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New General Split Feasibility Problems in Hilbert Spaces

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Abstract : In this paper, we establish the iterative algorithm for finding the solution of a general split feasibility problem (GSFPg) and show that the proposed algorithm converges strongly to solution of (GSFPg). Moreover, some numerical examples are presented to confirm our results.

Keywords : split feasibility problem; iteration method; Hilbert space
2000 Mathematics Subject Classification : 47H09; 47H10 (2000 MSC)

1 Introduction

Let H_1 and H_2 be infinite-dimensional real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Censor and Elfving was first introduced the split

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feasibility problem (SFP) in [1]. It can be formulated as the problem of finding a point x satisfying the property :

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.1}$$

where C and Q are nonempty, closed and convex subset in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The split feasibility problem (SFP) in the setting of finite-dimentional Hilbert spaces was introduce for modelling inverse problem which arise from phase retrievals and in medical image reconstruction [2]. Since then, a lot of work has been done on finding a solution of split feasibility problem (SFP). It has been found that the (SFP) can also be used to study the intensity-modulated radiation therapy. There are many algorithms invented to solve the (SFP), see e.g., [3, 4, 5] and references therein.

A special case of the SFP is the convexly constrained linear inverse problem (CLIP) in a finite dimensional real Hilbert space, that is to find $x^* \in C$ such that

$$Ax^* = b, \tag{1.2}$$

where C is a nonempty closed convex subset of a real Hilbert space H_1 and b is a given element of a real Hilbert space H_2, which has extensively been investigated to solve solution by using the well-known Landweber iterative method:

$$x_{n+1} = x_n + \gamma A^T(b - Ax_n), \qquad \forall n \in \mathbb{N}. \tag{1.3}$$

Otherwise, Mohammad and Abdul [6] considered a general split feasibility in infinite-dimensional real Hilbert spaces, that is to find x^* such that

$$x^* \in \bigcap_{i=1}^{\infty} C_i, \quad Ax^* \in \bigcap_{i=1}^{\infty} Q_i, \tag{1.4}$$

where $A : H_1 \to H_2$ and two sequences $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ are the families of nonempty closed convex subsets of H_1 and H_2, respectively.

In this paper, we consider a general split feasibility problem (for short GSFPg) which is different from [6], that is to find $x^* \in H_1$

$$g(x^*) \in \bigcap_{i=1}^{\infty} C_i \qquad \text{such that} \qquad Ag(x^*) \in \bigcap_{i=1}^{\infty} Q_i, \tag{1.5}$$

where $g : H_1 \to H_2$ is a continuous mapping. We denote the solution set of (1.5) by Ω. The GSFPg can be reduced to the following problem;

find a point $x^* \in H_1$ such that

$$g(x^*) \in C \qquad \text{and} \qquad Ag(x^*) \in Q. \tag{1.6}$$

In 2013, Mohammad and Abdul [6] purposed the cyclic algorithm to solve GSFP 1.4 as follows;

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n, \qquad n \geq 0 \quad (1.7)$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. They proved that the sequences $\{x_n\}$ converges strongly to solution of GSFP.

In this paper, we establish the iterative algorithm for finding the solution of a general split feasibility problem (GSFPg) and show that the proposed algorithm converges strongly to solution of (GSFPg). Moreover, some numerical examples are presented to confirm our results.

2 Preliminaries

Throughout the paper, we denote H by a real Hilbert space with inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H and $x \in H$. Weak convergence and strong convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

The operator P_C is called the metric projection or the nearest point mapping of H onto C. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \leq 0, \quad \forall x \in H, y \in C. \tag{2.2}$$

Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.3}$$

It is well known that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. *[6] Let H be a Hilbert space. Then, for all $x, y \in H$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.5}$$

Lemma 2.2. *([7]) Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integer i, j with $i < j$,*

$$\|\sum_{n=1}^{\infty} \lambda_n x_n\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{2.6}$$

