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Algorithm for system of equilibrium problems and Bregman totally asymptotically quasi-nonexpansive mappings in Banach spaces

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Abstract : In this paper, we introduce a new algorithm for finding common fixed points of two Bregman totally asymptotically quasi-nonexpansive mappings together with the solutions of equilibrium problems in Banach spaces. Moreover, we prove some strong convergence theorems under suitable control conditions. Our results extend and improve the recent ones of some others in the literature.

Keywords : Bregman totally asymptotically quai-nonexpansive; system of equilibrium problems; Banach space
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1 Introduction

In 1967, Bregman [10] discovered an effective technique for using of the so-called Bregman distance function $D_f(\cdot)$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique was applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed point of nonlinear mapping.

On the framework of a reflexive Banach space E, let C be a nonempty, closed and convex subset of E. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

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for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

Let C be a nonempty, closed and convex subset of int(dom f) and $T : C \to C$ be a mapping. We said that a point p in C is an *asymptotic fixed point of* T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotically fixed points of T is denoted by $\widehat{F}(T)$. The mapping T is said to be *closed* if $x_n \to x$ and $Tx_n \to y$ implies $Tx = y$.

Definition 1.1. *[10] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function D_f : dom $f\times$ int(dom f) $\to [0, +\infty)$ defined by*

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle$$

is called Bregman distance *with respect to f.*

By the definition, we know the following properties: the *three point identity*, for any $x \in$ dom f and $y, z \in$ int (dom f),

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle;$$

the *four point identity*, for any $y, w \in$ dom f and $x, z \in$ int(dom f),

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w\rangle.$$

Next, we will recall the necessary notation of the nonlinear mapping related to Bregman distance as shown in the following:

(1) T is called *Bregman quasi-nonexpansive* [6] if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

(2) T is called *Bregman strongly nonexpansive* (BSNE for short) (see [6]) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x)$$

for all $p \in \widehat{F}(T)$ and $x \in C$, and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and $\lim_{n\to+\infty}(D_f(p, x_n) - D_f(p, Tx_n)) = 0$, it follows that $\lim_{n\to+\infty} D_f(Tx_n, x_n) = 0$.

(3) T is called *Bregman relatively nonexpansive* if $\widehat{F}(T) = F(T)$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T).$$

(4) T is called *Bregman firmly nonexpansive* (BFNE for short) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \forall x, y \in C,$$

or, equivalently

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x), \forall x, y \in C.$$

(5) T is called *Bregman asymptotically quasi-nonexpansive* it there exists a sequence $\{k_n\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ such that for every $n \geq 1$,

$$D_f(p, T^n x) \leq k_n D_f(p, x), \forall x \in C, p \in F(T). \tag{1.1}$$

(6) T is said to be Bregman totally asymptotically quasi-nonexpasive, if $F(T) \neq \emptyset$ and there exist non-negative real sequences $\{v_n\}, \{\mu_n\}$ with $v_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$D_f(p, T^n x) \leq D_f(p, x) + v_n \zeta(D_f(p, x)) + \mu_n, \forall n \geq 1, \forall x \in C, p \in F(T).$$

Remark 1.2. *According to the definitions, it is obvious that*

(1) each Bregman relatively nonexpansive mapping is Bregman quasi-nonexpansive mapping;

(2) each Bregman asymptotically nquasi-onexpansive mapping is Bregman totally asymptotically quasi-nonexpansive mapping, but the converse may be not true. If taking, $\zeta(t) = t, t \geq 0, v_n = k_n - 1$ *and* $\mu_n = 0$, *then* (1.1) *can be rewritten as*

$$D_f(p, T^n x) \leq D_f(p, x) + v_n \zeta(D_f(p, x)) + \mu_n, \forall n \geq 1, \forall x \in C, p \in F(T).$$

Let C be a nonempty subset of Banach space E. The mapping $T : C \to C$ is said to be *uniformly asymptotically regular* on C if

$$\lim_{n \to \infty} (\sup_{x \in C} \|T^{n+1}x - T^n x\|) = 0.$$

Let E^* be the dual space of E, the norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. Denote the domain of f by dom f, that is, dom $f = \{x \in E : f(x) < +\infty\}$. The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x \rangle - f(x) : x \in E\}.$$

Let $H : C \times C \to \mathbb{R}$ be a bifunction, the *equilibrium problem for* H, denoted by $EP(H)$, is to find $u \in C$ such that

$$H(u, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

In 2008, Takahashi and Zembayashi [12] introduced the following shrinking projection method of closed relatively nonexpansive mappings as follow:

$$\begin{cases} x_0 = x \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) JS(x_n)), \\ u_n \in C \ \text{ such that } \ H(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \end{cases} \tag{1.3}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, they shows the proof which guarantee that their defined sequence $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap EP(H)} x$.

