



# A New General Iterative Methods for Solving the Equilibrium Problems, Variational Inequality Problems and Fixed Point Problems of Nonexpansive Mappings

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**Abstract :** In this paper, we introduce the new general iterative methods for finding a common solution set of equilibrium problems and the set of fixed points of nonexpansive mappings which is a solution of a certain optimization problem related to a strongly positive linear operator. Under suitable control conditions, we prove the strong convergence theorems of such iterative scheme in a real Hilbert space. The main result extends various results existing in the current literature.

**Keywords :** equilibrium problem; convex problem; nonexpansive mapping; Hilbert space; fixed point.

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## 1 Introduction

Let  $H$  be a real Hilbert space and  $C$  a nonempty closed convex subset of  $H$  with inner product  $\langle \cdot, \cdot \rangle$  and its norm  $\| \cdot \|$ . Recall that a mapping  $T : C \rightarrow C$  is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The set of all fixed points of  $T$  is denoted by  $F(T) = \{x \in C : Tx = x\}$ . It is well-known that the fixed point set of a nonexpansive mapping is a closed convex subset of  $C$ . A mapping  $g : C \rightarrow C$  is a *contraction* on  $C$  if there is a constant  $\alpha \in (0, 1)$  such that

$$\|g(x) - g(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Banach's contraction principle [1] guarantees the uniqueness of fixed point of a contraction mapping. Now, we recall that a bifunction  $f : C \times C \rightarrow \mathbb{R}$  is said to be :

(a) *monotone* on  $C$  if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;$$

(b) *pseudomonotone* on  $C$  if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0, \quad \forall x, y \in C;$$

(c) *pseudomonotone* on  $C$  with respect to  $x \in C$  if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0, \quad \forall y \in C;$$

(d) *Lipschitz-type continuous* on  $C$  with two constants  $c_1 > 0$  and  $c_2 > 0$  if

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2, \quad \forall x, y, z \in C.$$

**Remark 1.1.** It is clear that (a)  $\implies$  (b)  $\implies$  (c). The example of  $f$  is pseudomonotone on  $C$  with respect to the  $EP(C, f)$  but  $f$  is not pseudomonotone on  $C$  can be found in [2].

Suppose that  $\Delta$  is an open convex set containing  $C$  and  $f : \Delta \times \Delta \rightarrow \mathbb{R}$  is a bifunction such that  $f(x, x) = 0$  for all  $x \in C$ . Such a bifunction is called an *equilibrium bifunction*. We consider the following equilibrium problem ( $EP(C, f)$ ):

$$\text{Find } \tilde{x} \in C \text{ such that } f(\tilde{x}, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The set of solution of problem (1.1) is denoted by  $Sol(C, f)$ . This problem is also called the *Ky Fan inequality* was first discovered by Ky Fan [3] in 1972. It is well-known that the equilibrium problem covers many important problems in optimization and nonlinear analysis as well as has found many applications in

economic, transportation and engineering. Let  $\Omega := F(T) \cap \text{Sol}(C, f)$  denote the set of common elements of the solution set of the equilibrium problem  $\text{Sol}(C, f)$  and the set of fixed points  $F(T)$ . The theory and methods for finding an element of  $\Omega$  have been well developed by many authors (see [4, 5, 6, 7]). Takahashi and Takahashi [8] introduced the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a real Hilbert space.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [9, 10, 11, 12] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \quad (1.2)$$

where  $A$  is a linear bounded operator,  $C$  is the fixed point set of a nonexpansive mapping  $T$  and  $b$  is a given point in  $H$ . In [13] Marino and Xu considered a general iterative method for a nonexpansive mapping in a Hilbert space  $H$ . Starting with arbitrary initial  $x_0 \in H$ , define a sequence  $\{x_n\}$  by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 0, \quad (1.3)$$

where  $A$  is a strongly positive bounded linear operator on  $H$ , i.e.,

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2 \text{ for all } x \in H. \quad (1.4)$$

They proved that if the sequence  $\{\alpha_n\}$  of parameters satisfies the appropriate conditions, then the sequence  $\{x_n\}$  generated by (1.3) converges strongly to the unique solution  $x^*$  in  $F(T)$  of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(T), \quad (1.5)$$

which is the optimality condition for the minimization problem:  $\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x)$ , where  $h$  is a potential function for  $\gamma f$  (i.e.,  $h'(x) = \gamma f(x)$  for  $x \in H$ ). Recently, Wangkeeree and Preechasilp [14] introduced the new algorithm for solving the common element of the set of fixed points of a nonexpansive mapping, the solution set of equilibrium problems and the solution set of the variational inequality problems for an inverse strongly monotone mapping. Let  $f$  be monotone, Lipschitz-type continuous on  $C$  with two constants  $c_1 > 0$  and  $c_2 > 0$ ,  $A$  a strongly linear bounded operator with coefficient  $\bar{\gamma}$  and  $B$  a  $\beta$ -inverse strongly monotone mapping. Let  $g : C \rightarrow C$  be a contraction with coefficient  $\alpha$  such that  $0 < \gamma < \bar{\gamma}/\alpha$  and  $T : C \rightarrow C$  a nonexpansive mapping. The algorithm is now described as follows.

