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Strong Convergence of Multi-step Iterations with Errors for a Family of Finite Non-Lipschitzian Mappings

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Abstract : In this paper, we established strong convergence theorems for a multi-step iterative scheme with errors for finite mappings of asymptotically nonexpansive in the intermediate sense in Banace spaces. Our results extend and improve the recent ones announced by Plubtieng and Wangkeeree [Strong convergence theorems for multi-step iterations with errors in Banach spaces, J. Math. Anal. Appl. 321 (2006) 10-23], Cho, Zhou and Guo [Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Computers Math. Applic. 47(4/5), 707-717, (2004)], and many others.

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1 Introduction

Let C be a subset of real normed linear space X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{r_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \le (1 + r_n)\|x - y\|, \quad \forall n \ge 1.$$

If $r_n \equiv 0$, then T is known as a nonexpansive mapping. T is called *asymptoticall nonexpansive* in the intermediate sense [18] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense.

Fixed-point iterations process for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequations; see [1, 5–9, 11–14, 16–17].

In 2000, Noor [12] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Glowinski and Le Tallec [4] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [4] that the three-step iterative scheme gives better numberical results then the two-step and one-step approximate iterations. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences.

Recently, Xu and Noor [18] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho, Zhou and Guo [2] extended the work of Xu and Noor to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in Banach space. Moreover, Plubtieng and Wangkeeree [15] introduced strong convergence theorems of a multi-step scheme with errors of asymptotically nonexpansive in the intermediate sense. Inspired motivated by these fact, we introduce and study a multi-step scheme with errors for finite family of asymptotically nonexpansive in the intermediate sense.

Let C be a nonempty subset of normed space X and $T_1, ..., T_N : C \to C$ be mappings. For a given $x_1 \in C$, and a fixed $N \in \mathbb{N}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_n^{(1)} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1^n x_n + \gamma_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} = \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2^n x_n^{(1)} + \gamma_n^{(2)} u_n^{(2)}, \\ \qquad \vdots \\ x_{n+1} = x_n^{(N)} = \alpha_n^{(N)} x_n + \beta_n^{(N)} T_N^n x_n^{(N-1)} + \gamma_n^{(N)} u_n^{(N)}, n \geq 1, \end{cases} \tag{1.1}$$

where, $\{u_n^{(1)}\}, \ldots, \{u_n^{(N)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in [0, 1] such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, N\}$.

The iterative schemes (1.1) are called the multi-step iteratives with errors.

If $T_1 = T_2 = \cdots = T_N = T$, then (1.1) reduces to multi-step Noor iterations with errors introduced by Plubtieng and Wangkeeree [15] defined by

$$\begin{cases} x_1 = x \in C, \\ x_n^{(1)} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T^n x_n + \gamma_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} = \alpha_n^{(2)} x_n + \beta_n^{(2)} T^n x_n^{(1)} + \gamma_n^{(2)} u_n^{(2)}, \\ \qquad \vdots \\ x_{n+1} = x_n^{(N)} = \alpha_n^{(N)} x_n + \beta_n^{(N)} T^n x_n^{(N-1)} + \gamma_n^{(N)} u_n^{(N)}, n \geq 1, \end{cases} \tag{1.2}$$

where, $\{u_n^{(1)}\}, \ldots, \{u_n^{(N)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in [0, 1] such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \ldots, N\}$.

The purpose of this paper is to establish several strong convergence theorems of multi-step iterative scheme with errors for finite family of asymptotically nonexpansive in the intermediate sense in a uniformly convex Banach space. These results presented in this paper extend and improve the corresponding ones announced by Plubtieng and Wangkeeree [15], and many others.