In 2010, Reich and Sabach [6] presented the following algorithm for Bregman strongly nonexpansive mapping T_i in reflexive Banach space E, the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in E, \\ y_n^i = T_i(v_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n := \cap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle z - x_n, \nabla f(x_0) - \nabla f(x_n) \rangle \leq 0\}, \\ x_{n+1} = proj_{C_n \cap Q_n}^f (x_0), n = 0, 1, 2, \ldots, \end{cases} \tag{1.4}$$

and

$$
\begin{array}{l}
x_0 \in E, \\
C_0^i = E, i = 1, 2, \ldots, N, \\
y_n^i = T_i(v_n + e_n^i), \\
C_{n+1}^i = \{z \in C_n^i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\
C_{n+1} := \cap_{i=1}^N C_{n+1}^i, \\
x_{n+1} = proj_{C_{n+1}}^f(x_0), n = 0, 1, 2, \ldots,
\end{array}
\tag{1.5}
$$

where $proj_K^f$ is Bregman projection with respect to f from E onto a closed and convex subset K of E. They prove that $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Recently, Chen et al. [5] devoted to investigate the shrinking projection method for finding common element of solutions to the equilibrium problem and fixed point problems in Banach spaces,

$$
\begin{array}{l}
z_n = \nabla f^*(\beta_n \nabla f(T(x_n)) + (1 - \beta_n)\nabla f(x_n)), \\
y_n = \nabla f^*(\alpha_n \nabla f(x_0) + (1 - \alpha_n)\nabla f(z_n)), \\
u_n = Res_H^f(y_n), \\
C_n^i = \{z \in C_{n-1} \cap Q_{n+1} : D_f(z, u_n) \leq \alpha_n D_f(z, x_0) + (1 - \alpha_n) D_f(z, x_n)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\
x_{n+1} = proj_{C_n \cap Q_n}^f(x_0), \forall n \geq 0.
\end{array}
\tag{1.6}
$$

They show that the sequence $\{x_n\}$ converges storngly to the point $proj_{EP(H) \cap F(T)} x_0$.

Moreover, in 2011, Cholamjiak et al. [7] presented their results on the convergence investigation of the following scheme in the framework of a real reflexive Banach space:

$$
\begin{array}{l}
x_1 \in E, \\
C_1 \in E \\
y_n = Res_{\lambda_n^N A_N}^f Res_{\lambda_n^{N-1} A_{N-1}}^f \ldots Res_{\lambda_n^1 A_1}^f (x_n + e_n), \\
C_{n+1} = \{z \in C_n : D_f(z, y_n) \leq D_f(z, x_n + e_n), \\
x_{n+1} = proj_{C_{n+1}}^f(x_1), \forall n \geq 1.
\end{array}
\tag{1.7}
$$

Their proof can claim that the above defined sequence $\{x_n\}$ converges strongly to a point $P_F^f(x_1)$.

In this paper, motivated and inspired by Reich and Sabach [6], Chen et al. [5] and Witthayarat et al. [1], we introduce the new algorithm defined by:

$$
\begin{array}{l}
x_1 = u \in C \ \text{ chosen arbitrarily}, \\
y_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n - \beta_n)\nabla f(x_n) + \beta_n \nabla f(S^n(x_n))), \\
u_n = Res_{H_N}^f Res_{H_{N-1}}^f \ldots Res_{H_2}^f Res_{H_1}^f (T^n y_n), \\
C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) + \xi_n\}, \\
x_{n+1} = proj_{C_{n+1}}^f x, \quad \forall n \geq 1,
\end{array}
\tag{1.8}
$$

where T, S be two closed Bregman totally asymptotically quasi-nonexpansive mappings. Under appropriate difference conditions, we will prove that the sequence $\{x_n\}$ generated by algorithms (1.8) converges strongly to the point $proj_\Omega^f u$, $\Omega = F(T) \cap F(S) \cap (\cap_{k=1}^N EP(H_k))$.

