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Strong Convergence Theorems for the Modified Variational Inclusion Problems and Various Nonlinear Mappings in Hilbert space

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Abstract : In this paper, we prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of κ-strictly pseudononspreading mappings and the set of solutions of a finite family of variational inclusion problems and the set of solution of generalized equilibrium problem in Hilbert space. By using our main result, we give the numerical example to support some of our results.

Keywords : variational inclusion problems; κ-strictly pseudononspreading mapping; generalized equilibrium problem; resolvent operator; fixed point problem
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1 Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if and only if

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$Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

The mapping $T : C \to C$ is called a *κ-strictly pseudononspreading mapping* if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle, \forall x, y \in C. \tag{1.1}$$

This mapping was introduced by Osilike and Isiogugu [1] in 2011. It is shown in [11] that (1.1) is equivalent to

$$\frac{1-\kappa}{2}\|(I-T)x-(I-T)y\|^2 \leq \langle (I-T)x-(I-T)y, x-y\rangle + \langle (I-T)x, (I-T)y\rangle, \tag{1.2}$$

for all $x, y \in C$.

The mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Let $G : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The *variational inclusion problem* is to find $z \in H$ such that

$$\theta \in Gz + Mz, \tag{1.3}$$

where θ is a zero vector in H. The set of the solutions of (1.3) is denoted by $VI(H, G, M)$. Variational inclusion problem has a great impact and influence in the classes of mathematical problems and it is widely studied in many fields of pure and applied sciences. This problem is a useful and important generalization of the classical variational principles that includes variational, quasi-variational, variational-like inequalities as special cases. Many research papers have increasingly investigated such problems, see for instance [2, 3, 4] and references therein.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(z) = (I + \lambda M)^{-1}(z), \forall z \in H,$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is an identity mapping, see [10].

In 2008, Zhang *et al.*[10] introduced iterative scheme for finding a common element of the set of solutions of the variational inclusion problem with multi-valued maximal monotone mapping and inverse strongly monotone mappings and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n) S y_n, \forall n \geq 0, \end{aligned}$$

and proved strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and λ.

In 2014, Khuangsatung and Kangtunyakarn [11] have modified (1.3) as follows: For $i = 1, 2, ..., N$, let $A_i : H \to H$ be a single valued mapping and let $M : H \to 2^H$ be a multi-valued mapping. From the concept of (1.3), they introduced a new problem for finding $z \in H$ such that

$$\theta \in \sum_{i=1}^{N} a_i A_i z + Mz, \tag{1.4}$$

for all $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$ and θ is a zero vector. This problem is called *the modified variational inclusion.* The set of solutions (1.4) is denoted by $VI\left(H, \sum_{i=1}^{N} a_i A_i, M\right)$. If $A_i \equiv A$ for all $i = 1, 2, ..., N$, then (1.4) reduces to (1.3). They also introduced an iterative method for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems as follows:

$$\begin{cases} \displaystyle\sum_{i=1}^{N} a_i F_i\left(u_n, y\right) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \geq 0, \forall y \in C, \\ \displaystyle x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} b_i A_i) x_n \\ \qquad + \eta_n (I - \rho_n (I - T)) x_n + \delta_n u_n, \forall n \geq 1, \end{cases}$$

where $F_i : C \times C \to \mathbb{R}$ is a bifunction satisfying $(A1)$-$(A4)$ for all $i = 1, 2, ...N$, $A_i : H \to H$ is α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$, and $T : H \to H$ is a κ-strictly pseudononspreading mapping. Under suitable conditions of all parameters, they proved that $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}} u$, where $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction and let $A : C \to H$ be a nonlinear mapping. Now, we consider the following *generalized equilibrium problem*:

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Ax, y - x\rangle \geq 0, \tag{1.5}$$

for all $y \in C$. The set of solutions of this generalized equilibrium problem is denoted by

$$EP(F, A) = \{x \in C : F(x,y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

If $A \equiv 0$, then problem (1.5) reduces to *the equilibrium problem.* The set of solution of the equilibrium point is denoted by $EP(F)$. Several iterative methods have been proposed to solve the solution sets of these problems; see [5, 6] and the references therein.

In 2008, Takahashi and Takahashi [7] introduced an iterative method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space as follows:

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T(a_n u + (1 - a_n) u_n), \forall n \geq 1, \end{cases}$$

where A is an α-inverse strongly monotone mapping of C into H. Then, they proved that $\{x_n\}$ converges strongly to $z = P_{F(T) \cap EP(F,A)} u$ under some suitable conditions.

In 2012, Kangtunyakarn [15] have modified (1.3) as follows: Let $A, B : C \to H$ be two mappings. By modification of (1.5), we have

$$\begin{cases} EP(F, aA + (1 - b)B) = & \{x \in C : F(x, y) + \langle (aA + (1 - b)B)x, y - x \rangle \geq 0, \\ & \forall y \in C, \text{ and } a \in (0, 1)\}. \end{cases} \tag{1.6}$$

He also introduced an iterative method for finding a common element of the set of fixed points of strictly pseudo-contractive mapping and the set of solution of a modification of generalized equilibrium problem as follows:

$$\begin{cases} F(u_n, y) + \langle (aA + (1 - b)B)x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C (I - \lambda(I - T)) u_n, \forall n \geq 1, \end{cases}$$

where A, B are an α and β-inverse strongly monotone mapping, respectively and T is a κ-strictly pseudo contractive mapping. Under suitable conditions of the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, λ, he proved that $\{x_n\}$ converges strongly to $z = P_{F(T) \cap EP(F, aA + (1-b)B)} u$ under some suitable conditions.

Questions

1. Is it possible to prove a strong convergence theorem for finding a common element of the set of fixed point of a finite family of κ-strictly pseudononspreading mappings without using W-mapping, K-mapping, or S-mapping defined by [16], [18], [17], respectively ?

2. How can we give an iterative method for finding a common element in

$$\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \cap EP(F, \sum_{i=1}^{N} b_i B_i) \neq \emptyset?$$

In this paper, motivated by the work of Khuangsatung and Kangtunyakarn [11], [15], and the related research papers, we prove a strong convergence theorem for finding a common element of the fixed point sets of a finite family of κ-strictly pseudononspreading mappings and the solution sets of a finite family of variational inclusion problems and the solution sets of generalized equilibrium problem in Hilbert space. Moreover, we also give a numerical example to support our main results in the last section.

2 Preliminaries

In this paper, we denote weak and strong convergence by the notations "$\rightharpoonup$" and "$\rightarrow$", respectively. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. It is well known P_C is a firmly nonexpansive mapping from H onto C, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle,$$

for all $x, y \in H$.

For a proof of the main theorem in the next section, we will use the following lemmas.

Lemma 2.1 ([8]). *Given $x \in H$ and $y \in C$, then $P_C x = y$ if and only if we have the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.2 ([9]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\displaystyle\lim_{n\to\infty} s_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfy the following conditions:
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) For each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.3 ([12]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.4 ([14]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle\,;$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 2.5. *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x, y \in H$.

Remark 2.6 ([11]). *Let $T : H \to H$ be a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Define $S : H \to H$ by $Sx := ((1-\lambda)I + \lambda T)x$, where $\lambda \in (0, 1-\kappa)$. Then the following hold:*

(i) $F(T) = F(S) = F(I - \lambda(I - T))$;

(ii) for every $x \in H$ and $y \in F(T)$,

$$\|Sx - y\| \leq \|x - y\|.$$

Lemma 2.7 ([10]). *$u \in H$ is a solution of variational inclusion (1.3) if and only if $u = J_{M,\lambda}(u - \lambda Bu), \forall \lambda > 0$,* i.e.,

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \forall \lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is a closed convex subset in H.