2 Preliminaries

In this section, we recall the well known conceot results. Let C be a nonempty subset of normed space X and $T_1, \dots, T_N : C \to C$ be mappings. A family $\{T_i : i = 1, 2, \dots, N\}$ of N self-mappings of C $(i.e., T_i : C \longrightarrow C)$ with $F = \cap_{i=1}^{N} F(T_i) \neq \phi$ is said to satisfy condition (B) on C if there is a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \le i \le N} \{\|x - T_i x\|\} \ge f(d(x, F)). \tag{2.1}$$

A mapping $T : C \to C$ is called *semi − compact* if any sequence $\{x_n\}$ in C satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 2.1 ([17]) *Suppose that X is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r$$

and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.2 ([10]) *Let $\{a_n\}, \{b_n\}$ and $\{\gamma_n\}$ be nonnegative sequences of real number satisfying*

$$a_{n+1} \le (1 + \gamma_n) a_n + b_n \quad \text{for all } n \ge 1.$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$ then

(a) $\lim_{n\to\infty} a_n$ *exists;*

(b) $\lim_{n\to\infty} a_n = 0$, *whenever* $\liminf_{n\to\infty} a_n = 0$.

3 Main Theorems

This section we prove two strong convergence theorems for finite mappings of asymptotically nonexpansive in the intermediate sense.

Lemma 3.1 *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X. Let $T_1, T_2, \dots, T_N$ be finite mappings of asymptotically nonexpansive in the intermediate sense with $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Put*

$$G_n^{(i)} = \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \quad \forall n \ge 1,$$

so that $\sum_{n=1}^{\infty} G_n^{(i)} < \infty$ *for all* $i \in \{1, 2, \dots, N\}$. *Let* $\{x_n\}$ *be the sequence defined by (1.1) with the following restrictions :*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1, 2, 3, \dots, N\}$ *and for all* $n \geq 1$*;*

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i \in \{1, 2, 3, \dots, N\}$.

If $p \in F$*, then* $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. Let $p \in F$ for each $n \geq 1$, we note that

$$\begin{aligned}
\|x_n^{(1)} - p\| &\leq \alpha_n^{(1)}\|T_1^n x_n - p\| + \beta_n^{(1)}\|x_n - p\| + \gamma_n^{(1)}\|u_n^{(1)} - p\| \\
&\leq \alpha_n^{(1)}(\|x_n - p\| + \|T_1^n x_n - p\| - \|x_n - p\|) + \beta_n^{(1)}\|x_n - p\| \\
&\quad + \gamma_n^{(1)}\|u_n^{(1)} - p\| \\
&\leq \alpha_n^{(1)}\|x_n - p\| + \alpha_n^{(1)} G_n^{(1)} + \beta_n^{(1)}\|x_n - p\| + \gamma_n^{(1)}\|u_n^{(1)} - p\| \\
&\leq (\alpha_n^{(1)} + \beta_n^{(1)})\|x_n - p\| + d_n^{(1)} \\
&\leq \|x_n - p\| + d_n^{(1)},
\end{aligned} \tag{3.1}$$

where $d_n^{(1)} = \alpha_n^{(1)} G_n^{(1)} + \gamma_n^{(1)}\|u_n^{(1)} - p\|$. Since $\sum_{n=1}^{\infty} G_n^{(1)} < \infty$, we see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (3.1) that

$$\begin{aligned}
\|x_n^{(2)} - p\| &\leq \alpha_n^{(2)}\|T_2^n x_n^{(1)} - p\| + \beta_n^{(2)}\|x_n - p\| + \gamma_n^{(2)}\|u_n^{(2)} - p\| \\
&\leq \alpha_n^{(2)}(\|x_n^{(1)} - p\| + \|T_2^n x_n^{(1)} - p\| - \|x_n^{(1)} - p\|) + \beta_n^{(2)}\|x_n - p\| \\
&\quad + \gamma_n^{(2)}\|u_n^{(2)} - p\| \\
&\leq \alpha_n^{(2)}\|x_n^{(1)} - p\| + \alpha_n^{(2)} G_n^{(2)} + \beta_n^{(2)}\|x_n - p\| + \gamma_n^{(2)}\|u_n^{(2)} - p\| \\
&\leq \alpha_n^{(2)}(\|x_n - p\| + d_n^{(1)}) + \alpha_n^{(2)} G_n^{(2)} + \beta_n^{(2)}\|x_n - p\| + \gamma_n^{(2)}\|u_n^{(2)} - p\| \\
&\leq (\alpha_n^{(2)} + \beta_n^{(2)})\|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_n^{(2)} + \gamma_n^{(2)}\|u_n^{(2)} - p\| \\
&\leq \|x_n - p\| + d_n^{(2)}
\end{aligned} \tag{3.2}$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_n^{(2)} + \gamma_n^{(2)}\|u_n^{(2)} - p\|$. Since $\sum_{n=1}^{\infty} G_n^{(2)} < \infty$ and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we see that

