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S-Iterative Process for a Pair of Single Valued and Multi Valued Mappings in Banach Spaces

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Abstract : Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : E \to E$ and $T : E \to KC(E)$ be a single valued and a multivalued mappings , both satisfying the conditions (C). Assume in addition that $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. We prove that the sequence of the modified S-iteration method generated from an arbitrary $x_0 \in E$ by

$$y_n = (1-\beta_n)x_n + \beta_n z_n$$
$$x_{n+1} = (1-\alpha_n)z_n + \alpha_n t y_n$$

where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$, converges strongly to a common fixed point of t and T, i.e., there exists $x \in E$ such that $x = tx \in Tx$.

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1 Introduction

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) = \max\{ \sup_{a\in A} \mathrm{dist}(a,B), \sup_{b\in B} \mathrm{dist}(b,A) \}, \quad A,B \in FB(X),$$

where $\mathrm{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B.

A mapping $t : E \to E$ is said to be *nonexpansive* if

$$\|tx - ty\| \leq \|x - y\| \ \text{ for all } x, y \in E.$$

A point x is called a fixed point of t if $tx = x$.

Definition 1.1 ([1])**.** *Let t be a mapping on a subset E of a Banach space X. Then is said to satisfy condition (C) if*

$$\frac{1}{2}\|x - t(x)\| \leq \|x - y\| \Rightarrow \|t(x) - t(y)\| \leq \|x - y\| \quad for\ all\ x, y \in E.$$

The propositions and lemma following have property on condition (C).

Proposition 1.2 ([1])**.** *Every nonexpansive mapping satisfies condition (C).*

Proposition 1.3 ([1])**.** *Assume that a mapping t satisfies condition (C) and has a fixed point.Then t is a quasi nonexpansive mapping.*

Lemma 1.4 ([1])**.** *Let t be a mapping on a subset E of Banach space X. Assume that t satisfies condition (C). Then $\|x - t(y)\| \leq 3\|t(x) - x\| + \|x - y\|$ holds for all $x, y \in E$.*

Let $T : X \to 2^X$ be a multivalued mapping. A point x is a fixed point for a multivalued mapping T if $x \in T(x)$.

Definition 1.5. *A multivalued mapping $T : X \to FB(X)$ is said to be* nonexpansive *if*

$$H(T(x), T(y)) \leq \|x - y\| \qquad \forall x, y \in X.$$

Suzuki's condition can be modified to incorporate multivalued mappings.

Definition 1.6 ([2])**.** *A multivalued mapping $T : X \to FB(X)$ is said to satisfy condition (C) provided that*

$$\frac{1}{2}dist(x, T(x)) \leq \|x - y\| \Rightarrow H(T(x), T(y)) \leq \|x - y\|, \quad \forall x, y \in X.$$

2 Preliminaries

We use the notation Fix(T) stands for the set of fixed points of a mapping T and Fix(t)$\cap$Fix(T) stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $x = tx \in Tx$.

Dhompongsa et al. [3] proved a common fixed point theorem for two nonexpansive commuting mappings.

Theorem 2.1 ([3], Theorem 4.2)**.** *Let E be a nonempty bounded closed convex subset of a uniformly Banach space X, $t : E \to E$, and $T : E \to KC(E)$ a nonexpansive mapping and a multivalued nonexpansive mapping respectively. Assume that t and T are commuting , i.e. if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds $tx \in Tty$. Then t and T have a common fixed point.*

Abkar and Eslamian [2] proved the existence of a common fixed point for a commuting pair consisting of a single valued and a multivalued mapping both satisfying the Suzuki's condition in a uniformly convex Banach space.

Theorem 2.2 ([2], Theorem 3.3)**.** *Let E be a nonempty closed convex bounded subset of a uniformly convex Banach space X. Let $t : E \to E$ be a single valued mapping, and let $T : E \to KC(E)$ be a multivalued mapping. If both t and T satisfy the condition (C) and if t and T are commuting, then they have a common fixed point, that is, there exists a point $z \in E$ such that $z = tz \in Tz$.*

In this paper, we introduce an iterative process in a new sense, called the modified S-iteration method with respect to a pair of single valued and multivalued mappings, both satisfying the conditions (C). We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a uniformly convex Banach space.

The important property of a uniformly convex Banach space we use is the following lemma proved by Schu [4] in 1991.