For a given point  $x_0 = x \in C$ :

$$\begin{cases} y_k = \operatorname{argmin}\{\lambda_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ t_k = \operatorname{argmin}\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ x_{k+1} = P_C(\alpha_k \gamma g(x_k) + (I - \alpha_k A)TP_C(t_k - \beta B t_k)), \end{cases} \quad (1.6)$$

where  $P_C$  is the metric projection of  $H$  onto  $C$ . They proved that under some control conditions the proposed sequences  $\{x_k\}$ ,  $\{y_k\}$ , and  $\{t_k\}$  defined by (1.6) converge strongly to a common element of solution set of monotone, Lipschitz-type continuous equilibrium problems and the set of fixed points of nonexpansive mappings which is a unique solution of some variational inequalities, (For more related result, see [15]).

In general, it is hard to find the constants  $c_1$  and  $c_2$  satisfying the assumed Lipschitz-type condition (d). Furthermore solving the strongly convex programs (1.6) is expensive excepts special cases when  $C$  has a simple structure. To avoid the advantages, Anh and Muu [2] introduced a new algorithm for solving problem  $F(T) \cap \operatorname{Sol}(C, f)$ . More precisely, by using the concepts of  $\varepsilon$ -subdifferential, they introduced a new algorithm for solving the problem of finding a common element of the solution set of the equilibrium problem and the set of fixed point problem, which is a combination of the well-known Mann's iterative scheme for fixed point and the projection method for equilibrium problems. Furthermore, the proposed algorithm uses only one projection and does not require any Lipschitz condition for the bifunctions.

In this paper, motivated by the idea in Marino and Xu [13], Anh and Muu [2], and Wangkeeree and Preechasilp [14], we propose a new general iterative scheme using the concepts of  $\varepsilon$ -subdifferential for approximating the common element in  $F(T) \cap \operatorname{Sol}(C, f)$  which is a solution of a certain optimization problem related to a strongly positive linear operator. Under suitable control conditions, we prove the strong convergence theorems of such iterative scheme in a real Hilbert space. The main result extends various results existing in the current literature.

## 2 Preliminaries

First of all, we introduce the concepts of  $\varepsilon$ -subdifferential. Notice that  $\partial\varphi(x)$  may turn out to be empty, even though  $x \in \operatorname{dom}\varphi$ . To overcome this aspect of subdifferentials, the concept of the  $\varepsilon$ -subdifferential came into existence; it not only overcomes the drawback of subdifferentials but is also important from the optimization point of view. The idea can be found in the work of Brønsted and Rockafellar [16] but the theory of  $\varepsilon$ -subdifferential calculus was given by Hiriart-Urruty [17].

**Definition 2.1.** [18] *Let  $\varphi : H \rightarrow \overline{\mathbb{R}}$  be a proper convex function. For  $\varepsilon > 0$ , the  $\varepsilon$ -subdifferential of  $\varphi$  at  $\bar{x} \in \operatorname{dom}\varphi$  is given by*

$$\partial_\varepsilon\varphi(\bar{x}) = \{\xi \in H : \varphi(x) - \varphi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \forall x \in H\}.$$

**Proposition 2.2.** [18] *Let  $\varphi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  be a proper lower semicontinuous convex function and let  $\varepsilon > 0$  be given. Then for every  $\bar{x} \in \text{dom } \varphi$ , the  $\varepsilon$ -subdifferential( $\partial_\varepsilon \varphi(\bar{x})$ ) is a nonempty closed convex set and*

$$\partial \varphi(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon \varphi(\bar{x}).$$

For  $\varepsilon_1 \geq \varepsilon_2$ ,  $\partial_{\varepsilon_2}(\bar{x}) \subseteq \partial_{\varepsilon_1}(\bar{x})$ .

Now, we recall the following technical lemmas which will be used in the sequel.

**Lemma 2.3.** [11] *Let  $\{a_k\}$  be a sequence of non-negative real number satisfying the property*

$$a_{k+1} \leq (1 - \delta_k)a_k + \delta_k \sigma_k + \gamma_k, \quad k \geq 0,$$

where  $\{\gamma_k\} \subseteq (0, 1)$  and  $\{b_k\} \subseteq \mathbb{R}$  such that

- (i)  $\sum_{k=0}^{\infty} \alpha_k = \infty$ ;
- (ii)  $\limsup_{k \rightarrow \infty} \sigma_k \leq 0$ , and  $\sum_{k=0}^{\infty} \gamma_k < \infty$ .

Then  $\{a_k\}$  converges to zero, as  $k \rightarrow \infty$ .

The metric (nearest point) projection  $P_C$  from a Hilbert space  $H$  to a closed convex subset  $C$  of  $H$  is defined as follows: given  $x \in H$ ,  $P_C x$  is the only point in  $C$  such that  $\|x - P_C x\| = \inf \{\|x - y\| : y \in C\}$ . In what follows lemma can be found in any standard functional analysis book.

**Lemma 2.4.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Given  $x \in H$  and  $y \in C$ , then*

- (i)  $y = P_C x$  if and only if  $\langle x - y, y - z \rangle \geq 0$  for all  $z \in C$ ,
- (ii)  $P_C$  is nonexpansive,
- (iii)  $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$  for all  $x, y \in H$ ,
- (iv)  $\langle x - P_C x, P_C x - y \rangle$  for all  $x \in H$  and  $y \in C$ .

**Lemma 2.5.** [19] *Let  $C$  be a closed convex subset of a Hilbert space  $H$  and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T) \neq \emptyset$ . If a sequence  $\{x_k\}$  in  $C$  such that  $x_k \rightarrow \tilde{x}$  and  $x_k - Tx_k \rightarrow 0$ , then  $T\tilde{x} = \tilde{x}$ .*

**Lemma 2.6.** [13] *Assume that  $A$  is a strongly positive linear bounded operator on a Hilbert space  $H$  with coefficient  $\bar{\gamma} > 0$  and  $0 < \rho \leq \|A\|^{-1}$ , then  $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$ .*

### 3 Main Results

We are in a position to state and investigate the new general iterative method for finding a common solution of the set of equilibrium problems and the set of fixed points of nonexpansive mappings which is a solution of a certain optimization problem related to a strongly positive linear operator.

