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Functions that Share Two Finite Values with Their Derivative

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Abstract : The purpose of this paper is to study a meromorphic functions which share two finite nonzero values with their derivatives, and the result is proved: Let f be a nonconstant meromorphic function, a, b be a nonzero distinct finite complex constant. If f and f' share a CM, and share b IM and $\overline{N}(r, f) = S(r, f)$ then $f = f'$.

Keywords : Meromorphic function; Shared value.

1 Introduction

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We use the standard notations in Nevanlinna theory of meromorphic functions such as the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$, and the proximity function $m(r, f)$ (see, e.g., [1]).

The notation $S(r, f)$ is used to define any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure.

Two meromorphic functions f and g share the value $a \in \hat{\mathbb{C}}$ if $f^{-1}\{a\} = g^{-1}\{a\}$. And one says that f and g share the value a CM if the value a is shared by f and by g and moreover if $f(z_0) = a$ with multiplicity p implies that $g(z_0) = a$ with multiplicity p, here $p = p(z_0)$.

The usual sharing is also denoted by sharing IM (IM =ignoring multiplicity). In sharing CM the abbreviation CM stands for counting multiplicities. Obviously, sharing CM $\Longrightarrow$ sharing IM. But the converse must not be true.

In 1976, Rubel and Yang [2] proved that if f is an entire function and shares two finite values CM with f', then $f = f'$.

E. Mues and N. Steinmetz [3] have shown that "CM" can be replaced by "IM". (another proof of this result for nonzero shared values is in [4]). On the other hand, the meromorhic function [3]

$$f(z) = \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\, i \tan\left(\frac{\sqrt{5}}{4}\, iz\right)\right)^2 \tag{1.1}$$

shares 0 by DM and 1 by DM with f' ; while the meromorhic function [4]

$$f(z) = \frac{2A}{1 - Be^{-2z}}, A \neq 0, B \neq 0 \tag{1.2}$$

shares 0 (picard value) and A by DM with f' ; and $f \neq f'$ in both (1.1) and (1.2).

Mues and Steinmetz [6], and Gundersen [5] improved this result and proved the following theorem.

Theorem 1.1 *Let f be a nonconstant meromorphic function, a and b be two distinct finite values. If f and f' share the values a and b CM, then $f = f'$.*

Q. C. Zhang [7], results that (1.2) is unique in some sense.

Theorem 1.2 *Let f be a nonconstant meromorphic function, b be a nonzero finite complex constant. If f and f' share 0 CM, and share b IM, then $f = f'$ or $f = \frac{2b}{1 - ce^{-2z}}$, where c is a nonzero finite complex constant.*

2 Preliminaries

Theorem 2.1 (First fundamental theorem of value distribution theory) *Let $0 < R_0 \leq \infty$ and let $a \in \mathbb{C}$. Let f be meromorphic in the disk $\{z \in \mathbb{C} : |z| < R_0\}$. Assume that at the point $z = 0$ the function $f - a$ has the expansion*

$$f(z) - a = c_k(a)z^k + \cdots, \ c_k(a) \neq 0.$$

Then we have for $0 < r < R_0$, $m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) = T(r, f) + \eta(r, a)$ with the estimate. We can write the first fundamental theorem in the form

$$m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1) \qquad \text{if} \quad r \to R_0.$$

Theorem 2.2 *If f is meromorphic in $\mathbb{C}$, $f \not\equiv$ constant, then*

$$m\left(r, \frac{f'}{f}\right) = S(r, f).$$

Theorem 2.3 (Left main-inequality)
Let f be meromorphic in $\mathbb{C}$ and not constant. Let $a_1, a_2, \ldots, a_q$ be $q \geq 1$ pairwise different complex numbers. Then we have

$$\sum_{\nu=1}^{q} m\left(r, \frac{1}{f - a_\nu}\right) \leq m\left(r, \frac{1}{f'}\right) + S(r, f).$$

Theorem 2.4 (Second fundamental theorem of value distribution theory)
Let f be meromorphic in $\mathbb{C}$ and not constant. Let $a_1, a_2, \ldots, a_q$ be $q \geq 1$ pairwise different complex numbers. Then we have

$$(q-1)T(r,f) \leq \overline{N}(r,f) + \sum_{\nu=1}^{q} \overline{N}\left(r, \frac{1}{f-a_\nu}\right) + S(r,f).$$

3 Main Results

Theorem 3.1 *Let f be a nonconstant meromorphic function, a, b be a nonzero distinct finite complex constant. If f and f' share a CM, and share b IM and $\overline{N}(r,f) = S(r,f)$ then $f = f'$.*

Proof. Suppose $f \neq f'$. By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + S(r,f) \\
&= \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + S(r,f) \\
&= \overline{N}(r, \frac{1}{f'-a}) + \overline{N}(r, \frac{1}{f'-b}) + S(r,f) \\
&\leq N(r, \frac{1}{\frac{f'}{f}-1}) + S(r,f) \\
&\leq T(r, \frac{1}{\frac{f'}{f}-1}) + S(r,f) \\
&= N(r, \frac{f'}{f}) + S(r,f) \\
&= \overline{N}(r, \frac{1}{f}) + \overline{N}(r,f) + S(r,f) \\
&= \overline{N}(r, \frac{1}{f}) + S(r,f) \\
&\leq T(r,f) + S(r,f).
\end{aligned}$$