Lemma 2.3. *([8]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n, \qquad n \geq 0 \tag{2.7}$$

where $\{\gamma_n\}, \{\beta_n\}$, and $\{\delta_n\}$ satisfy the following conditions:

(i) $\gamma_n \subset [0,1], \sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$;

(iii) $\beta_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *[9] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then, T is demiclosed on C, that is, if $y_n \rightharpoonup z \in C$, and $(y_n - Ty_n) \to y$, then $(I-T)z = y$.*

Lemma 2.5. *[10] Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{\mu(n)\} \subset \mathbb{N}$ such that $\mu(n) \to \infty$, and the following properties are satisfied by all (sufficiently large) number $n \in \mathbb{N}$:*

$$t_{\mu(n)} \leq t_{\mu(n)+1}, \qquad t_n \leq t_{\mu(n)+1}. \tag{2.8}$$

In fact

$$\mu(n) = \max\{k \leq n : t_k < t_{k+1}\}. \tag{2.9}$$

Proposition 2.1. *[8] For given $x \in H_1$ and $z \in C, z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \qquad \text{for all} \quad y \in C. \tag{2.10}$$

Proposition 2.2. *Given $x^* \in H_1$. Then $g(x^*)$ solves the GSFPg (1.5) if only if $g(x^*)$ solves the fixed point equation*

$$P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)g(x^*) = g(x^*), \tag{2.11}$$

for all $i \in \mathbb{N}$

Proof. Let $\lambda_{n,i} > 0$, assume that $x^* \in \Omega$. Thus $Ag(x^*) \in \bigcap_{i=1}^{\infty} Q_i$ which implies that $(I - P_{Q_i})Ag(x^*) = 0$ and implies the equation $\lambda_{n,i}A^*(I - P_{Q_i})Ag(x^*) = 0$. Then

$(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x^*) = g(x^*)$. Requiring that $g(x^*) \in \bigcap_{i=1}^{\infty} C_i$, we obtain the fixed point equation:

$$P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x^*) = g(x^*).$$

Conversely, assume that $g(x^*)$ solves the fixed point equation (2.11). Then, for all $y \in \bigcap_{i=1}^{\infty} C_i$ by proposition 2.1 we obtain that

$$\langle (I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x^*) - g(x^*), y - g(x^*)\rangle \leq 0. \tag{2.12}$$

That is

$$\begin{array}{ll} \langle A^*(I-P_{Q_i})Ag(x^*), y-g(x^*)\rangle & \geq 0 \\ \langle Ag(x^*) - P_{Q_i}Ag(x^*), Ay - Ag(x^*)\rangle & \geq 0 \\ \langle Ag(x^*) - P_{Q_i}Ag(x^*), Ag(x^*) - Ay\rangle & \leq 0. \end{array} \tag{2.13}$$

Since $g(x^*) = P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x^*)$, then we have

$$P_{Q_i}[Ag(x^*)] = P_{Q_i}[P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x^*)].$$

By proposition 2.1 for all $v \in \bigcap_{i=1}^{\infty} Q_i$,

$$\langle Ag(x^*) - P_{Q_i}Ag(x^*), v - P_{Q_i}Ag(x^*)\rangle \leq 0. \tag{2.14}$$

Adding two equations (2.13) and (2.14), then we have

$$\langle Ag(x^*) - P_{Q_i}Ag(x^*), Ag(x^*) - P_{Q_i}Ag(x^*) + v - Ay\rangle \leq 0,$$

for all $v \in \bigcap_{i=1}^{\infty} Q_i$ and $y \in \bigcap_{i=1}^{\infty} C_i$.

So, we see that

$$\langle Ag(x^*) - P_{Q_i}Ag(x^*), v - Ay\rangle \leq 0.$$

Since $g(x^*) \in \bigcap_{i=1}^{\infty} C_i$, then we can put $z = Ag(x^*) \in \bigcap_{i=1}^{\infty} C_i$. For all $v \in \bigcap_{i=1}^{\infty} Q_i$ we obtain that

$$\langle Ag(x^*) - P_{Q_i}Ag(x^*), v - Ag(x^*)\rangle \leq 0,$$

and so $Ag(x^*) = P_{Q_i}Ag(x^*) \in \bigcap_{i=1}^{\infty} Q_i$.