**Assumption 3.1.** The bifunction  $f : C \times C \rightarrow \mathbb{R}$  satisfies the following conditions:

- (E<sub>1</sub>) For each  $x$ ,  $f(x, x) = 0$  and  $f(x, \cdot)$  is lower semicontinuous convex on  $C$ ;  
 (E<sub>2</sub>) If  $\{x^k\} \subseteq C$  is bounded and  $\varepsilon_k \downarrow 0$  as  $k \rightarrow \infty$ , then the sequence  $\{w^k\}$  with  $w^k \in \partial_{\varepsilon_k} f(x^k, \cdot)(x^k)$  is bounded, where  $\partial_{\varepsilon} f(x, \cdot)(x)$  stands for  $\varepsilon$ -subdifferential of the convex function  $f(x, \cdot)$  at  $x$ ;  
 (E<sub>3</sub>)  $f$  is pseudomonotone on  $C$  with respect to every solution of  $EP(C, f)$ ;  
 (E<sub>4</sub>) For each  $x \in C$ ,  $f(\cdot, x)$  is weakly upper semicontinuous on  $C$ ;  
 (E<sub>5</sub>) The solution set  $\Omega$  of Problem  $EP(C, f) \cap F(T)$  is nonempty.

**Assumption 3.2. Initialization:** Now suppose that the sequences  $\{\lambda_k\}$ ,  $\{\beta_k\}$ ,  $\{\varepsilon_k\}$  and  $\{\delta_k\}$  of nonnegative numbers satisfy the following conditions

- (i)  $0 < \lambda < \lambda_k$ ;  
 (ii)  $\sum_{k=1}^{\infty} \delta_k = \infty$ ,  $\sum_{k=1}^{\infty} |\delta_{k+1} - \delta_k| < \infty$  and  $\lim_{k \rightarrow \infty} \delta_k = 0$ ;  
 (iii)  $\beta_k > 0$ ,  $\sum_{k=0}^{\infty} \beta_k < +\infty$ ;  
 (iv)  $\lim_{k \rightarrow \infty} \frac{\beta_k}{\delta_k} = 0$  and  $\lim_{k \rightarrow \infty} \alpha_k \varepsilon_k = 0$ .

**Algorithm 3.3.** Let  $g : C \rightarrow C$  be a contraction with coefficient  $\alpha$  and  $A$  a strongly positive linear bounded operator on  $H$  with coefficient  $\bar{\gamma} > 0$  such that  $0 < \gamma < \bar{\gamma}/\alpha$ . Let  $\{\delta_k\}$  be a real sequence such that  $0 < \delta_k < \min\left\{1, \frac{1}{\bar{\gamma} - \gamma\alpha}\right\}$ . Now the general iterative scheme for finding a common point in the set of solutions of Problem  $EP(C, f)$  and the set of fixed points of the nonexpansive mapping  $T$  can be generated as follows:

$$\left\{ \begin{array}{l} \text{choose } x_0 \in C; \\ \text{compute } w_k \in \partial_{\varepsilon_k} f(x_k, \cdot)(x_k); \\ \text{take } \gamma_k := \max\{\lambda_k, \|w_k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ \text{compute } y_k = P_C(x_k - \alpha_k w_k) \text{ and;} \\ \text{let } x_{k+1} = P_C(\delta_k \gamma g(x_k) + (I - \delta_k A) T y_k), k = 0, 1, \dots \end{array} \right. \quad (3.1)$$

**Remark 3.4.** [2]

1. If  $f$  is pseudomonotone on  $C$  with respect to the solution set  $Sol(C, f)$  of Problem  $EP(C, f)$ , then under Assumptions  $E_1$  and  $E_4$ , the set  $Sol(C, f)$  is closed and convex.
2. Assumption  $E_3$  holds true if  $f$  is pseudomonotone on  $C$  and satisfies the paramonotonicity property :

$$x \in Sol(C, f), y \in C, f(x, y) = f(y, x) = 0 \Rightarrow y \in Sol(C, f).$$

3. Assumption  $E_2$  holds true whenever  $E_1$  is satisfied and the function  $f : \Delta \times \Delta \rightarrow \mathbb{R}$  is continuous on  $\Delta \times \Delta$ .

**Remark 3.5.** .

1. For an example of control sequences, let  $\delta_k = \frac{1}{k}$ ,  $\beta_k = \frac{1}{k^2}$  and  $\varepsilon_k = \gamma_k$  for all  $k \in \mathbb{N}$ .
2. Since  $f(x, \cdot)$  is a lower semicontinuous convex function and  $C \subseteq \text{dom}f(x, \cdot)$  for every  $x \in C$ , by Proposition 2.2, the  $\varepsilon_k$ -diagonal subdifferential  $\partial_{\varepsilon_k} f(x_k, \cdot)(x_k) \neq \emptyset$  for every  $\varepsilon_k > 0$ . Thus the Algorithm (3.3) is well defined.

