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On Characterizations of General Helices for Ruled Surfaces in the Pseudo-Galilean Space G_3^1-(PART II)

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Abstract : In this paper, we obtained characterizations of a curve with respect to the Frenet frame of Ruled surfaces in the 3-dimensional Pseudo-Galilean space G_3^1.

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1 Introduction

T. Ikawa obtained in [6] the following characteristic ordinary differential equation

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \qquad K = k^2 - \tau^2$$

for the circular helix which corresponds to the case that the curvatures k and τ of a time-like curve α on the Lorentzian manifold M are constant.

N. Ekmekçi and H. H. Hacisalihoglu generalized in [4]. T. Ikawa's this result, i.e. k and τ are variable, but $\frac{k}{\tau}$ is constant.

Recently, N. Ekmekçi and K. İlarslan obtained characterizations of timelike null helices in terms of principalnormal or binormal vector fields [5].

Furthermore, M. Bektas [1] obtained characterizations of a curve with respect to the Frenet frame of ruled surfaces in the 3-dimensional pseudo-Galilean space G_3^1.

In this paper, we obtained characterizations of helices in terms of principal normal vector fields and another two characterizations for a curve with respect to the Frenet frame of Ruled surfaces in the 3-dimensional Pseudo-Galilean space G_3^1.

2 Preliminaries

We will use the same notations and terminologies as in [3] unless otherwise stated. The pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature (0,0,+,-)). The absolute of the pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is a line in w and I is the fixed hyperbolic involution of points of f [2].

A vector $X(x, y, z)$ is said to *non isotropic* if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: space-like $(y^2 - z^2 > 0)$, time-like $(y^2 - z^2 < 0)$ and two types of lightlike $(y = \pm z)$ vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$.

A trihedron $(T_o;\, e_1, e_2, e_3)$ with a proper origin

$$T_o(x_o, y_o, z_o) \sim (1 : x_o : y_o : z_o),$$

is orthonormal in pseudo-Galilean sense iff the vectors e_1 ,e_2, e_3 are of following form :$e_1 = (1, y, z)$, $e_2 = (0, y_2, z_2)$, $e_3 = (0, \epsilon z_2, \epsilon y_2)$, with $y_2^2 - z_2^2 = \delta$, where ϵ, δ is +1 or -1.

Such trihedron $(T_o; e_1, e_2, e_3)$ is called *positively oriented* if for its vectors $\det(e_1, e_2, e_3) = 1$ holds i.e. if $y_2^2 - z_2^2 = \epsilon$.

3 Ruled Surfaces in the Galilean Space

A general equation of a ruled surface G_3^1 is

$$x(u, v) = r(u) + va(u),\ v \in \mathbb{R};\ r, a \in \mathbf{C}^3 \tag{3.1}$$

where the curve r does not line in a pseudo-Euclidean plane and is called a *directix*. The curve r is given by

$$r(u) = (u, y(u), z(u)). \tag{3.2}$$

This means that the curve r is parametrizd by the pseudo-Galilean arc length. Further, the generator vector field is of the form

$$a(u) = (1, a_2(u), a_3(u)). \tag{3.3}$$

Notice that under the given assumptions all tangent planes of ruled surfaces are isotropic.

According to the absolute figure, we distinguish two types of ruled surfaces in G_3^1. More about ruled surface in G_3^1 can be found in [3].

Type I : The equation of a ruled surface of type I in G_3^1 is

$$\begin{cases} x(u, v) = (u, y(u), z(u)) + v(1, a_2(u), a_3(u)), \\ \quad y, z, a_2, a_3 \in C^3, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R}. \end{cases} \tag{3.4}$$

The ruled surfaces of type I are non-conoidal and conoidal surfaces whose directional straight line at in finity is not the absolute line. The striction curve of these surfaces does not lie in a pseudo-Euclidean space.

The associated trihedron of a ruled surface of type I in G_3^1 is defined by

$$T(u) = a(u), \quad N(u) = \frac{1}{k(u)} a'(u), \quad B(u) = \frac{1}{k(u)} (0, a_3'(u), a_2'(u)).$$

The curvature is given by $k(x) = \sqrt{|a_2'^2 - a_3'^2|}$.

Type II : The equation of ruled surface of type II in G_3^1 is

$$\begin{cases} x(u,v) = (0, y(u), z(u)) + v(1, a_2(u), a_3(u)), \\ y, z, a_2, a_3 \in C^3, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R}, \\ \left| y'^2 - z'^2 \right| = 1, \quad y'a_2' - z'a_3' = 0. \end{cases} \tag{3.5}$$

A ruled surface of type II is a surface whose striction curve lies in a pseudo-Euclidean plane.

The associated trihedron of ruled surface of type II in G_3^1 is defined by

$$\begin{aligned} T(u) &= a(u) = (1, a_2, a_3), \\ N(u) &= (0, z'(u), y'(u)), \\ B(u) &= (0, y'(u), z'(u)), \end{aligned}$$

where

$$k(u) = \frac{a_2(u)}{z'(u)}, \qquad \tau(u) = \frac{y''(u)}{z'(u)}.$$

The Frenet formulas are in type I or type II as follows.

$$\begin{aligned} \nabla_{T_{(u)}} T(u) &= k(u) N(u), \\ \nabla_{N_{(u)}} N(u) &= \tau(u) B(u), \\ \nabla_{B_{(u)}} B(u) &= \tau(u) N(u). \end{aligned} \tag{3.6}$$

4 The Characterizations of Curves on Ruled Surfaces

Definition 4.1 Let α be a curve of a ruled surface of type I or II and $\{T(u), N(u), B(u)\}$ be the Frenet frame on ruled surface of type I or II along α. If k and τ are positive constants along α, then α is called a *circular helix* with respect to the frenet frame.