Lemma 2.8 ([10]). *The resolvent operator $J_{M,\lambda}$ associated with M is single valued, nonexpansive for all $\lambda > 0$ and 1-inverse strongly monotone.*

Lemma 2.9 ([11]). *Let H be a real Hilbert space and let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, ..., N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,...,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N VI(H, A_i, M) \neq \emptyset$. Then*

$$VI(H, \sum_{i=1}^N a_i A_i, M) = \bigcap_{i=1}^N VI(H, A_i, M),$$

where $\sum_{i=1}^N a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \ldots, N$. Moreover, we have $J_{M,\lambda}(I - \lambda \sum_{i=1}^N a_i A_i)$ is a nonexpansive mapping, for all $0 < \lambda < 2\eta$.

3 Main Results

In this section, we introduce the following iterative algorithm and prove a strong convergence theorem for finding a common element of the fixed point sets of a finite family of κ-strictly pseudononspreading mappings and the solution sets of a finite family of variational inclusion problems and the solution sets of combination of generalized equilibrium problems in Hilbert space.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ to $\mathbb{R}$ satisfing $A1)$-$A4)$. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be η_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,\ldots,N}\{\eta_i\}$, $B_i : H \to H$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,\ldots,N}\{\mu_i\}$, and let $\{T\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of H into itself with $\kappa = \max_{i=1,2,\ldots,N}\{\kappa_i\}$.

Algorithm 3.1. *The sequence $\{x_n\}$ is generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \langle \sum_{i=1}^N b_i B_i x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \delta_n x_n + (1-\delta_n) J_{M,\lambda}(I - \lambda \sum_{i=1}^N a_i A_i) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N c_i (I - \rho_n(I - T_i)) y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq (0,1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n = 1$, $0 \leq a_i, b_i, c_i \leq 1$, for every $i = 1, 2, ..., N$, $0 < p \leq \beta_n, \gamma_n, \delta_n \leq q < 1$, $r_n \in [c, d] \subset (0, 2\mu)$, and $\rho_n \in (0, 1-\kappa)$ for all $n \geq 1$.

If $A_i \equiv A$, $B_i \equiv B$, and $T_i \equiv T$, for all $i = 1, 2, ..., N$, then Algorithm 3.1 reduces to Algorithm 3.2.

Algorithm 3.2. *The sequence $\{x_n\}$ is generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \langle B x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \delta_n x_n + (1-\delta_n) J_{M,\lambda}(I - \lambda A) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n (I - \rho_n(I - T)) y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq (0,1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n = 1$, for every $i = 1, 2, ..., N$, $0 < p \leq \beta_n, \gamma_n, \delta_n \leq q < 1$, $r_n \in [c, d] \subset (0, 2\mu)$, and $\rho_n \in (0, 1-\kappa)$ for all $n \geq 1$.

Under the condition of parameters above, we give a strong convergence theorem for the Algorithm 3.1.

Theorem 3.3. *Suppose that* $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N VI(H, A_i, M) \cap EP(F, \sum_{i=1}^N b_i B_i) \neq \emptyset$. *Assume the following conditions hold:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \lambda < 2\eta$,

(iii) $\sum_{n=1}^{\infty} \rho_n < \infty$,

(iv) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = \sum_{i=1}^{N} c_i = 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ *defined by Algorithm 3.1 converges strongly to* $x_0 = P_{\mathcal{F}} f(x_0)$.

Proof. First, we will show that $\sum_{i=1}^N b_i B_i$ is a μ-inverse strongly monotone mapping. For $x, y \in H$, we have

$$
\begin{aligned}
\langle \sum_{i=1}^N b_i B_i x - \sum_{i=1}^N b_i B_i y, x - y \rangle &= \sum_{i=1}^N b_i \langle B_i x - B_i y, x - y \rangle \\
&\geq \sum_{i=1}^N b_i \mu_i \|B_i x - B_i y\|^2 \\
&\geq \sum_{i=1}^N b_i \mu \|B_i x - B_i y\|^2 \\
&\geq \mu \| \sum_{i=1}^N b_i B_i x - \sum_{i=1}^N b_i B_i y \|^2. \qquad (3.3)
\end{aligned}
$$

Then $\sum_{i=1}^N b_i B_i$ is a μ-inverse strongly monotone mapping.
Next, show that $I - r_n \sum_{i=1}^N b_i B_i$ is a nonexpansive mapping. For every $n \in \mathbb{N}$,

from (3.3), we have

$$\begin{aligned}\|(I-r_n\sum_{i=1}^{N}b_iB_i)x-(I-r_n\sum_{i=1}^{N}b_iB_i)y\|^2 &= \|x-y-r_n(\sum_{i=1}^{N}b_iB_ix-\sum_{i=1}^{N}b_iB_iy)\|^2\\
&= \|x-y\|^2-2r_n\langle x-y,\sum_{i=1}^{N}b_iB_ix-\sum_{i=1}^{N}b_iB_iy\rangle\\
&\quad+\ r_n^2\|\sum_{i=1}^{N}b_iB_ix-\sum_{i=1}^{N}b_iB_iy\|^2\\
&\le \|x-y\|^2-2r_n\mu\|\sum_{i=1}^{N}b_iB_ix-\sum_{i=1}^{N}b_iB_iy\|^2\\
&\quad+\ r_n^2\|\sum_{i=1}^{N}b_iB_ix-\sum_{i=1}^{N}b_iB_iy\|^2\\
&= \|x-y\|^2+r_n(r_n-2\mu)\|\sum_{i=1}^{N}b_iB_ix-\sum_{i=1}^{N}b_iB_iy\|^2\\
&\le \|x-y\|^2. \qquad (3.4)\end{aligned}$$

Then $I-r_n\sum_{i=1}^{N}b_iB_i$ is a nonexpansive mapping.
Now, we divide the proof into five steps:
Step 1. We show that the sequence $\{x_n\}$ is bounded. From (3.1) and Lemma 2.4, we have $u_n=T_{r_n}(I-r_n\sum_{i=1}^{N}b_iB_i)x_n$, for all $n\in\mathbb{N}$. Let $x^*\in\mathcal{F}$. Then $F(x^*,y)+\langle y-x^*,\sum_{i=1}^{N}b_iB_ix^*\rangle\ge 0$ for all $y\in C$. So,

$$F(x^*,y)+\frac{1}{r_n}\langle y-x^*,x^*-x^*+r_n\sum_{i=1}^{N}b_iB_ix^*\rangle\ge 0,$$

for all $n\in\mathbb{N}$ and $y\in C$. From Lemma 2.4, we have $x^*=T_{r_n}(I-r_n\sum_{i=1}^{N}b_iB_i)x^*$, for all $n\in\mathbb{N}$. Since $x^*\in\bigcap_{i=1}^{N}VI(H,A_i,M)$, from Lemma 2.7 and Lemma 2.9, we have

$$x^*=J_{M,\lambda}(I-\lambda\sum_{i=1}^{N}a_iA_i)x^*.$$

From the nonexpansiveness of $J_{M,\lambda}(I-\lambda\sum_{i=1}^N a_iA_i)$, we have

$$\begin{aligned}\|J_{M,\lambda}(I-\lambda\sum_{i=1}^N a_iA_i)u_n-x^*\| &\le \|u_n-x^*\|\\ &= \|T_{r_n}(I-r_n\sum_{i=1}^N b_iB_i)x_n-x^*\|\\ &\le \|x_n-x^*\|. \end{aligned}\tag{3.5}$$