2 Preliminaries

Let E be a real Banach space. For any $x \in \text{int}(\text{dom } f)$, the *right-hand derivative* of f at x in the direction $y \in E$ is defined by

$$f'(x,y) := \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

The function f is called *Gâteaux differentiable at* x if $\lim_{t \to 0^+} \dfrac{f(x+ty) - f(x)}{t}$ exists for all $y \in E$. In this case, $f'(x,y)$ coincides with the value of the *gradient* (∇f) of f at x. Furthermore, if f is Gâteaux differentiable for any $x \in \text{int}(\text{dom } f)$, we can said that f is *Gâteaux differentiable.* f is called *Fréchet differentiable* at x if this limit is attained uniformly for $\|y\| = 1$. Moreover, f is *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let E be a reflexive Banach space. The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [8]). The function f is *Legendre function* if and only if it satisfies the following conditions:

(L1) The interior of the domain of f denoted by $\text{int}(\text{dom } f)$, is nonempty, f^* is Gâteaux differentiable on $\text{int}(\text{dom } f)$ and $\text{dom } f = \text{int}(\text{dom } f)$;

(L2) The interior of the domain f^* denote by $\text{int}(\text{dom } f^*)$, is nonempty, f^* is Gâteaux differentiable on $\text{int}(\text{dom } f^*)$ and $\text{dom } f^* = \text{int}(\text{dom } f^*)$.

Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see [9]). This, by (L1) and (L2), implies

$$\nabla f = (\nabla f)^{-1}, \quad \text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom} f^*)$$

and

$$\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom} f).$$

By [8], the condition (L1) and (L2) also yield that the functions f and f^* are strictly convex on their respective domains. From now on we assume that the function $f : E \to (-\infty, +\infty]$ is Legendre.

Definition 2.1. *[10] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in$ int(dom f) onto the nonempty, closed and convex subset $C \subset$ dom f is the necessarily unique vector $proj_C^f(x) \in C$ satisfying*

$$D_f(proj_C^f(x), x) = inf\{D_f(y,x) : y \in C\}.$$

Remark 2.2. *[5]*

(1) If E is a Hilbert space and $f(y) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the Bregman projection $proj_C^f(x)$ is reduced to the metric projection of x onto C;

(2) If C is a smooth Banach space and $f(y) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the Bregman projection $proj_C^f(x)$ is reduced to the generalized projection $\Pi_C(x)$, which is defined by

$$\phi(\Pi_C(x), x) = min_{y \in C} \phi(y,x),$$

where $\phi(y,x) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2$, J is the normalized duality mapping from $E \to 2^{E^*}$.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function f is called:

(1) *totally convex* at $x \in \text{int}(\text{dom } f)$ if its modulus of total convexity at x, that is, the function ν_f : $\text{int}(\text{dom } f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \text{dom} f, \|y - x\| = t\},$$

is positive whenever $t > 0$;

(2) *totally convex* if it is totally convex at every point $x \in \text{int}(\text{dom } f)$;

(3) *totally convex on bounded sets* if $\nu_f(B,t)$ is positive for any nonempty bounded subset B is the function $\nu_f : \text{int}(\text{dom } f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \text{dom} f\}.$$

Definition 2.3. *[3, 6] The function $f : E \to (\infty, +\infty)$ is called:*

(1) cofinite *if dom $f^* = E^*$;*

(2) coercive *if $\lim_{\|x\| \to +\infty}(f(x)/\|x\|) = +\infty$;*

(3) sequentially consistent *if for any two sequence $\{x_n\}$ and $\{yn\}$ in E such that $\{x_n\}$ is bounded,*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.4. *[6] If $f : E \to (\infty, +\infty)$ is Fréchet differentiable totally convex, then f is cofinite.*

Lemma 2.5. *[4] Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then, the following statements hold:*

(1) f is sequence consistent if and only if it is totally convex on bounded sets,

(2) If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets,

(3) If f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain, and the Fréchet derivative ∇f is uniformly continuous on bounded sets.

Lemma 2.6. *[14] If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subset of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.7. *[3] The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.8. *[6] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 2.9. *[4] Suppose that f is Gâteaux differentiable and totally convex on int(dom f). Let $x \in$ int dom f and C be a nonempty, closed and vonvex subset of int(dom f). If $\hat{x} \in C$, then the following conditions are equivalent:*

(1) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f;

(2) The vector $\hat{x}$ is the unique solution of the variational inequality:

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \quad \forall y \in C;$$

(3) The vector $\hat{x}$ is the unique solution of the inequality:

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in C.$$

Lemma 2.10. *[11] Let E be a real reflexive Banach space and C be a nonempty, closed and convex subset of E and $f : X \to (-\infty, +\infty]$ be a Legendre function which is total convex on bounded subsets of E. Let $T : C \to C$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous functions $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$, then the fixed point set $F(T)$ of T is a closed and convex subset of C.*

For solving the equilibrium problem, let us assume that the bifunction $H : C \times C \to \mathbb{R}$ is convex and lower semi-continuous satisfies the following conditions:

(A1) $H(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $H(x, y) + H(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \to 0} H(tz + (1 - t)x, y) \leq H(x, y);$$

(A4) for each $x \in C$, $H(x, \cdot)$ is convex and lower semi-continuous.