□

3 Main Results

In the following result, we propose an algorithm and prove that the sequence generated by the proposed algorithm converges strongly to a solution of GSFPg.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and let $A : H_1 \to H_2$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ be two families of nonempty closed convex subsets of H_1 and H_2, respectively and assume that $C = \bigcap_{i=1}^{\infty} C_i$ and $Q = \bigcap_{i=1}^{\infty} Q_i$. Let $M \subset H_1$ and suppose that $g : M \to C$ is a bijection continuous function and $g^{-1} : C \to M$ is a continuous function. Assume that GSFPg has a nonempty solution set Ω and f is a self k-contraction mapping of H_1. Let $\{x_n\}$ be a sequence generated by $x_0 \in M$ as*

$$g(x_{n+1}) = \alpha_n g(x_n) + \beta_n f g(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A) g(x_n) \quad (3.1)$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_{n,i}\} \in (0,1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) *for each* $i \in \mathbb{N}, \liminf_{n\to\infty} \alpha_n \gamma_{n,i} > 0$,

(iii) *for each* $i \in \mathbb{N}, \{\lambda_{n,i}\} \subset (0, \frac{2}{\|A\|^2})$ *and* $0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \frac{2}{\|A\|^2}$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $g(x^*)$ solves the following variational inequality;

$$\langle (f - I) g(x^*), g(x) - g(x^*) \rangle \leq 0, \quad \forall g(x) \in C \cap A^{-1}(Q). \quad (3.2)$$

Proof. First, we will show that $g(x_n)$ is bounded and let $p \in \Omega$. For $i = 1, 2, 3, ..., n$, we see that $P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)$ are nonexpansive mappings and so we have

$$\begin{aligned}
& \|g(x_{n+1})-g(p)\| \\
= \ & \|[\alpha_n g(x_n)+\beta f g(x_n)+\sum_{i=1}^{\infty}\gamma_{n,i}P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)]-g(p)\| \\
\leq \ & \alpha_n\|g(x_n)-g(p)\|+\beta_n\|fg(x_n)-g(p)\| \\
& +\|\sum_{i=1}^{\infty}\gamma_{n,i}P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)+\alpha_n g(p)+\beta_n g(p)-g(p)\| \\
\leq \ & \alpha_n\|g(x_n)-g(p)\|+\beta_n\|fg(x_n)-g(p)\| \\
& +\|\sum_{i=1}^{\infty}\gamma_{n,i}P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)+(\alpha_n+\beta_n-1)g(p)\| \\
\leq \ & \alpha_n\|g(x_n)-g(p)\|+\beta_n\|fg(x_n)-g(p)\| \\
& +\sum_{i=1}^{n}\gamma_{n,i}\|P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)-g(p)\|.
\end{aligned}$$

Since $P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A$ is nonexpensive for all $i=1,2,3,...,n$, then we get that