From the definition of y_n and (3.5), we have

$$\begin{aligned}\|y_n-x^*\| &= \|\delta_n x_n+(1-\delta_n)J_{M,\lambda}(I-\lambda\sum_{i=1}^N a_iA_i)u_n-x^*\|\\ &\le \delta_n\|x_n-x^*\|+(1-\delta_n)\|J_{M,\lambda}(I-\lambda\sum_{i=1}^N a_iA_i)u_n-x^*\|\\ &\le \delta_n\|x_n-x^*\|+(1-\delta_n)\|x_n-x^*\|\\ &= \|x_n-x^*\|. \end{aligned}\tag{3.6}$$

Since $x^*\in\bigcap_{i=1}^N F(T_i)$, by Remark 2.6 and (3.6), we have

$$\begin{aligned}\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x^*\| &\le \sum_{i=1}^N c_i\|(I-\rho_n(I-T_i))y_n-x^*\|\\ &\le \|y_n-x^*\|\\ &\le \|x_n-x^*\|. \end{aligned}\tag{3.7}$$

From the definition of x_n and (3.7), we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x^*\|\\ &= \|\alpha_n(f(x_n)-x^*)+\beta_n(x_n-x^*)+\gamma_n(\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x^*)\|\\ &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x^*\|\\ &\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x^*\|\\ &\le (1-\alpha_n(1-\alpha))\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|\\ &\le \max\{\|x_1-x^*\|,\frac{\|f(x^*)-x^*\|}{1-\alpha}\}. \end{aligned}$$

By mathematical induction, we can prove that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Put $G=\sum_{i=1}^N a_iA_i$. By the definition of y_n, and Lemma 2.9, we obtain

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|\delta_n x_n+(1-\delta_n)J_{M,\lambda}(I-\lambda G)u_n\\ &\quad -\ \delta_{n-1}x_{n-1}-(1-\delta_{n-1})J_{M,\lambda}(I-\lambda G)u_{n-1}\|\\ &\le \delta_n\|x_n-x_{n-1}\|+|\delta_n-\delta_{n-1}|\|x_{n-1}\|\\ &\quad +\ (1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-J_{M,\lambda}(I-\lambda G)u_{n-1}\|\\ &\quad +\ |\delta_{n-1}-\delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\|\\ &\le \delta_n\|x_n-x_{n-1}\|+|\delta_n-\delta_{n-1}|\|x_{n-1}\|+(1-\delta_n)\|u_n-u_{n-1}\|\\ &\quad +\ |\delta_{n-1}-\delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\|. \end{aligned} \tag{3.8}$$

Putting $V=\sum_{i=1}^N b_iB_i$, then $v_n=x_n-r_nVx_n$ and $u_n=T_{r_n}(I-r_nV)x_n=T_{r_n}v_n$. By continuing in the same direction as in Step 2 of Theorem 3.1 in [15], we have

$$\|u_n-u_{n-1}\|\le\|x_n-x_{n-1}\|+|r_n-r_{n-1}|\,\|Vx_n\|+\frac{1}{d}\,|r_n-r_{n-1}|\,\|u_n-v_n\|\,. \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\begin{aligned}\|y_n-y_{n-1}\| &\le \delta_n\|x_n-x_{n-1}\|+|\delta_n-\delta_{n-1}|\|x_{n-1}\|+(1-\delta_n)\big(\,\|x_n-x_{n-1}\|\\ &\quad +\ \ |r_n-r_{n-1}|\,\|Vx_n\|+\frac{1}{d}\,|r_n-r_{n-1}|\,\|u_n-v_n\|\,\big)\\ &\quad +\ |\delta_{n-1}-\delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\|\\ &\le \|x_n-x_{n-1}\|+|\delta_n-\delta_{n-1}|\|x_{n-1}\|+|r_n-r_{n-1}|\,\|Vx_n\|\\ &\quad +\ \frac{1}{d}\,|r_n-r_{n-1}|\,\|u_n-v_n\|+|\delta_{n-1}-\delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\|. \end{aligned} \tag{3.10}$$

Putting $L_n=\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n$, we have

$$\begin{aligned}\|L_n-L_{n-1}\| &= \|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-\sum_{i=1}^N c_i(I-\rho_{n-1}(I-T_i))y_{n-1}\|\\ &\le\ \sum_{i=1}^N c_i\|(I-\rho_n(I-T_i))y_n-(I-\rho_{n-1}(I-T_i))y_{n-1}\|\\ &=\ \sum_{i=1}^N c_i\|(y_n-y_{n-1})-\rho_n(I-T_i)y_n+\rho_{n-1}(I-T_i)y_{n-1}\|\|\\ &=\ \sum_{i=1}^N c_i\|(y_n-y_{n-1})-\rho_n(I-T_i)y_n+\rho_n(I-T_i)y_{n-1}-\rho_n(I-T_i)y_{n-1}\\ &\quad +\ \rho_{n-1}(I-T_i)y_{n-1}\| \end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{N} c_i\big(\|y_n - y_{n-1}\| + \rho_n\|(I-T_i)y_n - (I-T_i)y_{n-1}\| \\
&\quad + |\rho_n - \rho_{n-1}|\|(I-T_i)y_{n-1}\|\big) \\
&= \|y_n - y_{n-1}\| + \sum_{i=1}^{N} c_i\rho_n\|(I-T_i)y_n - (I-T_i)y_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i|\rho_n - \rho_{n-1}|\|(I-T_i)y_{n-1}\|. \qquad (3.11)
\end{aligned}$$

Substituting (3.10) into (3.11), we obtain

$$\begin{aligned}
\|L_n - L_{n-1}\| &\le \|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + |r_n - r_{n-1}|\,\|Vx_n\| \\
&\quad + \frac{1}{d}|r_n - r_{n-1}|\,\|u_n - v_n\| + |\delta_{n-1} - \delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i\rho_n\|(I-T_i)y_n - (I-T_i)y_{n-1}\| + \sum_{i=1}^{N} c_i|\rho_n - \rho_{n-1}|\|(I-T_i)y_{n-1}\|. \qquad (3.12)
\end{aligned}$$