Lemma 2.11. *[14] Let C be a closed and convex subset of E. Let H be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$, let $f : E \to (-\infty, +\infty)$ be a coercive and Gâtraux differentiable function and $x \in E$, then there exists $z \in C$ such that*

$$H(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

Define the operator $Res_H^f : E \to 2^C$ as follows:

$$Res_H^f(x) = \{z \in C : H(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \quad \forall y \in C\}, \tag{2.2}$$

for all $x \in E$. Then, the followings hold:

(1) Res_H^f is single-valued;

(2) Res_H^f is BFNE operator;

(3) $F(Res_H^f) = EP(H)$;

(4) $EP(H)$ is closed and convex;

(5) $D_f(p, Res_H^f x) + D_f(Res_H^f x, x) \leq D_f(p, x)$, $\forall p \in F(Res_H^f)$, $x \in E$.

3 Main Results

Theorem 3.1. *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E, $f : E \to \mathbb{R}$ be a coercive Legengre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E and ∇f^* be bounded on bounded subsets of E^* and $T, S : C \to C$, be two closed Bregman totally asymptotically quasi-nonexpansive mappings with sequence $\{v_n\}, \{\mu_n\}, v_n, \mu_n \to 0$(as $n \to \infty$) and let there be a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$. Let $H : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A4)$. Assume that each T, S are uniformly asymptotically regular and $\Omega = F(T) \cap S(T) \cap (\cap_{k=1}^{N} EP(H_k))$ is nonempty and bounded. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_1 = u \in C \quad \textit{chosen arbitrarily}, \\ y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n-\beta_n)\nabla f(x_n) + \beta_n \nabla f(S^n(x_n)), \\ u_n = Res^f_{H_N} Res^f_{H_{N-1}} \dots Res^f_{H_2} Res^f_{H_1}(T^n y_n), \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \le D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = proj^f_{C_{n+1}} u, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\limsup_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\xi_n = 2l_n,\ l_n = v_n \sup_{p\in\Omega} \zeta(D_f(p, x_n)) + \mu_n$. Then, the sequence $\{x_n\}$ converges strongly to $proj^j_\Omega u$, where $proj^j_\Omega u$ is the Bregman projection of C into Ω.

***Proof*.** Firstly, we show that C_n is closed and convex for all $n \ge 1$. Note that

$$D_f(z, u_n) \le D_f(z, x_n) + \xi_n$$

is

$$f(z) - f(u_n) - \langle \nabla f(u_n), z - u_n\rangle \le f(z) - f(x_n) - \langle \nabla f(x_n), z - x_n\rangle + \xi_n$$

that is

$$\langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(u_n), z - u_n\rangle \le f(u_n) - f(x_n) + \xi_n.$$

It clearly shows that C_n is closed and convex for all $n \ge 1$.

Next, we show that $\Omega \subset C_n$ for all $n \ge 1$. For any given $p \in \Omega := F(T) \cap S(T) \cap (\cap_{k=1}^{N} EP(H_k))$ and let $\Theta^f_j = Res^f_{H_j} Res^f_{H_{j-1}} \dots Res^f_{H_2} Res^f_{H_1}$, $j = 1, 2, \dots, N$ and $\Theta^f_0 = I$. We note that $u_n = \Theta^f_N T^n y_n$. From (3.1), we have

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n-\beta_n)\nabla f(x_n)) + \beta_n \nabla f(S(x_n))) \\ &\le \alpha_n D_f(p, u) + (1-\alpha_n-\beta_n) D_f(p, x_n) + \beta_n D_f(p, S(x_n)) \\ &\le \alpha_n D_f(p, u) + (1-\alpha_n-\beta_n) D_f(p, x_n) + \beta_n (D_f(p, x_n) + v_n \zeta(D_f(p, x_n)) + \mu_n) \\ &\le \alpha_n D_f(p, u) + (1-\alpha_n-\beta_n) D_f(p, x_n) + \beta_n D_f(p, x_n) + v_n \sup_{p\in\Omega} \zeta(D_f(p, x_n)) + \mu_n \\ &= \alpha_n D_f(p, u) + (1-\alpha_n-\beta_n) D_f(p, x_n) + \beta_n D_f(p, x_n)) + l_n \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) + l_n, \end{aligned} \tag{3.2}$$

where $l_n = v_n \sup_{p\in\Omega} \zeta(D_f(p, x_n)) + \mu_n$. From (3.1), (3.2) and Lemma 2.11, we note that

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \Theta_N^f T^n y_n) \\ &\leq D_f(p, T^n y_n) \\ &\leq D_f(p, y_n) + v_n \zeta(D_f(p, y_n)) + \mu_n \\ &\leq D_f(p, y_n) + v_n \sup_{p\in\Omega} \zeta(D_f(p, y_n)) + \mu_n \\ &= D_f(p, y_n) + l_n \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) + l_n + l_n \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) + \xi_n, \end{aligned} \tag{3.3}$$

where $\xi_n = 2l_n$. Hence, we have

$$D_f(p, u_n) \leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) + \xi_n. \tag{3.4}$$

Thus, we have $p \in C_n$ for all $n \geq 1$. That is, $p \in C_n$ for all $n \geq 1$. This implies that $\Omega \in C_n$ for all $n \geq 1$ and also $\{x_n\}$ is well define.