$$\begin{aligned}
\|g(x_{n+1})-g(p)\| \ & \leq \ \alpha_n\|g(x_n)-g(p)\|+\beta_n\|fg(x_n)-g(p)\| \\
& \quad +\sum_{i=1}^{n}\gamma_{n,i}\|g(x_n)-g(p)\| \\
& \leq \ (1-\beta_n)\|g(x_n)-g(p)\|+\beta_n\|fg(x_n)-g(p)\| \\
& \leq \ (1-\beta_n)\|g(x_n)-g(p)\|+\beta_n\|fg(x_n)-fg(p)\| \\
& \quad +\beta_n\|fg(p)-g(p)\| \\
& \leq \ (1-\beta_n)\|g(x_n)-g(p)\|+\beta_n k\|g(x_n)-g(p)\| \\
& \quad +\beta_n\|fg(p)-g(p)\| \\
& \leq \ (1-\beta_n+\beta_n k)\|g(x_n)-g(p)\|+\beta_n\|fg(p)-g(p)\| \\
& \leq \ (1-(1-k)\beta_n)\|g(x_n)-g(p)\|+\beta_n\|fg(p)-g(p)\| \\
& \leq \ (1-(1-k)\beta_n)\|g(x_n)-g(p)\| \\
& \quad +(1-k)\beta_n.\frac{1}{1-k}\|fg(p)-g(p)\| \\
& \leq \ \max\{\|g(x_n)-g(p)\|,\frac{1}{1-k}\|fg(p)-g(p)\|\} \\
& \ . \\
& \ . \\
& \ . \\
& \leq \ \max\{\|g(x_0)-g(p)\|,\frac{1}{1-k}\|fg(p)-g(p)\|\}.
\end{aligned}$$

Therefore $\{g(x_n)\}$ is bounded, and also $\{fg(x_n)\}$.

Next, we will show that for each $i\in\mathbb{N}$,

$$\lim_{n\to\infty}\|g(x_n)-P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)\|=0. \tag{3.3}$$

Since every $p\in\Omega$, by using Lemma 2.2, we obtain that

$$
\begin{aligned}
& \|g(x_{n+1}) - g(p)\|^2 \\
= \;& \|\alpha_n g(x_n) + \beta_n fg(x_n) + \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_j})A)g(x_n) - g(p)\|^2 \\
= \;& \|\alpha_n [g(x_n) - g(p)] + \beta_n [fg(x_n) - g(p)] \\
& + \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_j})A)g(x_n) - g(p) + \alpha_n g(p) + \beta_n g(p)\|^2 \\
= \;& \|\alpha_n [g(x_n) - g(p)] + \beta_n [fg(x_n) - g(p)] \\
& + \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_j})A)g(x_n) - (1 - \alpha_n - \beta_n) g(p)\|^2 \\
= \;& \alpha_n [g(x_n) - g(p)] + \beta_n [fg(x_n) - g(p)] \\
& + \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_j})A)g(x_n) - g(p)\|^2 \\
\le \;& \alpha_n \|g(x_n) - g(p)\|^2 + \beta_n \|fg(x_n) - g(p)\|^2 \\
& + \sum_{j=1}^{\infty} \gamma_{n,j} \|P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_j})A)g(x_n) - g(p)\|^2 \\
& - \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)g(x_n) - g(p) - [g(x_n) - g(p)]\|^2 \\
\le \;& \alpha_n \|g(x_n) - g(p)\|^2 + \beta_n \|fg(x_n) - g(p)\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j} \|g(x_n) - g(p)\|^2 \\
& - \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)g(x_n) - g(p) - [g(x_n) - g(p)]\|^2.
\end{aligned}
$$

Hence, for each $i \in \mathbb{N}$, we have

$$
\begin{aligned}
& \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)g(x_n) - g(p) - [g(x_n) - g(x_n)]\|^2 \\
& \le \|g(x_n) - g(p)\|^2 - \|g(x_{n+1}) - g(p)\|^2 + \beta_n \|fg(x_n) - g(p)\|^2.
\end{aligned} \tag{3.4}
$$

In order to prove that $x_n \to x^*$ as $n \to \infty$, we consider two possible cases.