From the definition of x_n and (3.12), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n L_n - \alpha_{n-1} f(x_{n-1}) - \beta_{n-1}x_{n-1} - \gamma_{n-1}L_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) + \beta_n x_n - \beta_n x_{n-1} + \beta_n x_{n-1} \\
&\quad + \gamma_n L_n - \gamma_n L_{n-1} + \gamma_n L_{n-1} - \alpha_{n-1} f(x_{n-1}) - \beta_{n-1}x_{n-1} - \gamma_{n-1}L_{n-1}\| \\
&\le \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\|L_n - L_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|L_{n-1}\| \\
&\le \alpha_n\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\big(\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + |r_n - r_{n-1}|\,\|Vx_n\| \\
&\quad + \frac{1}{d}|r_n - r_{n-1}|\,\|u_n - v_n\| + |\delta_{n-1} - \delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i\rho_n\|(I-T_i)y_n - (I-T_i)y_{n-1}\| + \sum_{i=1}^{N} c_i|\rho_n - \rho_{n-1}|\|(I-T_i)y_{n-1}\|\big) \\
&\quad + |\gamma_n - \gamma_{n-1}|\|L_{n-1}\| \\
&\le (1 - \alpha_n(1-\alpha))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| + |r_n - r_{n-1}|\,\|Vx_n\| \\
&\quad + \frac{1}{d}|r_n - r_{n-1}|\,\|u_n - v_n\| + |\delta_{n-1} - \delta_n|\|J_{M,\lambda}(I-\lambda G)u_{n-1}\| \\
&\quad + \sum_{i=1}^{N} c_i\rho_n\|(I-T_i)y_n - (I-T_i)y_{n-1}\| + \sum_{i=1}^{N} c_i|\rho_n - \rho_{n-1}|\|(I-T_i)y_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|L_{n-1}\|. \qquad (3.13)
\end{aligned}$$

Applying Lemma 2.2, (3.13), and the conditions (i), (iii), (v), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Step 3. We show that $\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|(I - \rho_n(I - T_i))y_n - y_n\| = \lim_{n\to\infty} \|J_{M,\lambda}(I - \lambda G)u_n - u_n\| = 0$ for all $i = 1, 2, \ldots, N$.

To show this, let $x^* \in \mathcal{F}$. Since $u_n = T_{r_n}(I - r_n V)x_n$, where $V = \sum_{i=1}^N b_i B_i$, and T_{r_n} is firmly nonexpansive mapping, then we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 =& \ \|T_{r_n}(I - r_n V)x_n - T_{r_n}(I - r_n V)x^*\|^2 \\
\leq& \ \langle (I - r_n V)x_n - (I - r_n V)x^*, u_n - x^* \rangle \\
=& \ \frac{1}{2}\big(\|(I - r_n V)x_n - (I - r_n V)x^*\|^2 + \|u_n - x^*\|^2 \\
&- \ \|(I - r_n V)x_n - (I - r_n V)x^* - u_n + x^*\|^2 \big) \\
\leq& \ \frac{1}{2}\big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 - r_n^2 \|Vx_n - Vx^*\|^2 \\
&+ \ 2r_n \langle x_n - u_n, Vx_n - Vx^* \rangle \big),
\end{aligned}$$

which follows that

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 - r_n^2 \|Vx_n - Vx^*\|^2 + 2r_n \langle x_n - u_n, Vx_n - Vx^* \rangle. \tag{3.15}$$

From the nonexpansiveness of T_{r_n} and (3.3), we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(I - r_n V)x_n - T_{r_n}(I - r_n V)x^*\|^2 \\
&\leq \|(I - r_n V)x_n - (I - r_n V)x^*\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n \langle x_n - x^*, Vx_n - Vx^* \rangle + r_n^2 \|Vx_n - Vx^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2r_n \mu \|Vx_n - Vx^*\|^2 + r_n^2 \|Vx_n - Vx^*\|^2 \\
&= \|x_n - x^*\|^2 - r_n(2\mu - r_n)\|Vx_n - Vx^*\|^2.
\end{aligned} \tag{3.16}$$

From Remark 2.6, Lemma 2.9, and (3.16), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|\sum_{i=1}^N c_i(I - \rho_n(I - T_i))y_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|y_n - x^*\|^2 \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n (\delta_n \|x_n - x^*\|^2 \\
&\quad + \ (1 - \delta_n)\|J_{M,\lambda}(I - \lambda G)u_n - x^*\|^2) \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n (\delta_n \|x_n - x^*\|^2 \\
&\quad + \ (1 - \delta_n)\|u_n - x^*\|^2) \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n (\delta_n \|x_n - x^*\|^2 \\
&\quad + \ (1 - \delta_n)(\|x_n - x^*\|^2 - r_n(2\mu - r_n)\|Vx_n - Vx^*\|^2)) \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - r_n \gamma_n (2\mu - r_n)(1 - \delta_n)\|Vx_n - Vx^*\|^2,
\end{aligned}$$

which implies that

$$r_n\gamma_n(2\mu - r_n)(1-\delta_n)\|Vx_n - Vx^*\|^2 \leq \alpha_n \left\|f(x_n) - x^*\right\|^2 + \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right)\|x_{n+1} - x_n\|.$$

From the condition (i) and (3.14), we have

$$\lim_{n\to\infty} \|Vx_n - Vx^*\| = 0. \tag{3.17}$$

From the definition of x_n, and (3.15), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|\sum_{i=1}^{N} c_i(I - \rho_n(I - T_i))y_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n(\delta_n\|x_n - x^*\|^2 \\
&\quad + (1-\delta_n)\|J_{M,\lambda}(I - \lambda G)u_n - x^*\|^2) \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n(\delta_n\|x_n - x^*\|^2 \\
&\quad + (1-\delta_n)\|u_n - x^*\|^2) \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\big(\delta_n\|x_n - x^*\|^2 \\
&\quad + (1-\delta_n)\big(\|x_n - x^*\|^2 - \|x_n - u_n\|^2 - r_n^2\|Vx_n - Vx^*\|^2 \\
&\quad + 2r_n\langle x_n - u_n, Vx_n - Vx^*\rangle\big)\big) \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\big(\|x_n - x^*\|^2 \\
&\quad - (1-\delta_n)\|x_n - u_n\|^2 + 2r_n(1-\delta_n)\|x_n - u_n\|\|Vx_n - Vx^*\|\big) \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\
&\quad - \gamma_n(1-\delta_n)\|x_n - u_n\|^2 + 2r_n\gamma_n(1-\delta_n)\|x_n - u_n\|\|Vx_n - Vx^*\| \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - \gamma_n(1-\delta_n)\|x_n - u_n\|^2 + 2r_n\gamma_n(1-\delta_n)\|x_n - u_n\|\|Vx_n - Vx^*\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
\gamma_n(1-\delta_n)\|x_n - u_n\|^2 &\leq \alpha_n \left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2r_n\gamma_n(1-\delta_n)\|x_n - u_n\|\|Vx_n - Vx^*\| \\
&\leq \alpha_n \left\|f(x_n) - x^*\right\|^2 + \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right)\|x_{n+1} - x_n\| \\
&\quad + 2r_n\gamma_n(1-\delta_n)\|x_n - u_n\|\|Vx_n - Vx^*\|.
\end{aligned}$$

From the condition (i), (3.14), and (3.17), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.18}$$

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\sum_{i=1}^{N}c_i(I-\rho_n(I-T_i))y_n-x^*\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-x^*\|^2\\
&= \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\big(\|\delta_n(x_n-x^*)\\
&\quad +\ (1-\delta_n)(J_{M,\lambda}(I-\lambda G)u_n-x^*)\|^2\big)\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\big(\delta_n\|x_n-x^*\|^2\\
&\quad +\ (1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x^*\|^2-\delta_n(1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|^2\big)\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\big(\delta_n\|x_n-x^*\|^2\\
&\quad +\ (1-\delta_n)\|u_n-x^*\|^2-\delta_n(1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|^2\big)\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\big(\delta_n\|x_n-x^*\|^2\\
&\quad +\ (1-\delta_n)\|x_n-x^*\|^2-\delta_n(1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|^2\big)\\
&= \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\
&\quad -\ \gamma_n\delta_n(1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|^2\\
&\le \alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2-\gamma_n\delta_n(1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|^2.
\end{aligned}$$