From $x_{n+1} = proj_{C_n}^f u$, by Lemma 2.9 (3), we have

$$D_f(x_{n+1}, u) = D_f(proj_{C_n}^f u, u) \leq D_f(p, u) - D_f(p, proj_{C_n}^f u) \leq D_f(p, u) \tag{3.5}$$

for all $p \in \Omega$. Then, the sequence $\{D_f(x_n, u)\}$ is also bounded. Thus by Lemma 2.8, the sequence $\{x_n\}$ is bounded.

Since $x_n = proj_{C_n}^f u$ and $x_{n+1} = proj_{C_{n+1}}^f u \in C_{n+1} \subset C_n$, we have

$$D_f(x_n, u) \leq D_f(x_{n+1}, u), \forall n \in \mathbb{N}. \tag{3.6}$$

Therefore, $\{D_f(x_n, u)\}$ is nondecreasing. Hence the limit of $\{D_f(x_n, u)\}$ exists.

By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = proj_{C_m}^f u \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} D_f(x_m, x_n) &= D_f(x_m, proj_{C_n}^f u) \\ &\leq D_f(x_m, u) - D_f(proj_{C_n}^f u, u) \\ &= D_f(x_m, u) - D_f(x_n, u). \end{aligned} \tag{3.7}$$

Letting $m, n \to \infty$ in (3.7), we get $D_f(x_m, x_n) \to 0$. It yields from Lemma 2.7, that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. It means that $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can assume that

$$x_n \to p^* \in C, \quad \text{as} \quad n \to \infty. \tag{3.8}$$

From (3.7), taking $m = n + 1$, we have

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.9}$$

By Lemma 2.7, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Since $x_{n+1} = proj^f_{C_n} u \in C_n$ and by the definition of C_n, we have

$$D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, x_n) + \xi_n.$$

From (3.9), we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0. \tag{3.11}$$

By Lemma 2.7, again

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.12}$$

Taking into account $\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$, we see that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.13}$$

This means that the sequence $\{u_n\}$ is bounded. As f is uniformly Fréchet differentiable on bounded subsets of E, and by Lemma 2.5, ∇f is norm-to-norm uniformly continuous on bounded subsets of E, that

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(x_n)\| = 0. \tag{3.14}$$

Because of f is uniformly Fréchet differentiable, it is also uniformly continuous, we have

$$\lim_{n\to\infty} \|f(u_n) - f(x_n)\| = 0. \tag{3.15}$$

From (3.1) and Lemma 2.9, we have

$$\begin{aligned}
D_f(x_n, y_n) &\leq D_f(p^*, y_n) - D_f(p^*, x_n)\\
&= D_f(p^*, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n-\beta_n)\nabla f(x_n) + \beta_n \nabla f(S^n(x_n))) - D_f(p^*, x_n)\\
&\leq \alpha_n D_f(p^*, u) + (1-\alpha_n-\beta_n) D_f(p^*, x_n) + \beta_n D_f(p^*, S(x_n)) - D_f(p^*, x_n)\\
&\leq \alpha_n D_f(p^*, u) + (1-\alpha_n-\beta_n) D_f(p^*, x_n)\\
&\quad + \beta_n (D_f(p^*, x_n) + v_n \zeta(D_f(p^*, x_n)) + \mu_n) - D_f(p^*, x_n)\\
&= \alpha_n (D_f(p^*, u) - D_f(p^*, x_n)) + \beta_n (v_n \zeta(D_f(p^*, x_n) + \mu_n)).
\end{aligned}$$

Since $v_n, \mu_n \to 0$, we get

$$D_f(x_n, y_n) \to 0. \tag{3.16}$$

By Lemma 2.7 show that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.17}$$

Moreover, we have

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{3.18}$$

By Lemma 2.11, we obtain

$$\begin{aligned} D_f(u_n, T^n x_n) &= D_f(\Theta_N^f T^n y_n - T^n x_n) \\ &\le D_f(p, T^n x_n) - D_f(p, \Theta_N^f T^n y_n) \\ &\le D_f(p, x_n) + \upsilon_n \zeta(D_f(p, x_n)) + \mu_n - D_f(p, u_n). \end{aligned} \tag{3.19}$$