$$\begin{aligned}
\|x_n^{(3)} - p\| &\leq \alpha_n^{(3)}\|T_3^n x_n^{(2)} - p\| + \beta_n^{(3)}\|x_n - p\| + \gamma_n^{(3)}\|u_n^{(3)} - p\| \\
&\leq \alpha_n^{(3)}(\|x_n^{(2)} - p\| + \|T_3^n x_n^{(2)} - p\| - \|x_n^{(2)} - p\|) + \beta_n^{(3)}\|x_n - p\| \\
&\quad + \gamma_n^{(3)}\|u_n^{(3)} - p\| \\
&\leq \alpha_n^{(3)}\|x_n^{(2)} - p\| + \alpha_n^{(3)} G_n^{(3)} + \beta_n^{(3)}\|x_n - p\| + \gamma_n^{(3)}\|u_n^{(3)} - p\| \\
&\leq \alpha_n^{(3)}(\|x_n - p\| + d_n^{(2)}) + \alpha_n^{(3)} G_n^{(3)} + \beta_n^{(3)}\|x_n - p\| + \gamma_n^{(3)}\|u_n^{(3)} - p\| \\
&\leq (\alpha_n^{(3)} + \beta_n^{(3)})\|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} + \alpha_n^{(3)} G_n^{(3)} + \gamma_n^{(3)}\|u_n^{(3)} - p\| \\
&\leq \|x_n - p\| + d_n^{(3)},
\end{aligned} \tag{3.3}$$

where $d_n^{(3)} = \alpha_n^{(3)} d_n^{(2)} + \alpha_n^{(3)} G_n^{(3)} + \gamma_n^{(3)} \|u_n^{(3)} - p\|$. So that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuing the above method, there are nonnegative real sequence $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(k)} < \infty$ and

$$\|x_n^{(k)} - p\| \le \|x_n - p\| + d_n^{(k)}, \tag{3.4}$$

for all $k = 1, 2, \dots, N$ and for all $n \in \mathbb{N}$. This together with Lemma 2.2 give that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2 *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X. Let $T_1, T_2, \dots, T_N$ be finite mappings of asymptotically nonexpansive in the intermediate sense with $F = \cap_{i=1}^{N} F(T_i) \neq \phi$. Put*

$$G_n^{(i)} = \sup_{x,y\in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \quad \forall n \ge 1,$$

so that $\sum_{n=1}^{\infty} G_n^{(i)} < \infty$ for all $i \in \{1, 2, \dots, N\}$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma 3.1 for each $i \in \{1, 2, \dots, N\}$ and the additional assumption that $0 < \alpha \le \alpha_n^{(i)} \le \beta < 1$ for all $n \ge n_0$, for some $n_0 \in \mathbb{N}$. Then $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \dots, N$.

Proof. For any $p \in F$, it follows from Lemma 3.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \ge 0$. We note that

$$\|x_n^{(N-1)} - p\| \le \|x_n - p\| + d_n^{(N-1)}, \quad \forall n \ge 1$$

where $\{d_n^{(N-1)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(N-1)} < \infty$. It follows that

$$\limsup_{n\to\infty} \|x_n^{(N-1)} - p\| \le \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = c,$$

from which we have

$$\begin{aligned}\limsup_{n\to\infty} \|T_N^n x_n^{(N-1)} - p\| &\le \limsup_{n\to\infty} (\|x_n^{(N-1)} - p\| + G_n^{(N)}) \\ &= \limsup_{n\to\infty} (\|x_n^{(N-1)} - p\|) \le c.\end{aligned}$$

Next, we observe that

$$\|T_N^n x_n^{(N-1)} - p + \frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}}(u_n^{(N)} - p)\| \le \|T_N^n x_n^{(N-1)} - p\| + \|\frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}}(u_n^{(N)} - p)\|.$$

Thus we have

$$\limsup_{n\to\infty} \|T_N^n x_n^{(N-1)} - p + \frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}}(u_n^{(N)} - p)\| \le c. \tag{3.5}$$