It implies that

$$\gamma_n\delta_n(1-\delta_n)\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|^2\le\alpha_n\left\|f(x_n)-x^*\right\|^2+\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\|x_{n+1}-x_n\|.$$

From the condition (i) and (3.14), we have

$$\lim_{n\to\infty}\|J_{M,\lambda}(I-\lambda G)u_n-x_n\|=0. \tag{3.19}$$

Observe that

$$\|J_{M,\lambda}(I-\lambda G)u_n-u_n\|\le\|J_{M,\lambda}(I-\lambda G)u_n-u_n\|+\|u_n-x_n\|.$$

From (3.18) and (3.19), we have

$$\lim_{n\to\infty}\|J_{M,\lambda}(I-\lambda G)u_n-u_n\|=0. \tag{3.20}$$

From the definition of y_n, we have

$$y_n-x_n=(1-\delta_n)(J_{M,\lambda}(I-\lambda G)u_n-x_n).$$

From (3.20), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.21}$$

From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x^*\|^2 \\ &\quad - \beta_n\gamma_n\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x_n\|^2 \\ &\leq \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2 \\ &\quad - \beta_n\gamma_n\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x_n\|^2.\end{aligned}$$

It implies that

$$\beta_n\gamma_n\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x_n\|^2 \leq \alpha_n\left\|f(x_n)-x^*\right\|^2+\left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\|x_{n+1}-x_n\|.$$

From the condition (i) and (3.14), we have

$$\lim_{n\to\infty}\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x_n\|=0, \tag{3.22}$$

for all $i=1,2,...,N$.
For every $i=1,2,...,N$, observe that

$$\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-y_n\| \leq \|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-x_n\|+\|x_n-y_n\|.$$

From (3.21) and (3.22), we have

$$\lim_{n\to\infty}\|\sum_{i=1}^N c_i(I-\rho_n(I-T_i))y_n-y_n\|=0, \tag{3.23}$$

for all $i=1,2,...,N$.

Step 4. We show that $\limsup_{n\to\infty}\langle f(x_0)-x_0,x_n-x_0\rangle \leq 0$ where $x_0=P_{\mathcal{F}}f(x_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x_0)-x_0,x_n-x_0\rangle = \lim_{k\to\infty}\langle f(x_0)-x_0,x_{n_k}-x_0\rangle. \tag{3.24}$$

Without loss of generality, we can assume that $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$, where $\omega\in C$. From (3.18) and (3.21), we obtain $y_{n_k}\rightharpoonup\omega$ and $u_{n_k}\rightharpoonup\omega$ as $k\to\infty$

First, we will show that $\omega\in\bigcap_{i=1}^N F(T_i)$. Assume $\omega\notin\bigcap_{i=1}^N F(T_i)$, then we have $\omega\notin F(T_{i_0})$, for some $i_0=1,2,...,N$. From Remark 2.6, we have $\omega\neq$

$\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))\omega$. From the Opial's condition, the condition (iii), and (3.23), we have

$$\begin{aligned}
\liminf_{k\to\infty}\|y_{n_k}-\omega\| < & \liminf_{k\to\infty}\|y_{n_k}-\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))\omega\| \\
\leq & \liminf_{k\to\infty}\big(\|y_{n_k}-\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))y_{n_k}\| \\
& + \|\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))y_{n_k}-\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))\omega\|\big) \\
\leq & \liminf_{k\to\infty}\big(\|y_{n_k}-\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))y_{n_k}\| \\
& + \sum_{i_0=1}^N c_{i_0}\|(I-\rho_{n_k}(I-T_{i_0}))y_{n_k}-(I-\rho_{n_k}(I-T_{i_0}))\omega\|\big) \\
\leq & \liminf_{k\to\infty}\big(\|y_{n_k}-\sum_{i_0=1}^N c_{i_0}(I-\rho_{n_k}(I-T_{i_0}))y_{n_k}\| \\
& + \|y_{n_k}-\omega\|+\rho_{n_k}\|(I-T_{i_0})y_{n_k}-(I-T_{i_0})\omega\|\big) \\
\leq & \liminf_{k\to\infty}\|y_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then

$$\omega\in\bigcap_{i=1}^N F(T_i). \tag{3.25}$$

Second , we will show that $\omega\in\bigcap_{i=1}^N VI(H,A_i,M)$. Assume that $\omega\notin\bigcap_{i=1}^N VI(H,A_i,M)$. By Lemma 2.7 and 2.9, $\bigcap_{i=1}^N VI(H,A_i,M)=F(J_{M,\lambda}((I-\lambda G)))$. Then $\omega\neq J_{M,\lambda}(I-\lambda G)\omega$, where $G=\sum_{i=1}^N a_iA_i$. By the nonexpansiveness of $J_{M,\lambda}((I-\lambda G))$, (3.20), and Opial's condition, we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\|u_{n_k}-\omega\| < & \liminf_{k\to\infty}\|u_{n_k}-J_{M,\lambda}((I-\lambda G))\omega\| \\
\leq & \liminf_{k\to\infty}(\|u_{n_k}-J_{M,\lambda}((I-\lambda G))u_{n_k}\| \\
& + \|J_{M,\lambda}((I-\lambda G))u_{n_k}-J_{M,\lambda}((I-\lambda G))\omega\|) \\
\leq & \liminf_{k\to\infty}\|u_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in\bigcap_{i=1}^N VI(H,A_i,M). \tag{3.26}$$

Next, we will show that $\omega \in EP(F,V)$ where $V = \sum_{i=1}^{N} b_i B_i$, $\forall b_i \in (0,1)$ and $\sum_{i=1}^{N} b_i = 1$. By continuing the same method of proof as Step 4 of Theorem 3.1 in [15], we obtain

$$\omega \in EP(F, \sum_{i=1}^{N} b_i B_i). \tag{3.27}$$

Hence, we can conclude that $\omega \in \mathcal{F}$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in \mathcal{F}$, (3.24) and Lemma 2.1, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(x_0) - x_0, x_n - x_0\rangle &= \lim_{k\to\infty} \langle f(x_0) - x_0, x_{n_k} - x_0\rangle \\ &= \langle f(x_0) - x_0, \omega - x_0\rangle \\ &\leq 0. \end{aligned} \tag{3.28}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = x_0$, where $x_0 = P_{\mathcal{F}} f(x_0)$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - x_0\|^2 &= \|\alpha_n(f(x_n) - x_0) + \beta_n(x_n - x_0) + \gamma_n(\sum_{i=1}^{N} c_i(I - \rho_n(I - T_i))y_n - x_0)\|^2 \\ &\leq \|\beta_n(x_n - x_0) + \gamma_n(\sum_{i=1}^{N} c_i(I - \rho_n(I - T_i))y_n - x_0)\|^2 + 2\alpha_n\langle f(x_n) - x_0, x_{n+1} - x_0\rangle \\ &\leq (\beta_n\|x_n - x_0\| + \gamma_n\|\sum_{i=1}^{N} c_i(I - \rho_n(I - T_i))y_n - x_0\|)^2 + 2\alpha_n\langle f(x_n) - x_0, x_{n+1} - x_0\rangle \\ &\leq (\beta_n\|x_n - x_0\| + \gamma_n\|y_n - x_0\|)^2 + 2\alpha_n\langle f(x_n) - x_0, x_{n+1} - x_0\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\langle f(x_n) - f(x_0), x_{n+1} - x_0\rangle \\ &\quad + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\|f(x_n) - f(x_0)\|\|x_{n+1} - x_0\| \\ &\quad + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\alpha\|x_n - x_0\|\|x_{n+1} - x_0\| \\ &\quad + 2\alpha_n\langle f(x_0) - z_0, x_{n+1} - x_0\rangle \\ &\leq (1-\alpha_n)^2\|x_n - x_0\|^2 + \alpha_n\alpha(\|x_n - x_0\|^2 + \|x_{n+1} - x_0\|^2) \\ &\quad + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle \\ &= (1-\alpha_n)^2\|x_n - x_0\|^2 + \alpha_n\alpha\|x_n - x_0\|^2 + \alpha_n\alpha\|x_{n+1} - x_0\|^2 \\ &\quad + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle. \end{aligned}$$