Case I: Assume that $\{\|g(x_n) - g(x^*)\|\}$ is monotone sequence. We have $\|g(x_n) - g(x^*)\|$ is convergent. Since $\lim_{n\to\infty} \beta_n = 0$ and $\{fg(x_n)\}$ is bounded, we get that

$$
\lim_{n\to\infty} \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)g(x_n) - g(x_n)\|^2 = 0. \tag{3.5}
$$

By assuming that $\lim_{n\to\infty} \inf \alpha_n \gamma_{n,i} > 0$, we obtain that

$$
\lim_{n\to\infty} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)g(x_n) - g(x_n)\| = 0, \qquad \forall i \in \mathbb{N}. \tag{3.6}
$$

Now, we will show that

$$
\limsup_{n\to\infty} \langle fg(x^*) - g(x^*), g(x_n) - g(x^*)\rangle \le 0. \tag{3.7}
$$

To show this inequality, we choose a subsequence $\{g(x_{n_k})\}$ of $\{g(x_n)\}$ such that

$$\lim_{k\to\infty}\langle fg(x^*)-g(x^*),g(x_{n_k})-g(x^*)\rangle=\limsup_{n\to\infty}\langle fg(x^*)-g(x^*),g(x_n)-g(x^*)\rangle. \tag{3.8}$$

Since $\{g(x_{n_k})\}$ is bounded, there exists a subsequence $\{g(x_{n_{k_j}})\}$ of $\{g(x_{n_k})\}$ which converges weakly to $g(\omega)$ where $\omega\in H_1$. Without loss of generality, we can assume that $g(x_{n_k})\rightharpoonup g(\omega)$

Notice that for each $i\in\mathbb{N}, P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)$ is nonexpansive. Thus, from Lemma 2.4, we have $P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(\omega)=g(\omega)$ and by proposition 2.2, we also have $\omega\in\Omega$. Therefore, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle fg(x^*)-g(x^*),g(x_n)-g(x^*)\rangle &=\lim_{k\to\infty}\langle fg(x^*)-g(x^*),g(x_{n_k})-g(x^*)\rangle\\ &=\langle fg(x^*)-g(x^*),g(\omega)-g(x^*)\rangle\\ &\leq 0.\end{aligned}$$

Finally, we show that $x_n\to x^*$. Apply Lemma 2.1, we have that

$$\begin{aligned}&\|g(x_{n+1})-g(x^*)\|^2\\ =\ &\|\alpha_n g(x_n)+\beta_n fg(x_n)+\sum_{i=1}^{\infty}\gamma_{n,i}P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)-g(x^*)\|^2\\ =\ &\|(\alpha_n g(x_n)-g(x^*))+\sum_{i=1}^{\infty}\gamma_{n,i}(P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)-g(x^*))\\ &+\alpha_n g(x^*)+\sum_{i=1}^{\infty}\gamma_{n,i}g(x^*)+\beta_n fg(x_n)-g(x^*)\|^2\\ \leq\ &\|\alpha_n(g(x_n)-g(x^*))+\sum_{i=1}^{\infty}\gamma_{n,i}(P_{C_i}(I-\lambda_{n,i}A^*(I-P_{Q_i})A)g(x_n)-g(x^*))\|^2\\ &+2\beta_n\langle fg(x_n)-g(x^*),g(x_{n+1})-g(x^*)\rangle\\ \leq\ &(1-\beta_n)^2\|g(x_n)-g(x^*)\|^2+2\beta_n\langle fg(x_n)-fg(x^*),g(x_{n+1})-g(x^*)\rangle\\ &+2\beta_n\langle fg(x^*)-g(x^*),g(x_{n+1})-g(x^*)\rangle\\ \leq\ &(1-\beta_n)^2\|g(x_n)-g(x^*)\|^2+2\beta_n k\|g(x_n)-g(x^*)\|\|g(x_{n+1})-g(x^*)\|\\ &+2\beta_n\langle fg(x^*)-g(x^*),g(x_{n+1})-g(x^*)\rangle\\ \leq\ &(1-\beta_n)^2\|g(x_n)-g(x^*)\|^2+\beta_n k\{\|g(x_n)-g(x^*)\|^2+\|g(x_{n+1})-g(x^*)\|^2\}\\ &+2\beta_n\langle fg(x^*)-g(x^*),g(x_{n+1})-g(x^*)\rangle.\end{aligned}$$