Lemma 2.3 ([4])**.** *Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|u_n x_n + (1 - u_n)y_n\| = a$ for some $a \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following observation will be used in proving our results and the proof is a straightforward.

Lemma 2.4. *Let X be a Banach space and E be a nonempty closed convex subset of X. Then,*

$$dist(y, Ty) \leq \|y - x\| + dist(x, Tx) + H(Tx, Ty),$$

where $x, y \in E$ and T is a multivalued nonexpansive mapping from E into $FB(E)$.

A fundamental principle which plays a key role in ergodic theory is the demiclosedness principle. A mapping t defined on a subset E of a Banach space X is said to be demiclosed if any sequence $\{x_n\}$ in E the following implication holds: $x_n \rightharpoonup x$ and $tx_n \to y$ implies $tx = y$.

Theorem 2.5 ([5]). *Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $t : E \to E$ be a nonexpansive mapping. If a sequence $\{x_n\}$ in E converges weakly to p and $\{x_n - tx_n\}$ converges to 0 as $n \to \infty$, then $p \in Fix(t)$.*

Agarwal et al. [6] introduced the S-iteration following well-known iteration.

For E a convex subset of a linear space X and t a mapping of E into itself, the ietrative sequence $\{x_n\}$ of the S-iteration process is generated from $x_1 \in E$ and is defined by

$$\begin{cases} x_1 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n t x_n \\ x_{n+1} = (1-\alpha_n)tx_n + \alpha_n t y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in (0, 1) satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

Sokhuma and Keawkhao [7] defined the modified Ishikawa iteration method scheme for a pair of single valued and multivalued nonexpansive mappings as follows:

Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ be a single valued nonexpansive mapping, and $T : E \to FB(E)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$y_n = (1-\beta_n)x_n + \beta_n z_n$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n$$

where $x_0 \in E, z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

They proved the strong convergence theorem of a sequence from this process in a nonempty compact convex subset of a uniformly convex Banach space.

Kumam, Saluja and Nashine [8] modified iteration for two mappings in a CAT(0) space as follows.

Let K be a nonempty closed convex subset of a complete CAT(0) space X, and let $S, T : K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F(S,T) = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 1$.
In this paper we introduce a new iteration method modifying the above ones and call it the modified S-iteration method.

Definition 2.6. *Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ be a single valued nonexpansive mapping, and $T : E \to FB(E)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified S-iteration is defined by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1-\alpha_n)z_n + \alpha_n t y_n \end{aligned} \tag{2.1}$$

where $x_0 \in E, z_n \in Tx_n$ and $0 < a \le \alpha_n, \beta_n \le b < 1$.

3 Main Results

We first prove the following lemmas, which play very important roles in this section.

Lemma 3.1. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multivalued mapping, respectively, such that satisfy the condition (C), and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (2.1). Then $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_0 \in E$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$.
Since $\frac{1}{2}\|w - tw\| = 0 \le \|((1-\beta_n)x_n + \beta_n z_n) - w\|$, we get

$$\|t((1-\beta_n)x_n + \beta_n z_n) - tw\| \le \|(1-\beta_n)x_n + \beta_n z_n - w\|.$$

Similarly, since $\frac{1}{2}\mathrm{dist}(w, Tw) = 0 \le \|x_n - w\|$ then we get

$$H(Tx_n, Tw) \le \|x_n - w\|.$$

Consider,

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)z_n + \alpha_n t((1-\beta_n)x_n + \beta_n z_n) - w\| \\ &\le (1-\alpha_n)\|z_n - w\| + \alpha_n \|t((1-\beta_n)x_n + \beta_n z_n) - w\| \\ &= (1-\alpha_n)\|z_n - w\| + \alpha_n \|t((1-\beta_n)x_n + \beta_n z_n) - tw\| \\ &\le (1-\alpha_n)\|z_n - w\| + \alpha_n \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &= (1-\alpha_n)\|z_n - w\| + \alpha_n \|(1-\beta_n)x_n + \beta_n z_n - (1-\beta_n)w - \beta_n w\| \\ &\le (1-\alpha_n)\|z_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \|z_n - w\| \\ &= (1-\alpha_n)\mathrm{dist}(z_n, Tw) + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \mathrm{dist}(z_n, Tw) \\ &\le (1-\alpha_n)H(Tx_n, Tw) + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n H(Tx_n, Tw) \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned}$$

Since $\{\|x_n - w\|\}$ is a decreasing and bounded sequence, we can conclude that the limit of $\{\|x_n - w\|\}$ exists. □

We can see how Lemma 2.3 is useful via the following lemma.