Also,

$$\|x_n - p + \frac{\gamma_n^{(N)}}{2\beta_n^{(N)}}(u_n^{(N)} - p)\| \leq \|x_n - p\| + \|\frac{\gamma_n^{(N)}}{2\beta_n^{(N)}}(u_n^{(N)} - p)\|,$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \frac{\gamma_n^{(N)}}{2\beta_n^{(N)}}(u_n^{(N)} - p)\| \leq c, \tag{3.6}$$

and note that

$$\begin{aligned}
c = \lim_{n\to\infty} \|x_n^{(N)} - p\| &= \lim_{n\to\infty} \|\alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)} - p\| \\
&= \lim_{n\to\infty} \|\alpha_n^{(N)}[T_N^n x_n^{(N-1)} - p + \frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}}(u_n^{(N)} - p)] \\
&\quad + \beta_n^{(N)}[x_n - p + \frac{\gamma_n^{(N)}}{2\beta_n^{(N)}}(u_n^{(N)} - p)]\| \\
&= \lim_{n\to\infty} \|\alpha_n^{(N)}[T_N^n x_n^{(N-1)} - p + \frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}}(u_n^{(N)} - p)] \\
&\quad + (1 - \alpha_n^{(N)})[x_n - p + \frac{\gamma_n^{(N)}}{2\beta_n^{(N)}}(u_n^{(N)} - p)]\|.
\end{aligned}$$

This together with (3.5), (3.6) and Lemma 2.1, give

$$\lim_{n\to\infty} \|T_N^n x_n^{(N-1)} - x_n + (\frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}} - \frac{\gamma_n^{(N)}}{2\beta_n^{(N)}})(u_n^{(N)} - p)\| = 0.$$

Since $\lim_{n\to\infty} \|(\frac{\gamma_n^{(N)}}{2\alpha_n^{(N)}} - \frac{\gamma_n^{(N)}}{2\beta_n^{(N)}})(u_n^{(N)} - p)\| = 0$, it follows that

$$\lim_{n\to\infty} \|T_N^n x_n^{(N-1)} - x_n\| = 0.$$

Moreover, for each $n \geq 1$, we note that

$$\begin{aligned}
\|x_n - p\| &\leq \|x_n - T_N^n x_n^{(N-1)}\| + \|T_N^n x_n^{(N-1)} - p\| \\
&\leq \|x_n - T_N x_n^{(N-1)}\| + \|x_n^{(N-1)} - p\| + G_n^{(N-1)}.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - T_N x_n^{(N-1)}\| = 0 = \lim_{n\to\infty} G_n^{(N-1)}$, we obtain that

$$c = \lim_{n\to\infty} \|x_n - p\| \leq \liminf_{n\to\infty} \|x_n^{(N-1)} - p\| \leq \limsup_{n\to\infty} \|x_n^{(N-1)} - p\| \leq c.$$

This implies that

$$\lim_{n\to\infty} \|x_n^{(N-1)} - p\| = c.$$

On the other hand, we note that

$$\|x_n^{(N-2)} - p\| \leq \|x_n - p\| + d_n^{(N-2)}, \quad \forall n \geq 1,$$

where $\{d_n^{(N-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(N-2)} < \infty$. So that

$$\limsup_{n\to\infty} \|x_n^{(N-2)} - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c,$$

and hence

$$\limsup_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - p\| \le \limsup_{n\to\infty} (\|x_n^{(N-2)} - p\| + G_n^{(N-2)}) \le c.$$

Next, we observe that

$$\begin{aligned}\|T_{N-1}^n x_n^{(N-2)} - p + \frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)\| &\le \|T_{N-1}^n x_n^{(N-2)} - p\| \\ &\quad + \|\frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)\|.\end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - p + \frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)\| \le c. \tag{3.7}$$