**Theorem 3.6.** Suppose that Assumptions 3.1-3.3 are satisfied. Further, assume that  $\|A\| = 1$ . Then the sequences  $\{x_k\}$  and  $\{y_k\}$  strongly converge to the same point  $q \in \Omega := \text{Sol}(C, f) \cap F(T)$ , where  $q = P_\Omega(\gamma g + (I - A))(q)$  which is a unique solution of the variational inequality

$$\langle (\gamma g - A)q, x - q \rangle \leq 0, \quad \forall x \in \Omega. \quad (3.2)$$

*Proof.* We first prove that  $P_\Omega(\gamma g + (I - A))$  is a contraction mapping with a coefficient  $(1 - (\bar{\gamma} - \gamma\alpha))$ . To this end, applying  $\|A\| = 1$  to Lemma 2.6, we can calculate the following, for any  $x, y \in H$ ,

$$\begin{aligned} \|P_\Omega(\gamma g + (I - A))x - P_\Omega(\gamma g + (I - A))y\| &\leq \|\gamma g + (I - A)x - (\gamma g + (I - A))y\| \\ &\leq \gamma \|g(x) - g(y)\| + \|I - A\| \|x - y\| \\ &\leq \gamma\alpha \|x - y\| + (1 - \bar{\gamma}) \|x - y\| \\ &= (1 - (\bar{\gamma} - \gamma\alpha)) \|x - y\|. \end{aligned}$$

By Banach's contraction principle guarantees that  $P_\Omega(\gamma g + (I - A))$  has a unique fixed point, say  $q \in H$ . That is,  $q = P_\Omega(\gamma g + (I - A))(q)$ . By Lemma 2.4 (i), we obtain that

$$\langle (\gamma g - A)q, x - q \rangle \leq 0, \quad \forall x \in \Omega. \quad (3.3)$$

Next, we show that  $\{x_k\}$  is bounded. Since, for all  $k \geq 0$ ,  $y_k = P_C(x_k - \alpha_k w_k)$  and  $x_k \in C$ , it follows from metric projection property that

$$\begin{aligned} \|x_k - y_k\|^2 &\leq \alpha_k \langle w_k, x_k - y_k \rangle \\ &\leq \alpha_k \|w_k\| \|x_k - y_k\| \\ &= \frac{\beta_k}{\max\{\lambda_k, \|w_k\|\}} \|w_k\| \|x_k - y_k\| \\ &\leq \beta_k \|x_k - y_k\|, \end{aligned} \quad (3.4)$$

which gives that

$$\|x_k - y_k\| \leq \beta_k, \quad \text{for all } k \geq 0. \quad (3.5)$$

From  $\lim_{k \rightarrow \infty} \beta_k = 0$ , one has  $\|x_k - y_k\| \rightarrow 0$  as  $k \rightarrow \infty$ . Further, for every  $p \in \Omega$ , we have

$$\begin{aligned} \|y_k - p\| &\leq \|x_k - p\| + \|y_k - x_k\| \\ &\leq \|x_k - p\| + \beta_k. \end{aligned} \quad (3.6)$$

Since  $\lim_{k \rightarrow \infty} \frac{\beta_k}{\delta_k} = 0 = \lim_{k \rightarrow \infty} \beta_k$ , we have that  $\lim_{k \rightarrow \infty} \frac{(1 - \delta_k \bar{\gamma})\beta_k}{\delta_k} = 0$ . Then, for any fixed  $\tau > 0$  there exists a number  $N > 0$  such that  $\frac{(1 - \delta_k \bar{\gamma})\beta_k}{\delta_k} < \tau$  for all  $k \geq N$ , one arrives that

$$(1 - \delta_k \bar{\gamma})\beta_k < \delta_k \tau, \quad \text{for all } k \geq N.$$

This together with Lemma 2.6 implies that

$$\begin{aligned} \|x_{k+1} - p\| &= \|P_C(\delta_k \gamma g(x_k) + (I - \alpha_k A)T(y_k)) - p\| \\ &\leq \|\delta_k \gamma g(x_k) + (I - \alpha_k A)T(y_k) - p\| \\ &\leq \delta_k \|\gamma g(x_k) - Ap\| + \|I - \delta_k A\| \|T(y_k) - T(p)\| \\ &\leq \delta_k \|\gamma g(x_k) - Ap\| + \|I - \delta_k A\| \|y_k - p\| \\ &\leq \delta_k \|\gamma g(x_k) - Ap\| + (1 - \delta_k \bar{\gamma})(\|x_k - p\| + \beta_k) \\ &\leq \delta_k \gamma \alpha \|x_k - p\| + \delta_k \|\gamma g(p) - Ap\| + (1 - \delta_k \bar{\gamma})(\|x_k - p\| + \beta_k), \\ &\leq \delta_k \gamma \alpha \|x_k - p\| + \delta_k \|\gamma g(p) - Ap\| + (1 - \delta_k \bar{\gamma})\|x_k - p\| + (1 - \delta_k \bar{\gamma})\beta_k, \\ &\leq \delta_k \gamma \alpha \|x_k - p\| + \delta_k \|\gamma g(p) - Ap\| + (1 - \delta_k \bar{\gamma})\|x_k - p\| + \delta_k \tau, \\ &\leq (1 - \delta_k(\bar{\gamma} - \gamma \alpha))\|x_k - p\| + \delta_k(\|\gamma g(p) - Ap\| + \tau). \end{aligned}$$

By induction, we get that

$$\|x_k - p\| \leq \max \left\{ \|x_N - p\|, \frac{(\|\gamma g(p) - Ap\| + \tau)}{\bar{\gamma} - \gamma \alpha} \right\}, \quad k \geq N.$$