This implies that

$$
\begin{aligned}
& \|g(x_{n+1}) - g(x^*)\|^2 \\
\leq\ & ((1-\beta_n)^2 + \beta_n k)\|g(x_n) - g(x^*)\|^2 + \beta_n k\|g(x_{n+1}) - g(x^*)\|^2 \\
& +2\beta_n\langle fg(x^*) - g(x^*), g(x_{n+1}) - g(x^*)\rangle \\
\leq\ & (\frac{(1-\beta_n)^2}{1-\beta_n k})\|g(x_n) - g(x^*)\|^2 \\
& +\frac{2\beta_n}{1-\beta_n k}\langle fg(x^*) - g(x^*), g(x_{n+1}) - g(x^*)\rangle \\
=\ & \frac{1-2\beta_n+\beta_n k}{1-\beta_n k}\|g(x_n) - g(x^*)\|^2 + \frac{\beta_n^2}{1-\beta_n k}\|g(x_n) - g(x^*)\|^2 \\
& +\frac{2\beta_n}{1-\beta_n k}\langle fg(x^*) - g(x^*), g(x_{n+1}) - g(x^*)\rangle \\
=\ & (1 - \frac{2(1-k)\beta_n}{1-\beta_n k})\|g(x_n) - g(x^*)\|^2 + \frac{2(1-k)\beta_n}{1-\beta_n k}\{\frac{\beta_n M}{2(1-k)} \\
& +\frac{1}{1-k}\langle fg(x^*) - g(x^*), g(x_{n+1}) - g(x^*)\rangle\} \\
\leq\ & (1-\eta_n)\|g(x_n) - g(x^*)\|^2 + \eta_n\delta_n,
\end{aligned}
$$

where

$$\delta_n = \frac{\beta_n M}{2(1-k)} + \frac{1}{1-k}\langle fg(x^*) - g(x^*), g(x_{n+1}) - g(x^*)\rangle, \tag{3.9}$$

$M = sup\{\|g(x_n) - g(x^*)\|^2 : n \geq 0\}$ and $\eta_n = \dfrac{2(1-k)\beta_n}{1-\beta_n k}$. It is easy to see that $\eta_n \to 0, \sum_{n=1}^{\infty}\eta_n = \infty$ and $\lim_{n\to\infty}\sup\delta_n \leq 0$. Hence, by Lemma 2.3, the sequence $\{g(x_n)\}$ converges strongly to $g(x^*)$. Since g^{-1} is continuous, we have $x_n \to x^*$.

Case II: Assume that $\{\|g(x_n) - g(x^*)\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\mu(n)\}$ for all $n \geq n_0$ by

$$\mu(n) = \max\{k \in \mathbb{N}; k \leq n : \|g(x_k) - g(x^*)\| < \|g(x_{k+1}) - g(x^*)\|\}. \tag{3.10}$$

Clearly, $\mu(n)$ is a nondecreasing sequence such that $\mu(n) \to \infty$ as $n \to \infty$ and $\forall n \geq n_0$,

$$\|g(x_{\mu(n)}) - g(x^*)\| < \|g(x_{\mu(n)+1}) - g(x^*)\|\}. \tag{3.11}$$

From (3.4), we obtain that

$$\lim_{n\to\infty}\|P_{C_i}(I - \lambda_{\mu(n),i}A^*(I - P_{Q_i})A)g(x_{\mu(n)}) - g(x_{\mu(n)})\| = 0. \tag{3.12}$$

Following an argument similar to I, we have

$$\limsup_{n\to\infty}\langle fg(x^*) - g(x^*), g(x_{\mu(n)+1}) - g(x^*)\rangle \leq 0. \tag{3.13}$$

And by similar argument, we have

$$\|g(x_{\mu(n)+1}) - g(x^*)\|^2 \le (1 - \eta_{\mu(n)})\|g(x_{\mu(n)}) - g(x^*)\|^2 + \eta_{\mu(n)}\delta_{\mu(n)}, \qquad (3.14)$$

where $\eta_{\mu(n)} \to \infty, \sum_{n=1}^{\infty} \eta_{\mu(n)} = \infty$ and $\lim_{n\to\infty} \sup \delta_{\mu(n)} \le 0$.