Lemma 3.2. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multivalued mapping, respectively, such that satisfy the condition (C), and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (2.1). If $0 < a \leq \alpha_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n\to\infty} \|ty_n - z_n\| = 0$.*

Proof. Let $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. Since $\frac{1}{2}\|w - tw\| = 0 \leq \|y_n - w\|$, we get

$$\|ty_n - tw\| \leq \|y_n - w\|.$$

Similarly, since $\frac{1}{2}\ \mathrm{dist}(w, Tw) = 0 \leq \|x_n - w\|$ then we get

$$H(Tx_n, Tw) \leq \|x_n - w\|.$$

By Lemma 3.1, we put $\lim_{n\to\infty} \|x_n - w\| = c$ and consider

$$\begin{aligned}
\|ty_n - w\| &= \|ty_n - tw\| \\
&\leq \|y_n - w\| \\
&= \|(1-\beta_n)x_n + \beta_n z_n - w\| \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|z_n - w\| \\
&= (1-\beta_n)\|x_n - w\| + \beta_n \mathrm{dist}(z_n, Tw) \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned}$$

Then we have

$$\limsup_{n\to\infty} \|ty_n - w\| \leq \limsup_{n\to\infty} \|y_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c. \tag{3.1}$$

Further, we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - w\| \\
&= \lim_{n\to\infty} \|(1-\alpha_n)z_n + \alpha_n ty_n - w\| \\
&= \lim_{n\to\infty} \|\alpha_n ty_n - \alpha_n w + z_n - \alpha_n z_n + \alpha_n w - w\| \\
&= \lim_{n\to\infty} \|\alpha_n(ty_n - w) + (1-\alpha_n)(z_n - w)\|.
\end{aligned}$$

By Lemma 2.3, we can conclude that

$$\lim_{n\to\infty} \|(ty_n - w) - (z_n - w)\| = \lim_{n\to\infty} \|ty_n - z_n\| = 0.$$

□

Lemma 3.3. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multivalued mapping, respectively, such that satisfy the condition (C), and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (2.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b \in \mathbb{R}$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof. Let $w \in \text{Fix}(t) \cap \text{Fix}(T)$. Since $\frac{1}{2}\|w - tw\| = 0 \leq \|y_n - w\|$ then we get

$$\|ty_n - tw\| \leq \|y_n - w\|.$$

Similarly, since $\frac{1}{2}\text{dist}(w, Tw) = 0 \leq \|x_n - w\|$ then we get

$$H(Tx_n, Tw) \leq \|x_n - w\|.$$

We put, as in Lemma 3.2, $\lim_{n\to\infty} \|x_n - w\| = c$. For $n \geq 0$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|(1-\alpha_n)z_n + \alpha_n ty_n - w\| \\
&= \|(1-\alpha_n)z_n + \alpha_n ty_n - (1-\alpha_n)w - \alpha_n w\| \\
&\leq (1-\alpha_n)\|z_n - w\| + \alpha_n \|ty_n - w\| \\
&= (1-\alpha_n)\text{dist}(z_n, Tw) + \alpha_n \|ty_n - w\| \\
&\leq (1-\alpha_n)H(Tx_n, Tw) + \alpha_n \|ty_n - w\| \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|y_n - w\|,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|x_{n+1} - w\| - \|x_n - w\| \leq -\alpha_n \|x_n - w\| + \alpha_n \|y_n - w\| \\
&\|x_{n+1} - w\| - \|x_n - w\| \leq \alpha_n(\|y_n - w\| - \|x_n - w\|) \\
&\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \leq \|y_n - w\| - \|x_n - w\|.
\end{aligned}$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\| \leq \|y_n - w\|.$$

Thus,

$$\liminf_{n\to\infty} \left\{\left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\|\right\} \leq \liminf_{n\to\infty} \|y_n - w\|.$$

It follows that

$$c \leq \liminf_{n\to\infty} \|y_n - w\|.$$

Since, from (3.1), $\limsup_{n\to\infty} \|y_n - w\| \leq c$, we have

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(z_n - w)\| . \end{aligned} \tag{3.2}$$

Recall that

$$\|z_n - w\| = \mathrm{dist}(z_n, Tw) \leq H(Tx_n, Tw) \leq \|x_n - w\| .$$

Hence we have

$$\limsup_{n\to\infty} \|z_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c.$$

Using the fact that $0 < a \leq \beta_n \leq b < 1$ and (3.2), we can conclude that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. □

The following lemma allows us to go on.