Also,

$$\|x_n - p + \frac{\gamma_n^{(N-1)}}{2\beta_n^{(N-1)}}(u_n^{(N-1)} - p)\| \le \|x_n - p\| + \|\frac{\gamma_n^{(N-1)}}{2\beta_n^{(N-1)}}(u_n^{(N-1)} - p)\|,$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \frac{\gamma_n^{(N-1)}}{2\beta_n^{(N-1)}}(u_n^{(N-1)} - p)\| \le c, \tag{3.8}$$

and note that

$$\begin{aligned} c = \lim_{n\to\infty} \|x_n^{(N-1)} - p\| &= \lim_{n\to\infty} \|\alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} x_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}[T_{N-1}^n x_n^{(N-2)} - p + \frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)] \\ &\quad + \beta_n^{(N-1)}[x_n - p + \frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)]\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}[T_{N-1}^n x_n^{(N-2)} - p + \frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)] \\ &\quad + (1 - \alpha_n^{(N-1)})[x_n - p + \frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}(u_n^{(N-1)} - p)]\|. \end{aligned}$$

This together with (3.7), (3.8) and Lemma 2.1, give

$$\lim_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - x_n + (\frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}} - \frac{\gamma_n^{(N-1)}}{2\beta_n^{(N-1)}})(u_n^{(N-1)} - p)\| = 0.$$

Since

$$\lim_{n\to\infty}\left\|\left(\frac{\gamma_n^{(N-1)}}{2\alpha_n^{(N-1)}}-\frac{\gamma_n^{(N-1)}}{2\beta_n^{(N-1)}}\right)(u_n^{(N-1)}-p)\right\|=0,$$

it follows that

$$\lim_{n\to\infty}\|T_{N-1}^n x_n^{(N-2)}-x_n\|=0.$$

Similarly, as in the proof above we can show that

$$\begin{aligned}\lim_{n\to\infty}\|T_{N-2}^n x_n^{(N-3)}-x_n\| &= \lim_{n\to\infty}\|T_{N-3}^n x_n^{(N-4)}-x_n\|\\ &\vdots\\ &= \lim_{n\to\infty}\|T_1^n x_n^{(1)}-x_n\|\\ &= \lim_{n\to\infty}\|T_1^n x_n-x_n\|=0.\end{aligned}$$

For any $i\in\{2,3,\dots,N\}$, we note that

$$\begin{aligned}\|T_i^n x_n-x_n\| &\le \|T_i^n x_n-T_i^n x_n^{(i-1)}\|+\|T_i^n x_n^{(i-1)}-x_n\|\\ &\le \|x_n-x_n^{(i-1)}\|+G_n^{(i)}+\|T_i^n x_n^{(i-1)}-x_n\|\\ &\le \alpha_n^{(i-1)}\|x_n-T_i^n x_n^{(i-1)}\|+G_n^{(i)}+\gamma_n^{(i-1)}\|u_n^{(i-1)}-x_n\|\\ &\quad+\|T_i^n x_n^{(i-1)}-x_n\|\to 0,\ \text{ as } n\to\infty\end{aligned}\tag{3.9}$$

and

$$\|x_{n+1}-x_n\|\le \alpha_n^{(N)}\|T_N x_n^{(i-1)}-x_n\|+\gamma_n^{(N)}\|u_n^{(N)}-x_n\|\to 0\tag{3.10}$$

as $n\to\infty$. Since

$$\begin{aligned}\|T_i x_n-x_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-T_i^{n+1}x_{n+1}\|\\ &\quad+\|T_i^{n+1}x_{n+1}-T_i^{n+1}x_n\|+\|T_i^{n+1}x_n-T_i x_n\|,\end{aligned}$$

it follows from (3.9), (3.10) and uniformly continuity of T_i that

$$\lim_{n\to\infty}\|T_i x_n-x_n\|=0,\ \text{ for all }\quad i=1,2,\dots,N.\tag{3.11}$$

This completes the proof. □

Theorem 3.3 *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X. Let $T_1,T_2,\dots,T_N$ be finite mappings of asymptotically nonexpansive in the intermediate sense with $F=\cap_{i=1}^N F(T_i)\neq\phi$ and satisfying condition (B). Put*

$$G_n^{(i)}=\sup_{x,y\in C}(\|T_i^n x-T_i^n y\|-\|x-y\|)\vee 0,\quad \forall n\ge 1,$$

so that $\sum_{n=1}^{\infty} G_n^{(i)} < \infty$. *Let the sequence* $\{x_n\}$ *be defined by (1.1) whenever* $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ *satisfy the same assumptions as in Lemma 3.1 and the additional assumption that* $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$ *for each* $i \in \{1, 2, 3, \ldots, N\}$ *for all* $n \geq n_0$, *for some* $n_0 \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to a common fixed point in* F.

