AJCEM). 1 (5) (2012) 232 - 240.

[10] M. S. Metwally, On p-Kummers matrix function of complex variable under differential operators and their properties, South. Asian Bullet. Math. 35 (2011) 1-16.

[11] M. S. Metwally, M. T. Mohamed, A. Shehata, On Hermite-Hermite matrix polynomials, Math. Bohemica. 133 (2008) 421-434.

[12] M. S. Metwally, A. Shehata, A study of a new properties of Hermite-Hermite matrix polynomials and their Applications, Inter. J. Mathematical Sci. Eng. Applications (IJMSEA). 7 (I) (2013) 19-33.

[13] R. A. Rashwan, M. S. Metwally, M. T. Mohamed, A. Shehata, Certain Kummer's matrix function of two complex variables under certain differential and integral operators, Thai Journal of Mathematics, 11 (3) (2013) 725 -740.

[14] A. Shehata, A Study of Some Special Functions and Polynomials of Complex Variables, Ph. D. Thesis, Assiut University, Assiut, Egypt, 2009.

[15] A. Shehata, On p and q-Horn's matrix function of two complex variables, *Appl. Math.*, 2 (2011), 1437-1442.

[16] A. Shehata, Certain $pl(m,n)$-Kummer matrix function of two complex variables under differential operator, Applied Mathematics. 4 (1) (2013) 91 - 96.

[17] A. Shehata, On Tricomi and Hermite-Tricomi matrix functions of complex variable, Communications Math. Applications. 2 (2-3) (2011) 97 - 109.

[18] A. Shehata, A new extension of Gegenbauer matrix polynomials and their properties, Bulletin of the International Mathematical Virtual Institute, 2 (2012) 29-42.

[19] A. Shehata, On pseudo Legendre matrix polynomials, Inter. J. Mathematical Sci. Eng. Applications (IJMSEA). 6 (VI) (2012) 251-258.

[20] A. Shehata, A new extension of Hermite-Hermite matrix polynomials and their properties, Thai Journal of Mathematics. 10 (2) (2012) 433-444.

[21] A. Shehata, On Rices matrix polynomials, *Afrika Matematika*, 25 (3) (2014) 757-777.

[22] A. Shehata, On Rainville's matrix polynomials, Sylwan Journal. 158 (9) (2014) 158-178.

[23] A. Shehata, New kinds of Hypergeometric matrix functions, British Journal of Mathematics and Computer Science. 5 (1) (2015) 92-102.

[24] A. Shehata, On a new family of the extended generalized Bessel-type matrix polynomials, Mitteilungen Klosterneuburg J., 65 (2) (2015) 100-121.

[25] A. Shehata, Some relations on Humbert matrix polynomials, Mathematica Bohemica, (in press).

[26] A. Shehata, Some relations on Konhauser matrix polynomials, Miskolc Mathematical Notes, (in press).

[27] A. Shehata, Some Relations on Gegenbauer matrix polynomials, Review of Computer Engineering Research. 2 (1) (2015) 1-21.

[28] A. Shehata, Connections between Legendre with Hermite and Laguerre matrix polynomials. Gazi University Journal of Science(G. U. J. Sci.), (in press).

[29] A. Shehata, A new extension of Bessel matrix functions, Southeast Asian Bulletin of Mathematics, (in press).

[30] A. Shehata, B. Çekim, Some relations on Hermite-Hermite matrix polynomials, University Politechnica of Bucharest Scientific Bulletin- Series A- Applied Mathematics and Physics, (in press).

[31] A. Shehata, R. Bhukya, Some properties of Hermite matrix polynomials, J. Inter. math. Virtual Institute. 5 (2015) 1-17.

[32] L. M. Upadhyaya, A. Shehata, On Legendre matrix polynomials and its applications, Inter. Trans. Mathematical Sci. Computer (ITMSC). (2) (2011) 291-310.

[33] L. M. Upadhyaya, A. Shehata, A new extension of generalized Hermite matrix polynomials, Bulletin Malaysian Mathematical Sci. Soc. 38 (1) (2015) 165-179.

[34] N. Dunford, J. T. Schwartz, Linear Operators, part I, General Theory, Interscience Publishers, INC. New York, 1957.

[35] K. A. M. Sayyed, Basic Sets of Polynomials of two Complex Variables and Convergence Propertiess, Ph. D. Thesis, Assiut University, 1975.

(Received 8 November 2013)
(Accepted 23 March 2015)