By definition of the Bregman distance, we have

$$\begin{aligned} D_f(p, x_n) - D_f(p, u_n) &= [f(p) - f(x_n) - \langle \nabla f(x_n), p - x_n \rangle] - [f(p) - f(u_n) - \langle \nabla f(u_n), p - u_n \rangle] \\ &= f(u_n) - f(x_n) + \langle \nabla f(u_n), p - u_n \rangle - \langle \nabla f(x_n), p - x_n \rangle \\ &= f(u_n) - f(x_n) + \langle \nabla f(u_n), x_n - u_n \rangle + \langle \nabla f(u_n) - \nabla f(x_n), p - x_n \rangle, \end{aligned}$$

for any $p \in F$. Since every sequence $\{u_n\}$ is bounded, $\{\nabla f(u_n)\}$ is also bounded. From (3.13), (3.14) and (3.15), we obtain

$$\lim_{n\to\infty} [D_f(p, x_n) - D_f(p, u_n)] = 0. \tag{3.20}$$

From (3.19), we have

$$\lim_{n\to\infty} [D_f(u_n, T^n x_n)] = 0. \tag{3.21}$$

By Lemma 2.7, show that,

$$\lim_{n\to\infty} \|u_n - T^n x_n\| = 0, \tag{3.22}$$

and hence

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(T^n x_n)\| = 0. \tag{3.23}$$

Taking into account $\|x_n - T^n x_n\| \le \|x_n - u_n\| + \|u_n - T^n x_n\|$, we obtain

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.24}$$

Note that $\|T^n x_n - p^*\| \le \|T^n x_n - x_n\| + \|x_n - p^*\|$. It follows from (3.8) and (3.24), we have

$$\lim_{n\to\infty} \|T^n x_n - p^*\| = 0, \tag{3.25}$$

Further, we have

$$\|T^{n+1} x_n - p^*\| \le \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - p^*\|.$$

From (3.25) and T is uniformly asymptotically regular, we obtain that

$$\lim_{n\to\infty} \|T^{n+1} x_n - p^*\| \to 0.$$

This implies $TT^n y_n \to p^*$ as $n \to \infty$. From the closedness of T, we have $p^* \in F(T)$.

Further, we consider

$$\begin{aligned} \|\nabla f(x_n) - \nabla f(y_n)\| &= \|\nabla f(x_n) - \nabla f[\nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n-\beta_n)\nabla f(x_n) + \beta_n \nabla f(S^n(x_n)))]\| \\ &= \|\nabla f(x_n) - \alpha_n \nabla f(u) - (1-\alpha_n-\beta_n)\nabla f(x_n) - \beta_n \nabla f(S^n(x_n))\| \\ &= \|\alpha_n(\nabla f(x_n) - \nabla f(u)) + \beta_n(\nabla f(x_n) - \nabla f(S^n(x_n)))\|. \end{aligned}$$

From (3.18), we have

$$\lim_{n\to\infty} \|\alpha_n(\nabla f(x_n) - \nabla f(u)) + \beta_n(\nabla f(x_n) - \nabla f(S^n(x_n)))\| = 0 \tag{3.26}$$

and hence

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(S^n(x_n))\| = 0. \tag{3.27}$$

Since ∇f^* is uniformly continuous on bounded subset of E^* and thus

$$\|x_n - S^n(x_n)\| \to 0, \text{ as } n \to \infty. \tag{3.28}$$

Note that $\|S^n x_n - p^*\| \le \|S^n x_n - x_n\| + \|x_n - p^*\|$, we have

$$\lim_{n\to\infty} \|S^n x_n - p^*\| \to 0, \text{ as } n \to \infty. \tag{3.29}$$

In a similar way, one has $p^* \in F(S)$. Thus $p^* \in F(T) \cap F(S)$.

Next, we prove that $p^* \in \cap_{k=1}^N EP(H_k)$. By $u_n = \Theta_N^f T^n y_n$, we have

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \Theta_N^f T^n y_n) \\ &= D_f(p, Res_{H_N}^f \Theta_{N-1}^f T^n y_n) \\ &\le D_f(p, \Theta_{N-1}^f T^n y_n) \\ &= D_f(p, Res_{H_{N-1}}^f \Theta_{N-2}^f T^n y_n) \\ &\le D_f(p, \Theta_{N-2}^f T^n y_n) \le \ldots \le D_f(p, T^n y_n). \end{aligned} \tag{3.30}$$