Definition 4.2 Let α be a curve of a ruled surface of type I or II and $\{T(u), N(u), B(u)\}$ be the Frenet frame on ruled surface of type I or II along α. A curve α such that

$$\frac{k(u)}{\tau(u)} = const$$

is called a *general helix* with respect to Frenet Frame.

Theorem 4.3 *Let α be a curve of a ruled surface of type I or II in pseudo-Galilean space G_3^1 . α is a general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$ if and only if*

$$\nabla_{T_{(u)}}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u) - K(u)\nabla_{T_{(u)}}N(u) = \frac{3}{\lambda}\tau'(u)\nabla_{T_{(u)}}T(u) \tag{4.1}$$

where $K(u) = \frac{\tau''(u)}{\tau(u)} + \tau^2(u)$.

Proof. Suppose that α is general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$. Then from (3.6) , we have

$$\nabla_{T_{(u)}}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u) = (\tau''(u) + \tau^3(u))B(u) + (3\tau(u)\tau'(u))N(u). \tag{4.2}$$

Now, since α is general helix with respect to the Frenet Frame

$$\frac{k(u)}{\tau(u)} = \lambda = const. \tag{4.3}$$

If we substitute the equations

$$N(u) = \frac{1}{k(u)}\nabla_{T_{(u)}}T(u), \tag{4.4}$$

$$B(u) = \frac{1}{\tau(u)}\nabla_{T_{(u)}}N(u) \tag{4.5}$$

and (4.5) in (4.2), we obtain (4.1).

Conversely let us assume that the equation (4.1) holds. We show that the curve α is a general helix. Differentiating covariantly (4.5) we obtain

$$\nabla_{T_{(u)}}B(u) = -\frac{\tau'(u)}{\tau^2(u)}\nabla_{T_{(u)}}N(u) + \frac{1}{\tau(u)}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u) \tag{4.6}$$

and so

$$\begin{aligned}\nabla_{T_{(u)}}\nabla_{T_{(u)}}B(u) = &\left(-\frac{\tau'(u)}{\tau^2(u)}\right)'\nabla_{T_{(u)}}N(u) - 2\frac{\tau'(u)}{\tau^2(u)}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u)\\ &+ \frac{1}{\tau(u)}\nabla_{T_{(u)}}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u).\end{aligned} \tag{4.7}$$

If we use (4.1) in (4.7) and make some calculations, we have

$$\begin{aligned}\nabla_{T_{(u)}}\nabla_{T_{(u)}}B(u) = &\left[\left(-\frac{\tau'(u)}{\tau^2(u)}\right)' + \frac{K(u)}{\tau(u)}\right]\nabla_{T_{(u)}}N(u) - 2\frac{\tau'(u)}{\tau^2(u)}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u)\\ &+ \frac{3}{\lambda}\frac{\tau'(u)k(u)}{\tau(u)}N(u).\end{aligned} \tag{4.8}$$

Also we obtain

$$\nabla_{T_{(u)}}\nabla_{T_{(u)}}B(u) = \tau^2(u)B(u) + \tau'(u)N(u) \tag{4.9}$$

since (4.8) and (4.9) are equal, routine calculations show that α is a general helix.
□

Corollary 4.4 *Let α be a curve of ruled surface of type I or II in pseudo-Galilean space G_3^1. α is a circular helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$, if and only if*

$$\nabla_{T_{(u)}}\nabla_{T_{(u)}}\nabla_{T_{(u)}}N(u) = \tau^2(u)\nabla_{T_{(u)}}N(u). \tag{4.10}$$

Proof. From the hypotesis of corollary 4.4 and since α is a circular helix, we can show easily (4.10). □

Theorem 4.5 *Let α be a curve of a ruled surface of type I or II in pseudo-Galilean space G_3^1 . α is a general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$ if and only if $\nabla_{T_{(u)}}T(u)$ and $\nabla_{T_{(u)}}B(u)$ are linear independent.*

Proof. Suppose that α is a general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$. Then from (4.3), we have

$$k(u) = \lambda\tau(u). \tag{4.11}$$

If we product n with (4.11) equation and consider (3.6), we obtain

$$\nabla_{T_{(u)}}T(u) = \lambda\nabla_{T_{(u)}}B(u). \tag{4.12}$$

Conversely let us assume that the equation (4.12) holds. We show that the curve α is a general helix. From (4.12), we obtain

$$\frac{k(u)}{\tau(u)} = \lambda = const$$

That is α is a general helix. □

Theorem 4.6 *Let α be a curve of a ruled surface of type I or II in pseudo-Galilean space G_3^1 . α is a general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$ if and only if $\nabla_{T_{(u)}}\nabla_{T_{(u)}}T(u)$ and $\nabla_{T_{(u)}}\nabla_{T_{(u)}}B(u)$ are linear independent.*

Proof. It is similar to the proof of Theorem 4.5. □

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