Hence, by Lamma 2.3, we obtain $\lim_{n\to\infty} \|g(x_{\mu(n)}) - g(x^*)\| = 0$ and $\lim_{n\to\infty} \|g(x_{\mu(n)+1}) - g(x^*)\| = 0$. Now, from Lemma 2.5, we have

$$\begin{aligned} 0 &\le \|g(x_n) - g(x^*)\| \\ &\le \max\{\|g(x_{\mu(n)}) - g(x^*)\|, \|g(x_n) - g(x^*)\|\} \\ &\le \|g(x_{\mu(n)+1}) - g(x^*)\|. \end{aligned} \qquad (3.15)$$

Therefore, $\{g(x_n)\}$ converges to $g(x^*)$. Since g^{-1} is continuous, we have $\{x_n\}$ converges strongly to x^*. □

For all $i \in \mathbb{N}$, We put $C = C_i$, $Q = Q_i$ and $\lambda_{n,i} = \lambda_n$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \gamma_{n,i} = 1 - \alpha_n - \beta_n$. Then we have the following corollary.

Corollary 3.2. *Let H and K be a real Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Let C and Q be a closed convex subsets of H and K, respectively. Let $M \subset H$ suppose that $g : M \to C$ is a bijection continuous function and $g^{-1} : C \to M$ is continuous function. Suppose that f is a self k-contraction mapping of H, and let $\{x_n\}$ be a sequence generated by $x_0 \in M$ as*

$$g(x_{n+1}) = \alpha_n g(x_n) + \beta_n f g(x_n) + (1 - \alpha_n - \beta_n) P_C(I - \lambda_n A^*(I - P_Q)A)g(x_n), \qquad n \ge 0 \qquad (3.16)$$

If the nonnegative sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\} \in (0, 1)$ satisfy the following conditions :

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$,

(iii) $\{\lambda_n\} \subset (0, \frac{2}{\|A\|^2})$ *and* $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2}{\|A\|^2}$.

Then, the sequences $\{x_n\}$ converges strongly to x^*, where $g(x^*)$ solves the following variational inequality;

$$\langle (f - I)g(x^*), g(x) - g(x^*) \rangle \le 0, \quad \forall g(x) \in C \cap A^{-1}(Q). \qquad (3.17)$$

Setting a continuous operator $g \equiv I$. Then we have the following corollary

Corollary 3.3. *(Theorem 6 [6]) Let H and K be a real Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ be the families of nonempty closed convex subsets of H and K, respectively. Suppose that f is a self k-contraction mapping of H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ as*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n, \qquad n \geq 0 \quad (3.18)$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions :

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) for each $i \in \mathbb{N}$, $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$,

(iii) for each $i \in \mathbb{N}$, $\{\lambda_{n,i}\} \subset (0, \frac{2}{\|A\|^2})$ *and* $0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \frac{2}{\|A\|^2}$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where x^* solves the following variational inequality;

$$\langle (f - I)x^*, x - x^* \rangle \leq 0, \quad \forall x \in C \cap A^{-1}(Q). \quad (3.19)$$

4 The Nunerical Result

In this section, let us present the following numerical example to confirm the convergence of our theoretical results.