Hence  $\{x_k\}$  is bounded, and so are  $\{y_k\}$ ,  $\{g(x_{k-1})\}$  and  $\{AT(y_{k-1})\}$ . Now, we prove that

$$\lim_{n \rightarrow \infty} \|x_{k+1} - x_k\| = 0. \quad (3.7)$$



Using (3.13), we can calculate the following, for all  $k \geq 0$ ,

$$\begin{aligned}
& \|x_{k+1} - x_k\| \\
&= \|P_C(\delta_k \gamma g(x_k) + (I - \alpha_k A)T(y_k)) - P_C(\delta_{k-1} \gamma g(x_{k-1}) + (I - \alpha_{k-1} A)T(y_{k-1}))\| \\
&\leq \|\delta_k \gamma g(x_k) + (I - \delta_k A)T(y_k) - \alpha_{k-1} \gamma g(x_{k-1}) - (I - \delta_{k-1} A)T(y_{k-1})\| \\
&= \|\delta_k \gamma g(x_k) - \delta_k \gamma g(x_{k-1}) + \delta_k \gamma g(x_{k-1}) - \delta_{k-1} \gamma g(x_{k-1}) + (I - \delta_k A)T(y_k) \\
&\quad - (I - \delta_k A)T(y_{k-1}) + (I - \delta_k A)T(y_{k-1}) - (I - \alpha_{k-1} A)T(y_{k-1})\| \\
&\leq \delta_k \gamma \alpha \|x_k - x_{k-1}\| + |\delta_k - \delta_{k-1}| \|\gamma g(x_{k-1})\| \\
&\quad + \|I - \delta_k A\| \|T(y_k) - T(y_{k-1})\| + |\delta_k - \delta_{k-1}| \|AT(y_{k-1})\| \\
&\leq \delta_k \gamma \alpha \|x_k - x_{k-1}\| + |\delta_k - \delta_{k-1}| \|\gamma g(x_{k-1})\| \\
&\quad + \|I - \delta_k A\| (\|y_k - x_k\| + \|x_k - x_{k-1}\| + \|x_{k-1} - y_{k-1}\|) + |\delta_k - \delta_{k-1}| \|AS(y_{k-1})\| \\
&\leq \delta_k \gamma \alpha \|x_k - x_{k-1}\| + |\delta_k - \delta_{k-1}| \|\gamma g(x_{k-1})\| \\
&\quad + (1 - \delta_k \bar{\gamma}) (\|y_k - x_k\| + \|x_k - x_{k-1}\| + \|x_{k-1} - y_{k-1}\|) + |\delta_k - \delta_{k-1}| \|AT(y_{k-1})\| \\
&\leq \delta_k \gamma \alpha \|x_k - x_{k-1}\| + |\delta_k - \delta_{k-1}| \|\gamma g(x_{k-1})\| + |\delta_k - \delta_{k-1}| \|AT(y_{k-1})\| \\
&\quad + (1 - \delta_k \bar{\gamma}) \|y_k - x_k\| + (1 - \delta_k \bar{\gamma}) \|x_k - x_{k-1}\| + (1 - \delta_k \bar{\gamma}) \|x_{k-1} - y_{k-1}\| \\
&\leq (\delta_k \gamma \alpha + (1 - \delta_k \bar{\gamma})) \|x_k - x_{k-1}\| + |\delta_k - \delta_{k-1}| M + (1 - \delta_k \bar{\gamma}) \beta_k + (1 - \delta_k \bar{\gamma}) \beta_{k-1} \\
&\leq ((1 - \delta_k (\bar{\gamma} - \gamma \alpha)) \|x_k - x_{k-1}\| + |\delta_k - \delta_{k-1}| M + (1 - \delta_k \bar{\gamma}) (\beta_k + \beta_{k-1})),
\end{aligned}$$

where  $M$  is a constant such that

$$M \geq \sup_{k \geq 0} \{\gamma \|g(x_{k-1})\| + \|AT(y_{k-1})\|\}.$$

Since  $\sum_{k=1}^{\infty} |\delta_{k+1} - \delta_k| < \infty$  and  $\sum_{k=0}^{\infty} \beta_k < +\infty$ , by applying Lemma 2.3, we have  $\lim_{n \rightarrow \infty} \|x_{k+1} - x_k\| = 0$ . Next, we show that

$$\lim_{n \rightarrow \infty} \|x_k - Tx_k\| = 0. \quad (3.8)$$

From definition of  $x_k$ , we have

$$\|x_{k+1} - Ty_k\| = \delta_k \|\gamma f(x_k) - ATy_k\|.$$

Since  $\lim_{k \rightarrow \infty} \delta_k = 0$ , we have  $\lim_{k \rightarrow \infty} \|x_{k+1} - Ty_k\| = 0$ . This together with (3.7) and (3.8) implies that

$$\begin{aligned}
\|Tx_k - x_k\| &\leq \|Tx_k - Ty_k\| + \|Ty_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\
&\leq \|x_k - y_k\| + \|Ty_k - x_{k+1}\| + \|x_{k+1} - x_k\| \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

Now, we claim that, for any  $x \in \Omega$ ,

$$\limsup_{k \rightarrow \infty} \langle (\gamma g - A)q, Ty_k - q \rangle \leq 0. \quad (3.9)$$