Proof. It follows from Lemma 3.2 that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \;\; for\ all\ i = 1, 2, ..., N.$$

Now by the condition (B), there exists $f : [0, \infty) \to [0, \infty)$ is nondecreasing function such that

$$\max_{1\leq i\leq N} \{\|T_i x_n - x_n\|\} \geq f(d(x_n, F)).$$

Then $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing function and $f(0) = 0$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.12}$$

We next show that $\{x_n\}$ is a cauchy sequence. Let $\epsilon > 0$. By (3.4), we obtain

$$\|x_{n+1} - p\| \leq \|x_n - p\| + d_n^{(N)}, \quad \forall p \in F, \ \forall\ n \in \mathbb{N}.$$

Thus, we note that

$$\|x_{n+m} - p\| \leq \|x_{n+(m-1)} - p\| + d_{n+(m-1)}^{(N)}, \quad \forall p \in F.$$

By (3.12) and $\sum_{n=1}^{\infty} d_n^{(N)} < \infty$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, F) < \frac{\epsilon}{3} \;\; \text{and} \;\; \sum_{n=1}^{\infty} d_n^{(N)} < \frac{\epsilon}{3},$$

for all $n \geq N_1$. Let $n \geq N_1$. Then there exists $p_1 \in F$ such that $\|x_n - p_1\| < \frac{\epsilon}{3}$. Hence for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&\leq \|x_{n+(m-1)} - p_1\| + \|x_n - p_1\| + d_{n+(m-1)}^{(N)} \\
&\leq \|x_n - p_1\| + \|x_n - p\| + \Sigma_{i=1}^{m} d_{n+(m-i)}^{(N)} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus, we have $\{x_n\}$ is a cauchy sequence. Since X is complete, it follows that $\{x_n\}$ is converges, that is $x_n \to p$ as $n \to \infty$ for some $p \in X$. We now show that $p \in F$. For $i \in \{1, 2, ..., N\}$, we have

$$\|T_i p - p\| \leq \|T_i p - T_i x_n\| + \|T_i x_n - x_n\| + \|x_n - p\|,$$

it follows from Lemma 3.2 and uniform continuous of T_i that

$$\|T_ip - p\| = 0 \quad \text{as } n \to \infty.$$

Thus p is common fixed point in F. This completes the proof. □

Theorem 3.4 *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X. Let $T_1, T_2, \ldots, T_N$ be finite mappings of asymptotically nonexpansive in the intermediate sense with $F = \cap_{i=1}^{N} F(T_i) \neq \phi$ and one of mappings in $\{T_i : i = 1, 2, \ldots, N\}$ is $semi-compact$. Put*

$$G_n^{(i)} = \sup_{x,y\in C} (\|T_i^n x - T_i^n y\| - \|x - y\|) \vee 0, \quad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n^{(i)} < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma 3.1 for each $i \in \{1, 2, 3, \ldots, N\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point in F.

Proof. Suppose that T_{i_0} is *semi − compact* for some $i_0 = 1, 2, \ldots, N$. By Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - T_{i_0}x_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$ as $j \to \infty$. Now Lemma 3.2 guarantees that $\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l = 1, 2, \ldots, N$. These imply $\|p - T_l p\| = 0$ for all $l = 1, 2, \ldots, N$. This implies that $p \in F$. By Lemma 3.1 $\lim_{n\to\infty} \|x_n - p\|$ exists and then $\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_{n_j} - p\| = 0$. This completes the proof. □

If $T_1 = T_2 = \cdots = T_N = T$, then we obtain the following result.

Theorem 3.5 *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X. Let $T : C \longrightarrow C$ be a mappings of completely continuous asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \quad \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.2) whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma 3.1 for each $i \in \{1, 2, ..., N\}$ and the additional assumption that $0 < \alpha \leq \alpha_n^{(N-1)}, \alpha_n^{(N)} \leq \beta < 1$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T.

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