It implies that

$$\begin{aligned}
\|x_{n+1}-x_0\|^2 &\leq \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-x_0\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x_0)-x_0, x_{n+1}-x_0\rangle \\
&= \frac{1-2\alpha_n+\alpha_n^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-x_0\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x_0)-x_0, x_{n+1}-x_0\rangle \\
&= \frac{1-2\alpha_n+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-x_0\|^2+\frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-x_0\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x_0)-x_0, x_{n+1}-x_0\rangle \\
&= \left(1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)\|x_n-x_0\|^2+\frac{\alpha_n^2 2(1-\alpha)}{(1-\alpha_n\alpha)2(1-\alpha)}\|x_n-x_0\|^2 \\
&\quad + \frac{2\alpha_n(1-\alpha)}{(1-\alpha_n\alpha)(1-\alpha)}\langle f(x_0)-x_0, x_{n+1}-x_0\rangle \\
&= \left(1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)\|x_n-z_0\|^2+\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left(\frac{\alpha_n}{2(1-\alpha)}\|x_n-x_0\|^2\right. \\
&\quad \left. + \frac{1}{1-\alpha}\langle f(x_0)-x_0, x_{n+1}-x_0\rangle\right).
\end{aligned}$$

From the condition (i), (3.28) and Lemma 2.1, we can conclude that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathcal{F}}f(x_0)$. This completes the proof. □

Theorem 3.4. *Suppose that* $\mathcal{F} := F(T)\cap VI(H,A,M)\cap EP(F,B) \neq \emptyset$. *Assume the following conditions hold:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0<\lambda<2\eta$,

(iii) $\sum_{n=1}^{\infty}\rho_n < \infty$,

(iv) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$,
$\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Then the sequence $\{x_n\}$ *defined by Algorithm 3.2 converges strongly to* $x_0 = P_{\mathcal{F}}f(x_0)$.

Proof. Put $A_i \equiv A$, $B_i \equiv B$, and $T_i \equiv T$, for all $i = 1,2,...,N$ in Theorem 3.3. So, from Theorem 3.3, we obtain the desired result. □

Corollary 3.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be bifunctions from* $C\times C$ *to* $\mathbb{R}$ *satisfy* A1)-A4). *Let* $M: H\to 2^H$ *be a* *multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $A : H \to H$ be η-inverse strongly monotone mapping, $B_i : H \to H$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,\ldots,N}\{\mu_i\}$, and let $\{T\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of H itself with $\kappa = \max_{i=1,2,\ldots,N}\{\kappa_i\}$. Assume $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap VI(H, A, M) \cap EP(F, \sum_{i=1}^N b_iB_i) \neq \emptyset$. Let the sequences $\{x_n\}$ be generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \langle\sum_{i=1}^N b_iB_ix_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ y_n = \delta_n x_n + (1 - \delta_n)J_{M,\lambda}(I - \lambda A)u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N c_i(I - \rho_n(I - T_i))y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.29}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq (0, 1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n = 1$, $0 \leq a_i, b_i, c_i \leq 1$, for every $i = 1, 2, ..., N$, $0 < p \leq \beta_n, \gamma_n, \delta_n \leq q < 1$, $r_n \in [c, d] \subset (0, 2\mu)$, and $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $0 < \lambda < 2\eta$,

(iii) $\sum_{n=1}^\infty \rho_n < \infty$,

(iv) $\sum_{i=1}^N b_i = \sum_{i=1}^N c_i = 1$,

(v) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^\infty |\rho_{n+1} - \rho_n| < \infty$, $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\mathcal{F}}f(x_0)$.

Proof. Put $A_i \equiv A$ for all $i = 1, 2, ..., N$ in Theorem 3.3. So, from Theorem 3.3, we obtain the desired result. □

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ to $\mathbb{R}$ satisfy A1)-A4). Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be η_i-inverse strongly monotone mapping with $\eta = min_{i=1,2,\ldots,N}\{\eta_i\}$, $B_i : H \to H$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,\ldots,N}\{\mu_i\}$.*

Assume $\mathcal{F} := \bigcap_{i=1}^{N} VI(H, A_i, M) \cap EP(F, \sum_{i=1}^{N} b_i B_i) \neq \emptyset$. *Let the sequences* $\{x_n\}$ *be generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \langle \sum_{i=1}^{N} b_i B_i x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \delta_n x_n + (1 - \delta_n) J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_i A_i) u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.30}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq (0,1)$ *and* $\lambda > 0$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $0 \leq a_i, b_i, c_i \leq 1$, *for every* $i = 1, 2, ..., N$, $0 < p \leq \beta_n, \gamma_n, \delta_n \leq q < 1$, *and* $r_n \in [c, d] \subset (0, 2\mu)$ *for all* $n \geq 1$. *Suppose the following conditions hold:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \lambda < 2\eta$,

(iii) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$,

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x_0 = P_{\mathcal{F}} f(x_0)$.

Proof. Put $T_i \equiv I$ for all $i = 1, 2, ..., N$ in Theorem 3.3. So, from Theorem 3.3, we obtain the desired result. □

4 Applications

In this section, we utilize our main theorem to prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of κ-strictly pseudononspreading mappings and a common element of the set of fixed points of a finite family of κ-strictly pseudo-contractive mappings and the set of solution of generalized equilibrium problem in Hilbert space.

Recall that let $S : C \to C$ be a mapping. Then S is said to be *ξ-strictly pseudo-contractive* if there exists a constant $\xi \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \xi \|(I - S)x - (I - S)y\|^2, \forall x, y \in C.$$

Now, we consider a property of finite family of strictly pseudo-contractive mappings in Hilbert space as follows:

Proposition 4.1. *[19] Let C be a nonempty closed convex subset of a real Hilbert space H.*

(i) Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $S_i : C \to H$ is a ξ_i-strict pseudo-contraction for some $0 \leq \xi_i < 1$. Assume $\{a_i\}_i^N$ is a positive sequence such that $\sum_{i=1}^N a_i = 1$ .Then $\sum_{i=1}^N a_i S_i$ is a ξ-strict pseudo-contraction, with $\xi = \max_{i=1,2,\ldots,N}\{\xi_i\}$.