Indeed, We may assume without loss of generality that there exists a subsequence  $\{x_{k_i}\}$  of  $\{x_k\}$  such that

$$\limsup_{k \rightarrow \infty} \langle (\gamma g - A)q, Ty_k - q \rangle = \lim_{i \rightarrow \infty} \langle (\gamma g - A)q, Ty_{k_i} - q \rangle$$

and  $x_{k_i} \rightharpoonup x^*$  as  $i \rightarrow \infty$  for some  $x^* \in C$ . Since  $\lim_{k \rightarrow \infty} \|x_k - Tx_k\| = 0$ , by demiclosedness of a mapping  $T$ , we get that  $x^* \in F(T)$ . Next, we show that  $x^* \in \text{Sol}(C, f)$ . Since  $y_k = P_C(x_k - \alpha_k w_k)$  and  $w^k \in \partial_{\varepsilon_k} f(x^k, \cdot)(x^k)$ ,  $x^* \in C$ , we can have

$$\begin{aligned} \langle x_k - y_k, y - y_k \rangle &\leq \alpha_k \langle w_k, x_k - y_k \rangle \leq \alpha_k (f(x_k, y) - f(x_k, x_k) + \varepsilon_k) \\ &\leq \alpha_k f(x_k, y) + \alpha_k \varepsilon_k \end{aligned}$$

since and  $f(x, x) = 0$  for all  $x \in C$ . Since limit of  $\beta_k$  exists, then there exists  $\beta > 0$  such that  $\beta_k < \beta$  for all  $k$ . Since  $0 < \lambda < \lambda_k$  for all  $k$  and  $\gamma_k = \max\{\lambda_k, \|w_k\|\}$ , we get that  $\frac{1}{\gamma_k} < \frac{1}{\lambda}$ . Hence  $\alpha_k = \frac{\beta_k}{\gamma_k} < \frac{\beta}{\lambda}$ , and so

$$\langle x_k - y_k, y - y_k \rangle \leq \alpha_k f(x_k, y) + \alpha_k \varepsilon_k \leq \frac{\beta}{\lambda} f(x_k, y) + \alpha_k \varepsilon_k, \text{ for all } k \geq 0.$$

Taking  $k := k_i$  in the last inequality, we arrive that

$$\langle x_{k_i} - y_{k_i}, y - y_{k_i} \rangle \leq \alpha_{k_i} f(x_{k_i}, y) + \alpha_{k_i} \varepsilon_{k_i} \leq \frac{\beta}{\lambda} f(x_{k_i}, y) + \alpha_{k_i} \varepsilon_{k_i}.$$

Since  $f(\cdot, y)$  is weakly upper semicontinuous on  $C$ ,  $\lim_{k \rightarrow \infty} \alpha_k \varepsilon_k = 0$  and  $\lim_{k \rightarrow \infty} \|x_k - y_k\| = 0$ , we have that

$$\begin{aligned} \frac{\beta}{\lambda} f(x^*, y) &\geq \limsup_{i \rightarrow \infty} \frac{\beta}{\lambda} f(x_{k_i}, y) + \limsup_{i \rightarrow \infty} \alpha_{k_i} \varepsilon_{k_i} \\ &\geq \limsup_{k_i \rightarrow \infty} \left( \frac{\beta}{\lambda} f(x_{k_i}, y) + \alpha_{k_i} \varepsilon_{k_i} \right) \\ &\geq \limsup_{k_i \rightarrow \infty} \langle x_{k_i} - y_{k_i}, y - y_{k_i} \rangle = 0, \end{aligned}$$

which gives that

$$f(x^*, y) \geq 0, \text{ for all } y \in C,$$

and so  $x^* \in \text{Sol}(C, f)$ . Thus,  $x^* \in \Omega$ . This also implies that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \langle (\gamma g - A)q, Ty_{k_i} - q \rangle &= \lim_{i \rightarrow \infty} \langle (\gamma g - A)q, Ty_{k_i} - q \rangle \\ &= \langle (\gamma g - A)q, x^* - q \rangle \leq 0. \end{aligned}$$

Finally, we show that  $x_k \rightarrow q$  as  $k \rightarrow \infty$ . We observe that

$$\begin{aligned}
\|x_{k+1} - q\|^2 &= \|P_C(\delta_k \gamma g(x_k) + (I - \delta_k A)Ty_k) - P_C(q)\|^2 \\
&\leq \|\delta_k \gamma g(x_k) + (I - \delta_k A)Ty_k - q\|^2 \\
&\leq \delta_k^2 \|\gamma g(x_k) - Aq\| + \|(I - \delta_k A)(Ty_k - q)\|^2 \\
&\quad + 2\delta_k \langle (I - \delta_k A)(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq \delta_k^2 \|\gamma g(x_k) - Aq\| + (1 - \delta_k \bar{\gamma})^2 \|Ty_k - q\|^2 \\
&\quad + 2\delta_k \langle Ty_k - q, \gamma g(x_k) - Aq \rangle - 2\delta_k^2 \langle A(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq (1 - \delta_k \bar{\gamma})^2 \|Ty_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\delta_k \langle Ty_k - q, \gamma g(x_k) - \gamma g(q) \rangle + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad - 2\delta_k^2 \langle A(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq (1 - \delta_k \bar{\gamma})^2 \|y_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\delta_k \|y_k - q\| \|\gamma g(x_k) - \gamma g(q)\| + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad - 2\delta_k^2 \langle A(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq (1 - \delta_k \bar{\gamma})^2 \|x_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\gamma \alpha \delta_k \|y_k - q\| \|x_k - q\| + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad - 2\delta_k^2 \langle A(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq (1 - \delta_k \bar{\gamma})^2 \|x_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\gamma \alpha \delta_k (\|x_k - q\| + \beta_k) \|x_k - q\| + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad - 2\delta_k^2 \langle A(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq (1 - \delta_k \bar{\gamma})^2 \|x_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\gamma \alpha \delta_k \|x_k - q\|^2 + 2\gamma \alpha \delta_k \beta_k + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad - 2\delta_k^2 \langle A(Ty_k - q), \gamma g(x_k) - Aq \rangle \\
&\leq (1 - \delta_k \bar{\gamma})^2 \|x_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\gamma \alpha \delta_k \|x_k - q\|^2 + 2\gamma \alpha \delta_k \beta_k + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad + 2\delta_k^2 \|A(Ty_k - q)\| \|\gamma g(x_k) - Aq\| \\
&\leq ((1 - \delta_k \bar{\gamma})^2 + 2\gamma \alpha \delta_k) \|x_k - q\|^2 + \delta_k^2 \|\gamma g(x_k) - Aq\| \\
&\quad + 2\gamma \alpha \delta_k \beta_k + 2\delta_k \langle Ty_k - q, \gamma g(q) - Aq \rangle \\
&\quad + 2\delta_k^2 \|A(Ty_k - q)\| \|\gamma g(x_k) - Aq\| \\
&= (1 - 2\delta_k(\bar{\gamma} - \gamma \alpha)) \|x_k - q\|^2 \\
&\quad + \delta_k \left( \delta_k \|\gamma g(x_k) - Aq\| + 2\gamma \alpha \beta_k + 2\langle Ty_k - q, \gamma g(q) - Aq \rangle \right. \\
&\quad \left. + 2\delta_k \|A(Ty_k - q)\| \|\gamma g(x_k) - Aq\| + \delta_k \bar{\gamma}^2 \|x_k - q\|^2 \right) \\
&= (1 - 2\delta_k(\bar{\gamma} - \gamma \alpha)) \|x_k - q\|^2 \\
&\quad + \delta_k \left( 2\langle Ty_k - q, \gamma g(q) - Aq \rangle + \delta_k \|\gamma g(x_k) - Aq\|^2 \right. \\
&\quad \left. + 2\delta_k \|A(Ty_k - q)\| \|Aq - \gamma g(x_k)\| + \delta_k \bar{\gamma}^2 \|x_k - q\|^2 \right). \quad (3.10)
\end{aligned}$$