Therefore

$$\begin{aligned}
T(r,f) &= \overline{N}(r, \frac{1}{f}) + S(r,f) \\
&= \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + S(r,f).
\end{aligned}$$

Let $\phi = \dfrac{f'}{f-a} - \dfrac{f''}{f'-a}$. Using the theorem on the logarithmic derivative, we obtain

$$\begin{aligned} m(r,\phi) &\le m\left(r,\tfrac{f'}{f-a}\right)+m\left(r,\tfrac{f''}{f'-a}\right)+O(1)\\ &= S(r,f)+S(r,f'). \end{aligned}$$

Since, generally $T(r,f') \le 2T(r,f)+S(r,f)$ this gives $m(r,\phi)=S(r,f)$. And since ϕ is the logarithmic derivative $\dfrac{f-a}{f'-a}$ and f, f' share the value a CM, it follows that

$$N(r,\phi)=\overline{N}(r,f)=S(r,f).$$

Hence $T(r,\phi)=S(r,f)$.

Suppose that $\phi \not\equiv 0$. Then from

$$\frac{\phi}{f-b}=\frac{f'}{(f-a)(f-b)}-\frac{f''}{f'(f'-a)}\cdot\frac{f'}{(f-b)}$$

it follows that

$$\begin{aligned} m\left(r,\frac{1}{f-b}\right) &\le m\left(r,\frac{1}{\phi}\right)+m\left(r,\frac{f'}{(f-b)(f-a)}\right)+m\left(r,\frac{f''}{f'(f'-a)}\right)\\ &\quad +m\left(r,\frac{f'}{f-b}\right)+O(1)\\ &\le T(r,\phi)+S(r,f)+S(r,f')+S(r,f)\\ &= S(r,f), \end{aligned}$$

we have $m\left(r,\frac{1}{f-b}\right)=S(r,f)$. By the first fundamental theorem we have

$$T(r,f)=N\left(r,\frac{1}{f-b}\right)+S(r,f).$$

Since

$$T(r,f)=N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right)+S(r,f),$$

so

$$N\left(r,\frac{1}{f-a}\right)=S(r,f).$$

But from the assumption, we have $N\left(r,\frac{1}{f'-a}\right)=N\left(r,\frac{1}{f-a}\right)$. Hence

$$N\left(r,\frac{1}{f'-a}\right)=S(r,f).$$

Putting all together one has

$$\begin{aligned} m\left(r,\frac{1}{f'-b}\right)+N\left(r,\frac{1}{f'-b}\right) &= T(r,f')+O(1)\\ &\le T(r,f)+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{f'-b}\right)+S(r,f)\\ &\le N\left(r,\frac{1}{f'-b}\right)+S(r,f). \end{aligned}$$

Hence $m\left(r,\frac{1}{f'-b}\right)=S(r,f)$. Using the left main inequality we have

$$m\left(r,\frac{1}{f'}\right)+m\left(r,\frac{1}{f'-a}\right)+m\left(r,\frac{1}{f'-b}\right)\le m\left(r,\frac{1}{f''}\right)+S(r,f) \tag{3.1}$$

$$m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)\le m\left(r,\frac{1}{f'}\right)+S(r,f). \tag{3.2}$$

By the previous equalities for the counting functions

$$N\left(r,\frac{1}{f'-a}\right)+N\left(r,\frac{1}{f'-b}\right)=N\left(r,\frac{1}{f'-b}\right)+S(r,f) \tag{3.3}$$

$$N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right)=T(r,f)+S(r,f). \tag{3.4}$$

Sum four inequality above we have

$$\begin{aligned} &T\left(r,\frac{1}{f'-a}\right)+T\left(r,\frac{1}{f-a}\right)+T\left(r,\frac{1}{f-b}\right)+T\left(r,\frac{1}{f'-b}\right)\\ &\quad\le m\left(r,\frac{1}{f''}\right)+N\left(r,\frac{1}{f'-b}\right)+T(r,f)+S(r,f)\\ &\quad\le T(r,f'')+N\left(r,\frac{1}{f'-b}\right)+T(r,f)+S(r,f)\\ &\quad\le T(r,f')+N\left(r,\frac{1}{f'-b}\right)+T(r,f)+S(r,f). \end{aligned}$$

Since $T\left(r,\frac{1}{f'-b}\right)=N\left(r,\frac{1}{f'-b}\right)+S(r,f)$. By above, we have $T(r,f)=S(r,f)$ a contradiction.

And so our assumption $\phi\not\equiv 0$ cannot be true. Therefore $\phi\equiv 0$. Then from integration we get

$$\frac{f-a}{f'-a}=C$$

where C is some nonzero constant. If $C\neq 1$, then b is a Picard value for both f and f'. This is impossible because $b\neq 0$. Therefore $f=f'$. □

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