(ii) Let $\{S_i\}_i^N$ and $\{a_i\}_i^N$ be given as in (i) above. Suppose that $\{S_i\}_i^N$ has a common fixed point. Then

$$F(\sum_{i=1}^N a_i S_i) = \bigcap_{i=1}^N F(S_i).$$

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunctions from $C \times C$ to $\mathbb{R}$ satisfing A1)-A4). Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, \ldots, N$, $S_i : H \to H$ be ξ_i-strictly pseudo-contractive mappings with $\xi = \max_{i=1,2,\ldots,N}\{\xi_i\}$, $B_i : H \to H$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,\ldots,N}\{\mu_i\}$, and let $\{T\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of H itself with $\kappa = \max_{i=1,2,\ldots,N}\{\kappa_i\}$. Assume $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap EP(F, \sum_{i=1}^N b_i B_i) \neq \emptyset$. Let the sequences $\{x_n\}$ be generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \langle \sum_{i=1}^N b_i B_i x_n, y - u_n \rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n = \delta_n x_n + (1 - \delta_n)((1 - \lambda)u_n + \lambda \sum_{i=1}^N a_i S_i u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N c_i (I - \rho_n(I - T_i))y_n, \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq (0, 1)$ and $\lambda > 0$ with $\alpha_n + \beta_n + \gamma_n = 1$, $0 \leq a_i, b_i, c_i \leq 1$, for every $i = 1, 2, ..., N$, $0 < p \leq \beta_n, \gamma_n, \delta_n \leq q < 1$, $r_n \in [c, d] \subset (0, 2\mu)$, and $\rho_n \in (0, 1 - \kappa)$ for all $n \geq 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $0 < \lambda < 1 - \xi$,

(iii) $\sum_{n=1}^\infty \rho_n < \infty$,

(iv) $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i = \sum_{i=1}^N c_i = 1$,

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, $\sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty$,
$\sum_{n=1}^{\infty}|\rho_{n+1}-\rho_n|<\infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x_0 = P_{\mathcal{F}}f(x_0)$.

Proof. Let $A_i = I - S_i$ and $M = 0$ in Theorem 3.3, then we have that A_i is η_i-inverse strongly monotone with $\frac{1-\xi}{2}$. Now, we show that $\bigcap_{i=1}^{N} VI(H, A_i, M) = \bigcap_{i=1}^{N} F(S_i)$. Since $A_i = I - S_i$, $M = 0$, Lemma 2.9, and Proposition 4.1, then

$$\begin{aligned} x \in \bigcap_{i=1}^{N} VI(H, A_i, M) \Leftrightarrow x \in VI(H, \sum_{i=1}^{N} a_iA_i, M) &\Leftrightarrow 0 \in \sum_{i=1}^{N} a_iA_ix + Mx \\ &\Leftrightarrow 0 = \sum_{i=1}^{N} a_iA_ix \\ &\Leftrightarrow 0 = \sum_{i=1}^{N} a_i(I - S_i)x \\ &\Leftrightarrow x = \sum_{i=1}^{N} a_iS_ix \\ &\Leftrightarrow x \in F(\sum_{i=1}^{N} a_iS_i) \\ &\Leftrightarrow x \in \bigcap_{i=1}^{N} F(S_i). \end{aligned}$$

It implies that

$$\bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \cap EP(F, \sum_{i=1}^{N} b_iB_i) = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap EP(F, \sum_{i=1}^{N} b_iB_i)$$

From the definition of $J_{M,\lambda}$, we have

$$\begin{aligned} J_{M,\lambda}(I - \lambda \sum_{i=1}^{N} a_iA_i)u_n &= (I + \lambda M)^{-1}(I - \lambda \sum_{i=1}^{N} a_iA_i)u_n \\ &= u_n - \lambda \sum_{i=1}^{N} a_iA_iu_n \end{aligned}$$

$$= u_n - \lambda \sum_{i=1}^{N} a_i(I - S_i)u_n$$
$$= (1-\lambda)u_n + \lambda \sum_{i=1}^{N} a_i S_i u_n.$$

Since $\lambda \in (0, 1-\xi) \subset (0,1)$, then $(1-\lambda)u_n + \lambda \sum_{i=1}^{N} a_i S_i u_n \in H$. So, from Theorem 3.3, we obtain the desired result. □

5 Numerical results

The following example supports Theorem 3.3. Using these examples, we see that the Algorithm 3.1 converges faster than the Algorithm 3.2

Example 5.1. *Let $\mathbb{R}$ be the set of real numbers. For every $i = 1, 2, \ldots, N$, let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A_i, B_i : \mathbb{R} \to \mathbb{R}$ be defined by*

$$F(x,y) = -6x^2 + xy + 5y^2, A_i x = \frac{ix}{9}, B_i x = \frac{ix}{8}, f(x) = \frac{x}{6}, \text{ for all } x, y \in \mathbb{R}.$$

For all $x, y \in \mathbb{R}$ and for every $i = 1, 2, \ldots, N$, let $T_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_i x = \begin{cases} \frac{-(2i+1)x}{2i+2} & \text{if } x \in [0, \infty), \\ x & \text{if } x \in (-\infty, 0). \end{cases} \tag{5.1}$$

For every $i = 1, 2, \ldots, N$, suppose that $J_{M,\lambda} = I$, $\lambda = \frac{1}{2N}$, $a_i = \frac{1}{2^i} + \frac{1}{N2^N}$, $b_i = \frac{5}{6^i} + \frac{1}{N6^N}$, and $c_i = \frac{7}{8^i} + \frac{1}{N8^N}$. Let $\{x_n\}$ and $\{u_n\}$ be generated by (3.1), where $\alpha_n = \frac{1}{6n}$, $\beta_n = \frac{3(6n-1)}{30n}$, $\gamma_n = \frac{2(6n-1)}{30n}$, $\delta_n = \frac{n}{6n+5}$, $r_n = \frac{3n}{5n+6}$, and $\rho_n = \frac{1}{2n^2}$ for every $n \in \mathbb{N}$. Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Solution. *From (1.2), it is easy to see that T_i is a $\frac{2i-1}{2i+2}$-strictly pseudononspreading mapping, for all $i = 1, 2, ..., N$ and $F(T) = 0$. Since $b_i = \frac{5}{6^i} + \frac{1}{N6^N}$, we obtain*

$$\sum_{i=1}^{N} b_i B_i x = \sum_{i=1}^{N} \left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i\frac{x}{8} = S_1 \frac{x}{8}. \tag{5.2}$$

where $S_1 = \sum_{i=1}^{N} \left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i$. It is easy to check that $0 \in EP(F, \sum_{i=1}^{N} b_i B_i)$. Since $A_i x = \frac{ix}{9}$ and $a_i = \frac{1}{2^i} + \frac{1}{N2^N}$, then

$$\sum_{i=1}^{N} a_i A_i x = \sum_{i=1}^{N} \left(\frac{1}{2^i} + \frac{1}{N2^N}\right) \frac{ix}{9}.$$

From the definition of T_i, A_i, for all $i = 1, 2, ..., N$, and $0 \in EP(F, \sum_{i=1}^{N} b_i B_i)$, we have

$$\bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \cap EP(F, \sum_{i=1}^{N} b_i B_i) = \{0\}. \tag{5.3}$$