Since  $\{x_k\}$ ,  $\{g(x_k)\}$  and  $\|Tw_k\|$  are bounded, we can take a constant  $M > 0$  such that

$$M \geq \sup_{k \geq 0} \{ \|\gamma g(x_k) - Aq\|^2 + 2\|A(Tw_k - q)\| \|Aq - \gamma g(x_k)\| + \bar{\gamma}^2 \|x_k - q\|^2 \}.$$

This implies that

$$\|x_{k+1} - q\|^2 \leq (1 - 2(\bar{\gamma} - \gamma\alpha)\alpha_k) \|x_k - q\|^2 + \alpha_k \sigma_k, \quad (3.11)$$

where  $\sigma_k = 2\langle Ty_k - q, \gamma g(q) - Aq \rangle + M\alpha_k$ . From (3.9), we have  $\limsup_{n \rightarrow \infty} \sigma_k \leq 0$ . Applying Lemma 2.3 to (3.11), we obtain that  $x_k \rightarrow q$  as  $k \rightarrow \infty$ . This completes the proof.  $\square$

**Remark 3.7.** In contrast to results in [14], the function  $f$  as in Theorem 3.6 did not need to be Lipschitz-type continuous on  $C$ .

**Remark 3.8.** In general case, if  $g : C \rightarrow C$  is a contraction with coefficient  $\alpha$  and  $A$  is any strongly positive bounded linear operator with coefficient  $\bar{\gamma}$  and  $0 < \gamma < \bar{\gamma}/\alpha$ . We define a new bounded linear operator  $\bar{A}$  on  $H$  by

$$\bar{A} = \|A\|^{-1}A.$$

It is easy to see that  $\bar{A}$  is a strongly positive with coefficient  $\|A\|^{-1}\bar{\gamma} > 0$  such that  $\|\bar{A}\| = 1$  and

$$0 < \|A\|^{-1}\gamma < \|A\|^{-1}\bar{\gamma}/\alpha.$$

Let the bifunction  $f : C \times C \rightarrow \mathbb{R}$  be satisfied all conditions in Assumption 3.1. Now suppose that the sequences  $\{\lambda_k\}$ ,  $\{\beta_k\}$ ,  $\{\varepsilon_k\}$  and  $\{\delta_k\}$  of nonnegative numbers satisfy all conditions in Assumption 3.2. Let  $\{\delta_k\}$  be a real sequence such that  $0 < \delta_k < \min \left\{ 1, \frac{1}{\|A\|^{-1}(\bar{\gamma} - \gamma\alpha)} \right\}$  and  $\lim_{k \rightarrow \infty} \delta_k = 0$ . Now the new general iterative scheme for finding a common point in the set of solutions of Problem  $EP(C, f)$  and the set of fixed points of the nonexpansive mapping  $T$  can be generated as follows:

$$\left\{ \begin{array}{l} \text{choose } x_0 \in C; \\ \text{compute } w_k \in \partial_{\varepsilon_k} f(x_k, \cdot)(x_k); \\ \text{take } \gamma_k := \max\{\lambda_k, \|w_k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ \text{compute } y_k = P_C(x_k - \alpha_k w_k) \text{ and;} \\ \text{let } x_{k+1} = P_C(\delta_k \|A\|^{-1} \gamma g(x_k) + (I - \delta_k \bar{A}) T y_k), k = 0, 1, \dots \end{array} \right. \quad (3.12)$$

From Theorem 3.6, we have that  $\{x_k\}$  converges strongly, as  $k \rightarrow \infty$ , to a point  $q$  satisfying

$$q = P_{\Omega} (I - \|A\|^{-1}(A - \gamma g)) q,$$

which is a unique solution of the variational inequality:

$$\|A\|^{-1} \langle (A - \gamma g)q, x - q \rangle \geq 0, x \in C;$$

which gives that

$$\langle (A - \gamma g)q, x - q \rangle \geq 0, x \in C.$$

Using Theorem 3.6, we obtain the following two strong convergence theorems of new general iterative approximation methods for a nonexpansive mapping.