Lemma 3.4. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multivalued mapping, respectively, such that satisfy the condition (C), and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (2.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $\lim_{n\to\infty} \|tx_n - x_n\| = 0$.*

Proof. By Lemma 1.4, we have

$$\begin{aligned} \|tx_n - x_n\| &= \|tx_n - y_n + y_n - x_n\| \\ &\leq \|tx_n - y_n\| + \|y_n - x_n\| \\ &\leq 3\|y_n - ty_n\| + \|y_n - x_n\| + \|y_n - x_n\| \\ &= 3\|y_n - ty_n\| + 2\|y_n - x_n\| \\ &\leq 3\|y_n - x_n\| + 3\|ty_n - x_n\| + 2\|y_n - x_n\| \\ &= 5\|y_n - x_n\| + 3\|ty_n - x_n\| \\ &= 5\|x_n - (1-\beta_n)x_n - \beta_n z_n\| + 3\|ty_n - x_n\| \\ &= 5\|x_n - x_n + \beta_n x_n - \beta_n z_n\| + 3\|ty_n - x_n\| \\ &= 5\beta_n\|x_n - z_n\| + 3\|ty_n - x_n\| \\ &= 5\beta_n\|x_n - z_n\| + 3\|ty_n - z_n + z_n - x_n\| \\ &\leq 5\beta_n\|x_n - z_n\| + 3\|ty_n - z_n\| + 3\|x_n - z_n\| \\ &= (3+5\beta_n)\|x_n - z_n\| + 3\|ty_n - z_n\| . \end{aligned}$$

Then we have

$$\lim_{n\to\infty} \|tx_n - x_n\| \leq \lim_{n\to\infty} (3+5\beta_n)\|x_n - z_n\| + \lim_{n\to\infty} 3\|ty_n - z_n\| .$$

Hence, by Lemma 3.2 and Lemma 3.3, $\lim_{n\to\infty} \|tx_n - x_n\| = 0$. □

We give the sufficient conditions which imply the existence of common fixed points for single valued mappings and multivalued nonexpansive mappings, respectively, as follow:

Theorem 3.5. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multivalued mapping, respectively, such that satisfy the condition (C), and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by (2.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $x_{n_i} \to y \in Tx_{n_i}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in Fix(t) \cap Fix(T)$.*

Proof. Assumed that $\lim_{n\to\infty} \|x_{n_i} - y\| = 0$. From Lemma 3.4, we have

$$0 = \lim_{n\to\infty} \|tx_{n_i} - x_{n_i}\| = \lim_{n\to\infty} \|(I - t)(x_{n_i})\| .$$

Since $I - t$ is demiclosed at 0, we have $(I - t)(y) = 0$ and hence $y = ty$, i.e., $y \in \text{Fix}(t)$. Since $\frac{1}{2}\,\text{dist}(x_n, Tx_n) \leq \|x_n - y\|$ then we get

$$H(Tx_n, Ty) \leq \|x_n - y\|.$$

By Lemma 2.4 and by Lemma 3.3. we have

$$\begin{aligned}\text{dist}(y, Ty) &\leq \|y - x_{n_i}\| + \text{dist}(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty)\\ &\leq \|y - x_{n_i}\| + \|x_{n_i} - z_{n_i}\| + \|x_{n_i} - y\| \to 0 \quad \text{as } i \to \infty.\end{aligned}$$

It follows that $y \in \text{Fix}(T)$. Therefore $y \in \text{Fix}(t) \cap \text{Fix}(T)$ as desired. □

Hereafter, we arrive at the convergence theorem of the sequence of the modified S-iteration. We conclude this paper with the following theorem.

Theorem 3.6. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multivalued mapping, respectively,such that satisfy the condition (C), and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified S-iteration defined by 2.1 with $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. Since $\{x_n\}$ is contained in E which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in E$, i.e., $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. By Theorem 3.5, we have $y \in \text{Fix}(t) \cap \text{Fix}(T)$ and by Lemma 3.1, we have that $\lim_{n\to\infty} \|x_n - y\|$ exists. It must be the case that $\lim_{n\to\infty} \|x_n - y\| = \lim_{i\to\infty} \|x_{n_i} - y\| = 0$. Therefor $\{x_n\}$ converges strongly to a common fixed point y of t and T. □

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