Example 4.1. *Let $H_1 = H_2 = \mathbb{R}$, $C = [0, 1]$ and $Q_i = [-1 - \frac{1}{i}, 1 + \frac{1}{i}]$. Let $A : H_1 \to H_2$ be a operators defined by $Ax = 2x$, and let f be a contraction defined by $f(x) = \frac{1}{2}x$. Let $M = [-1, -0.5]$ and define a function $g : [-1, -0.5] \to [0, 1]$ by $g(x) = 2x + 2$ then g is a bijective continuous function and g^{-1} is a continuous function. Assume the parameters that $\{\beta_n\} = \frac{1}{n+1}, \{\alpha_n\} = \frac{1}{2}(\frac{n}{n+1}), \{\gamma_{n,i}\} = (1 - \frac{1}{n+1})\frac{1}{3^i}$ and $\{\lambda_{n,i}\} = (1 - \frac{1}{n+2})\frac{1}{3^{i+1}}$. From (3.1), we obtain the following algorithm;*

Algorithm *(The general split feasibility Problems in Hilbert Spaces)*

Step 1. *Choose the initial point* $x_0 \in M$ *and compute* $g(x_0)$*. Let* $n = 1$*.*

Step 2. *Given* $x_n \in H_1$ *and compute* $x_{n+1} \in H_1$ *as follows;*

$$\begin{cases} y_n = \sum_{i=1}^{\infty} \gamma_{n,i} P_C(I - \lambda_{n,i} A^*(I - P_{Q_i})A)g(x_n); \\ \\ g(x_{n+1}) = \alpha_n g(x_n) + \beta_n f g(x_n) + y_n \end{cases} \tag{4.1}$$

Step 3. *Put* $n := n + 1$ *and go to step 2.*

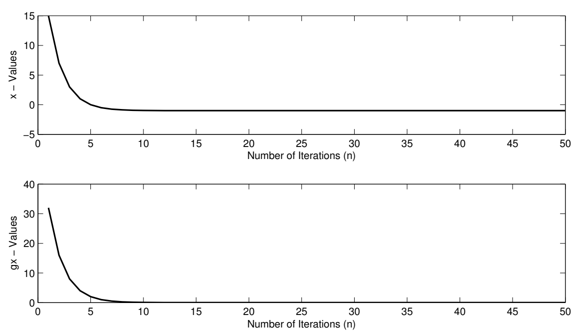

Figure 1: The convergent behaviors of $g(x_n)$ and x_n

It is easy to see that all of conditions in Theorem 3.1 are satisfied. First we take $x_0 = -1$, Figure 1 shows that the sequence x_n converge to 0 which is the solution of this example and the sequence $g(x_n)$ converge to 0, i.e., $g(-1) = 0 \in C \cap A^{-1}(Q)$ as a solution of this example where $Q = \bigcap_{i=1}^{\infty} Q_i$.

Next, we take three initial point randomly generated by Matlab. In this way, Figure 2 and Figure 3 indicate that x_n and $g(x_n)$ converge to the same points, respectively.

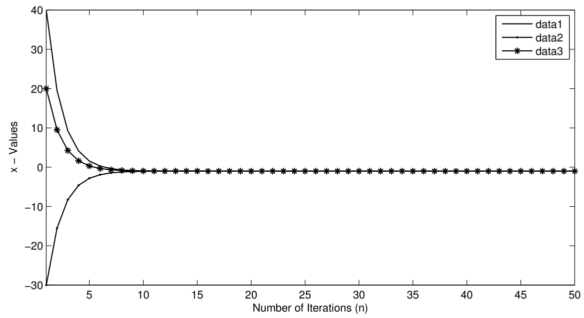

Figure 2: The convergent behaviors of x_n

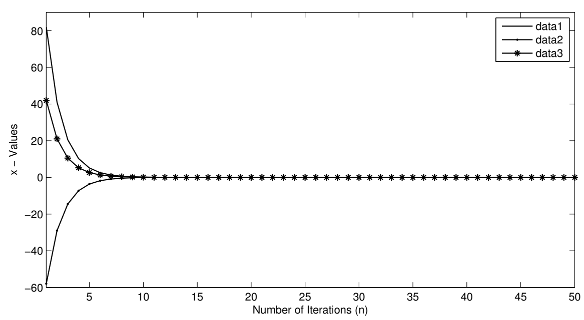

Figure 3: The convergent behaviors of $g(x_n)$

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