By the definition of F, we have

$$0 \le F(u_n, y) + \langle \sum_{i=1}^{N} b_i B_i x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle$$
$$= -6u_n^2 + u_n y + 5y^2 + (S_i \frac{x_n}{8})(y - u_n) + \frac{1}{r_n} (y - u_n)(u_n - x_n)$$
$$\Leftrightarrow$$
$$0 \le 40r_n y^2 + 8u_n y + 8r_n u_n y - 8x_n y + r_n S_1 x_n y - 8u_n^2$$
$$- 48r_n u_n^2 + 8u_n x_n - r_n S_1 u_n x_n$$
$$= 40r_n y^2 + (8u_n + 8r_n u_n - 8x_n + r_n S_1 x_n) y - 8u_n^2 - 48r_n u_n^2 + 8u_n x_n - r_n S_1 u_n x_n.$$

Let $G(y) = 40r_n y^2 + (8u_n + 8r_n u_n - 8x_n + r_n S_1 x_n) y - 8u_n^2 - 48r_n u_n^2 + 8u_n x_n - r_n S_1 u_n x_n$. $G(y)$ is a quadratic function of y with coefficient $a = 40r_n$, $b = 8u_n + 8r_n u_n - 8x_n + r_n S_1 x_n$, and $c = -8u_n^2 - 48r_n u_n^2 + 8u_n x_n - r_n S_1 u_n x_n$. Determine the discriminant Δ of G as follows:

$$\Delta = b^2 - 4ac$$
$$= (8u_n + 8r_n u_n - 8x_n + r_n S_1 x_n)^2 - 4(40r_n)\left(-8u_n^2 - 48r_n u_n^2 + 8u_n x_n - r_n S_1 u_n x_n\right)$$
$$= 64u_n^2 + 1408r_n u_n^2 + 7744r_n^2 u_n^2 - 128u_n x_n - 1408r_n u_n x_n + 16r_n S_1 u_n x_n$$
$$+ 176r_n^2 S_1 u_n x_n + 64x_n^2 - 16r_n S_1 x_n^2 + r_n^2 S_1^2 x_n^2$$
$$= ((8 + 88r_n)u_n + (-8 + r_n S_1)x_n)^2.$$

We know that $G(y) \ge 0, \forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \le 0$, so we obtain

$$u_n = \frac{(8 - r_n S_1)x_n}{8 + 88S_1 r_n}, \tag{5.4}$$

where $S_1 = \sum_{i=1}^{N} \left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i$.

For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{6n}$, $\beta_n = \frac{3(6n-1)}{30n}$, $\gamma_n = \frac{2(6n-1)}{30n}$, $\delta_n = \frac{n}{6n+5}$, $r_n = \frac{3n}{5n+6}$, and $\rho_n = \frac{1}{2n^2}$ for every $n \in \mathbb{N}$. It is easy to check that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{r_n\}$, and $\{\rho_n\}$ satisfy all the conditions of Theorem 3.3. For every $n \in \mathbb{N}$, from (5.4), we rewrite the Algorithm 3.1 as follows:

$$\begin{cases} u_n &= \frac{(8 - r_n S_1)x_n}{8 + 88S_1 r_n}, \\ y_n &= \left(\frac{n}{6n+5}\right) x_n + \left(1 - \frac{n}{6n+5}\right)\left(u_n - \frac{1}{2N}\sum_{i=1}^{N}\left(\frac{1}{2^i} + \frac{1}{N2^N}\right)\frac{iu_n}{9}\right), \\ x_{n+1} &= \left(\frac{1}{6n}\right)\frac{x_n}{6} + \left(\frac{3(6n-1)}{30n}\right) x_n + \left(\frac{2(6n-1)}{30n}\right)\sum_{i=1}^{N}\left(\frac{7}{8^i} + \frac{1}{N8^N}\right)\left(I - \frac{1}{2n^2}(I - T_i)\right) y_n. \end{cases} \tag{5.5}$$

Now, we consider the algorithm 3.2. That is, put $N = 1$, we have $S_1 = 1$. From (5.4), we have

$$u_n = \frac{(8 - r_n)x_n}{8 + 88r_n}. \tag{5.6}$$

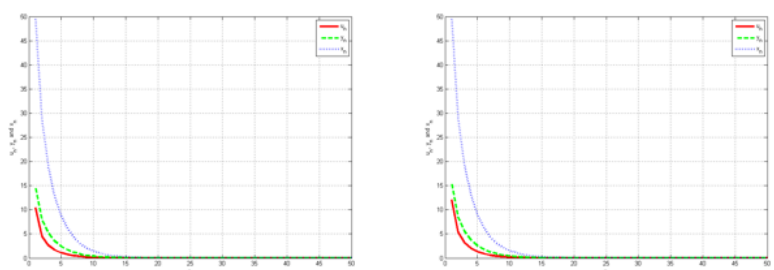
For every $n \in \mathbb{N}$, from (5.6), we rewrite the Algorithm 3.2 as follows:

$$\begin{cases} u_n &= \frac{(8-r_n)x_n}{8+88r_n}, \\ y_n &= \left(\frac{n}{6n+5}\right)x_n + \left(1-\frac{n}{6n+5}\right)\left(u_n - \frac{1}{2}\frac{u_n}{9}\right), \\ x_{n+1} &= \left(\frac{1}{6n}\right)\frac{x_n}{6} + \left(\frac{3(6n-1)}{30n}\right)x_n + \left(\frac{2(6n-1)}{30n}\right)\left(I - \frac{1}{2n^2}(I-T)\right)y_n. \end{cases} \quad (5.7)$$

Table 1 shows that the convergence of Algorithm 3.1 is faster than Algorithm 3.2.

	Algorithm 3.1			Algorithm 3.2		
n	u_n	y_n	x_n	u_n	y_n	x_n
1	10.424901	14.801542	50.000000	12.073864	15.691639	50.000000
2	4.590835	8.061049	28.943542	5.408636	8.533063	29.082503
3	2.660742	5.224570	18.929858	3.172962	5.548394	19.157504
4	1.676671	3.523448	12.775959	2.019204	3.759490	13.021807
5	1.098328	2.415289	8.748418	1.334222	2.590910	8.978801
⋮	⋮	⋮	⋮	⋮	⋮	⋮
45	0.000001	0.000002	0.000008	0.000001	0.000003	0.000010
46	0.000001	0.000002	0.000006	0.000001	0.000002	0.000007
47	0.000000	0.000001	0.000004	0.000001	0.000002	0.000005
48	0.000000	0.000001	0.000003	0.000000	0.000001	0.000004
49	0.000000	0.000001	0.000002	0.000000	0.000001	0.000003
50	0.000000	0.000000	0.000001	0.000000	0.000001	0.000002

Table 1: The values of u_n, y_n, and x_n with an initial value $x_1 = 50$.



(a) Algorithm 3.1 (b) Algorithm 3.2

Figure 1: The values of u_n, y_n, and x_n with an initial value $x_1 = 50$ for Algorithm 3.1 and Algorithm 3.2.

Conclusion

1. *Table 1 shows that the sequence $\{x_n\}$ and $\{u_n\}$ converge to 0, where* $\{0\} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N VI(H, A_i, M) \cap EP(F, \sum_{i=1}^N b_iB_i)$

2. *Theorem 3.3 guarantees the convergence of* $\{x_n\}$ *and* $\{u_n\}$ *to 0 in Example 5.1.*

3. *For case* $N = 1$*, Theorem 3.4 guarantees the convergence of* $\{x_n\}$ *and* $\{u_n\}$ *to 0 in Example 5.1.*

4. *The convergence of the Algorithm 3.1 is faster than the Algorithm 3.2.*

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