Since $p \in EP(H_N)$ that

$$\begin{aligned} D_f(\Theta_N^f T^n y_n, \Theta_{N-1}^f T^n y_n) &= D_f(Res_{H_N}^f \Theta_{N-1}^f T^n y_n, \Theta_{N-1}^f T^n y_n) \\ &\le D_f(p, \Theta_{N-1}^f T^n y_n) - D_f(p, \Theta_N^f T^n y_n) \\ &\le D_f(p, T^n y_n) - D_f(p, u_n) \\ &\le D_f(p, y_n) + v_n \zeta(D_f(p, y_n)) + \mu_n - D_f(p, u_n) \\ &\le D_f(p, y_n) + v_n \sup_{p\in\Omega} \zeta(D_f(p, y_n)) + \mu_n - D_f(p, u_n) \\ &\le D_f(p, y_n) + l_n - D_f(p, u_n) \\ &\le \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) + \xi_n - D_f(p, u_n) \\ &= \alpha_n (D_f(p, u) - D_f(p, x_n)) + D_f(p, x_n) - D_f(p, u_n) + \xi_n \end{aligned}$$

From (3.13), (3.14), (3.15) and (3.20), we have

$$\lim_{n\to\infty} D_f(\Theta_N^f T^n y_n, \Theta_{N-1}^f T^n y_n) = \lim_{n\to\infty} D_f(u_n, \Theta_{N-1}^f T^n y_n) = 0. \tag{3.31}$$

By Lemma 2.7, show that

$$\lim_{n\to\infty} \|\Theta_N^f T^n y_n - \Theta_{N-1}^f T^n y_n\| = \lim_{n\to\infty} \|u_n - \Theta_{N-1}^f T^n y_n\| = 0. \tag{3.32}$$

Since f is uniformly Fréchet differentiable, we have

$$\lim_{n\to\infty} \|\nabla f(\Theta_N^f T^n y_n) - \nabla f(\Theta_{N-1}^f T^n y_n)\| = \lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(\Theta_{N-1}^f T^n y_n)\| = 0. \tag{3.33}$$

Again, since $p \in EP(H_{N-1}) = F(Res^f_{H_{N-1}})$, by Lemma 2.11 and (3.30) that

$$\begin{aligned} D_f(\Theta_{N-1}^f T^n y_n, \Theta_{N-2}^f T^n y_n) &= D_f(Res^f_{H_{N-1}} \Theta_{N-2}^f T^n y_n, \Theta) \\ &\leq D_f(p, \Theta_{N-2}^f T^n y_n) - D_f(p, \Theta_{N-1}^f T^n y_n) \\ &\leq D_f(p, T^n y_n) - D_f(p, u_n) \\ &\leq D_f(p, x_n) - D_f(p, u_n) + \xi_n. \end{aligned}$$

Thus $\lim_{n\to\infty} D_f(\Theta_{N-1}^f T^n y_n, \Theta_{N-2}^f T^n y_n) = 0$. By Lemma 2.7, again

$$\lim_{n\to\infty} \|\Theta_{N-1}^f T^n y_n - \Theta_{N-2}^f T^n y_n\| = 0, \tag{3.34}$$

and

$$\lim_{n\to\infty} \|\nabla f(\Theta_{N-1}^f T^n y_n) - \nabla f(\Theta_{N-2}^f T^n y_n)\| = 0. \tag{3.35}$$

Similarly, we also have

$$\lim_{n\to\infty} \|\Theta_{N-2}^f T^n y_n - \Theta_{N-3}^f T^n y_n\| = \ldots = \lim_{n\to\infty} \|\Theta_1^f T^n y_n - T^n y_n\| = 0. \tag{3.36}$$

Therefore, we can conclude

$$\lim_{n\to\infty} \|\Theta_i^f T^n y_n - \Theta_{i-1}^f T^n y_n\| = 0, i = 1, 2, \ldots, N. \tag{3.37}$$

Note that $\|\Theta_N^f T^n y_n - T^n y_n\| \leq \|\Theta_N^f T^n y_n - \Theta_{N-1}^f T^n y_n\| + \ldots + \|\Theta_1^f T^n y_n - T^n y_n\|$. It follows from (3.37), we have

$$\lim_{n\to\infty} \|\Theta_N^f T^n y_n - T^n y_n\| = \lim_{n\to\infty} \|u_n - T^n y_n\| = 0. \tag{3.38}$$

and

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(T^n y_n)\| = 0. \tag{3.39}$$

By Lemma 2.11, we get

$$H_k(u_n, y) + \langle \nabla f(u_n) - \nabla f(T^n y_n), y - u_n \rangle \geq 0, \forall y \in C.$$

From $(A2)$, we obtain

$$\begin{aligned} \langle \nabla f(u_n) - \nabla f(T^n y_n), y - u_n \rangle &\geq -H_k(u_n, y) \\ &\geq H_k(y, u_n), \forall y \in C. \end{aligned} \tag{3.40}$$