**Theorem 3.9.** *Suppose that Assumptions 3.1-3.2 are satisfied. Let  $g : C \rightarrow C$  be a contraction with coefficient  $\alpha$  and  $A$  a strongly positive linear bounded operator on  $H$  with coefficient  $\bar{\gamma} > 0$  such that  $0 < \gamma < \bar{\gamma}/\alpha$ . Let  $\{\delta_k\}$  be a real sequence such that  $0 < \delta_k < \min\left\{1, \frac{1}{\bar{\gamma}-\gamma\alpha}\right\}$  and  $\lim_{k \rightarrow \infty} \delta_k = 0$ . Let the sequences  $\{x'_k\}$  and  $\{y'_k\}$  be generated by*

$$\left\{ \begin{array}{l} \text{choose } x'_0 \in C; \\ \text{compute } w'_k \in \partial_{\varepsilon_k} f(x'_k, \cdot)(x'_k); \\ \text{take } \gamma'_k := \max\{\lambda_k, \|w'_k\|\} \text{ and } \alpha'_k := \frac{\beta_k}{\gamma'_k}; \\ \text{compute } y'_k = P_C(x'_k - \alpha'_k w'_k) \text{ and;} \\ \text{let } x'_{k+1} = P_C(\delta_k \gamma g(Tx'_k) + (I - \delta_k A)Ty'_k), k = 0, 1, \dots \end{array} \right. \quad (3.13)$$

Then the sequences  $\{x'_k\}$  and  $\{y'_k\}$  strongly converge to the same point  $q$  that obtained in Theorem 3.6.

*Proof.* Define the sequence  $\{x_k\}$  by

$$\left\{ \begin{array}{l} \text{choose } x_0 = x'_0 \in C; \\ \text{compute } w_k \in \partial_{\varepsilon_k} f(x_k, \cdot)(x_k); \\ \text{take } \gamma_k := \max\{\lambda_k, \|w_k\|\} \text{ and } \alpha_k := \frac{\beta_k}{\gamma_k}; \\ \text{compute } y_k = P_C(x_k - \alpha_k w_k) \text{ and;} \\ \text{let } x_{k+1} = P_C(\delta_k \gamma g(x_k) + (I - \delta_k A)Ty_k), k = 0, 1, \dots \end{array} \right. \quad (3.14)$$

Applying Theorem 3.6, we can conclude that both sequences  $\{x_k\}$  and  $\{y_k\}$  strongly converge to the same point  $q \in \Omega := \text{Sol}(C, f) \cap F(T)$ , where  $q = P_\Omega(\gamma g + (I - A))(q)$ . Next, we prove that  $x'_k \rightarrow q$  and  $y'_k \rightarrow q$  as  $k \rightarrow \infty$ . By using the same proof as in (3.6), for each  $k \geq 0$ , we have

$$\begin{aligned} \|y'_k - y_k\| &\leq \|y'_k - x'_k\| + \|x'_k - x_k\| + \|x_k - y_k\| \\ &\leq \beta_k + \|x'_k - x_k\| + \beta_k \\ &\leq \|x'_k - x_k\| + 2\beta_k. \end{aligned}$$

It then follows that, for each  $k \geq 1$

$$\begin{aligned} \|x'_{k+1} - x_{k+1}\| &\leq \|\delta_k \gamma g(Tx'_k) + (I - \delta_k A)Ty'_k - \delta_k \gamma g(x_k) - (I - \delta_k A)Ty_k\| \\ &\leq \delta_k \gamma \|g(Tx'_k) - g(x_k)\| + (1 - \delta_k \bar{\gamma}) \|Ty'_k - Ty_k\| \\ &\leq \delta_k \gamma \alpha \|Tx'_k - x_k\| + (1 - \delta_k \bar{\gamma})(\|x'_k - x_k\| + \beta_k + \beta'_k) \\ &\leq \delta_k \gamma \alpha (\|Tx'_k - Tq\| + \|Tq - Tx_k\|) + (1 - \delta_k \bar{\gamma})(\|x'_k - x_k\| + \beta_k + \beta'_k) \\ &\leq \delta_k \gamma \alpha \|x'_k - q\| + \delta_k \gamma \alpha \|q - x_k\| + (1 - \delta_k \bar{\gamma})(\|x'_k - x_k\| + \beta_k + \beta'_k) \\ &\leq \delta_k \gamma \alpha \|x'_k - x_k\| + \delta_k \gamma \alpha \|x_k - q\| + \delta_k \gamma \alpha \|q - x_k\| \\ &\quad + (1 - \delta_k \bar{\gamma})(\|x'_k - x_k\| + \beta_k + \beta'_k) \\ &\leq (1 - \delta_k(\bar{\gamma} - \gamma\alpha))\|x'_k - x_k\| + 2\delta_k \gamma \alpha \|x_k - q\| + 2(1 - \delta_k \bar{\gamma})\beta_k. \end{aligned}$$

Since  $x_k \rightarrow q$ ,  $\sum_{k=0}^{\infty} \beta_k = \infty$ , by using Lemma 2.3, we obtain that  $\|x'_k - x_k\| \rightarrow 0$  as  $k \rightarrow \infty$ . Consequently,  $x'_n \rightarrow q$  as required.  $\square$

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