Taking limit as $n \to \infty$ in (3.40) together with conditions $(A4)$ and (3.39), we have

$$H_k(y, p^*) \leq 0, \forall y \in C.$$

For any $y \in C$ and $0 < t < 1$, let $y_t = ty + (1-t)p^*$. Note that $y, p \in C$, that is we can claim that $y_t \in C$ and $H_k(y_t, p^*) \leq 0$.
From $(A1)$,

$$0 \leq H_k(y_t, y_t) \leq tH_k(y_t, y) + (1-t)H_k(y_t, p^*) \leq tH_k(y_t, y).$$

Thus, $H_k(y_t, y) \geq 0$.
Letting $t \to 0$, therefore from $(A3)$, we obtain $\limsup_{t\to 0} H_k(ty+(1-t)p^*, y) \leq H_k(p^*, y)$. That is $H_k(p^*, y) \geq 0$, for all $y \in C$, which implies that $p^* \in EP(H_k), k = 1, 2, \ldots, N$.. Thus, $p^* \in \cap_{k=1}^{N} EP(H_k)$. Hence, we have $p^* \in \Omega$.

Finally, we now prove that $p^* = proj_{\Omega}^{f} u$. Since $\Omega \subset C_n$ for all $n \geq 1$, by Lemma 2.9 that

$$\langle \nabla f(u) - \nabla f(x_n), x_n - p \rangle \geq 0, \forall p \in \Omega. \tag{3.41}$$

Taking the limit as $n \to \infty$ in (3.41), we have

$$\langle \nabla f(u) - \nabla f(p^*), p^* - p \rangle \geq 0, \forall p \in \Omega, \tag{3.42}$$

and hence $p^* = proj_{\Omega}^{f} u$, by Lemma 2.9. This completes the proof. □

Corollary 3.2. *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E, $f : E \to \mathbb{R}$ be a coercive Legengre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E and ∇f^* be bounded on bounded subsets of E^* and $T, S : C \to C$ be two closed Bregman asymptotically quasi-nonexpansive mappings. Let $H : C \times C \to \mathbb{R}$ satisfying conditions $(A1)-(A4)$ and $\Omega = F(T) \cap S(T) \cap (\cap_{k=1}^{N} EP(H_k))$ is nonempty and bounded. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_1 = u \in C \quad \text{chosen arbitrarily}, \\ y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n-\beta_n)\nabla f(x_n) + \beta_n \nabla f(S^n(x_n))), \\ u_n = Res_{H_N}^{f} Res_{H_{N-1}}^{f} \ldots Res_{H_2}^{f} Res_{H_1}^{f}(Ty_n), \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n)\}, \\ x_{n+1} = proj_{D_{n+1}}^{f} u, \quad \forall n \geq 1, \end{cases} \tag{3.43}$$

where $\{\alpha_n\} \subset (0,1)$, $\limsup_{n\to\infty} \alpha_n < 1$. Then, the defined sequence $\{x_n\}$ converges strongly to $proj_{\Omega}^{j} u$, where $proj_{\Omega}^{j} u$ is the Bregman projection of C into Ω.

Corollary 3.3. *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E, $f : E \to \mathbb{R}$ be a coercive Legengre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E and ∇f^* be bounded on bounded subsets of E^* and $T, S : C \to C$ be* *two closed Bregman totally asymptotically quasi-nonexpansive mappings with sequence* $\{v_n\}, \{\mu_n\}, v_n, \mu_n \to 0$ *(as* $n \to \infty$*) and let there be a strictly increasing continuous function* $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\zeta(0) = 0$*. Let* $H : C \times C \to \mathbb{R}$ *satisfying conditions* $(A1) - (A4)$*. Assume that each* T, S *are uniformly asymptotically regular and* $\Omega = F(T) \cap S(T) \cap EP(H)$ *is nonempty and bounded. Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{cases} x_1 = u \in C \quad \text{chosen arbitrarily}, \\ y_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n - \beta_n)\nabla f(x_n) + \beta_n \nabla f(S^n(x_n))), \\ u_n = Res_H^f(T^n y_n), \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = proj_{D_{n+1}}^f u, \quad \forall n \geq 1, \end{cases} \tag{3.44}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$*,* $\limsup_{n\to\infty} \alpha_n < 1$*,* $\xi_n = 2l_n$*, and* $l_n = v_n \sup_{p\in\Omega} \zeta(D_f(p, x_n)) + \mu_n$*. Then, the defined sequence* $\{x_n\}$ *converges strongly to* $proj_\Omega^j u$*, where* $proj_\Omega^j u$ *is the Bregman projection of* C *into* Ω*.*

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