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Fixed Point Theorems for F_w-Contractions in Complete S-Metric Spaces

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Abstract : In this paper, we define a w-distance on a complete S-metric space, which is a generalization of the concept of the w-distance due to Kada, Suzuki and Takahashi. Also, we introduce the concept of the F_w-contraction in a complete S-metric space and extend the fixed point theorem.

Keywords : fixed point; F-contraction; F_w-contraction; S-metric space
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1 Introduction

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions; see [1],[4],[5] and [9] and the reference therein. In [6], Dhage introduced the D-metric space as a generalization of the metric space and proved some results in this setting. In [13], S. Sedghi, N. Shobe and A. Aliouche introduced the notion of S-metric space which is a generalization of G-metric space of [6] and D^*- metric space of [14] and proved some fixed point theorems on S-metric space. Later, S. Sedghi, N. V. Dung [12] proved generalized fixed point theorems in S-metric spaces which is a generalization of [13].

In [15], Wardowski introduce a new type of contractions called F-contraction and prove a new fixed point theorem concerning F-contractions. In this way, Wardowski [15] generalized the Banach contraction principle in a different manner from the well-known results from the literature. In [2], Batra and Vashistha generalized the concept of the F-contraction to the F_w-contraction and proved a fixed point theorem for the F_w-contraction in a complete metric space.

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In this paper, using the concept of the S-metric, we define a w-distance on a complete S-metric space, which is a generalization of the concept of the w-distance due to Kada, Suzuki and Takahashi [7]. Also, we introduce the concept of the F_w-contraction in a complete S-metric space and extend the fixed point theorem. In another way, we introduce the concept of the F_w-contraction of Hardy-Rogers-type in a complete S-metric space.

2 Preliminaries

In [13], S. Sedghi, N. Shobe and A. Aliouche have introduced a new structure of generalized metric spaces as follows.

Definition 2.1. [13] *Let X be a nonempty set. An S-*metric *on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.*

(i) $S(x, y, z) = 0$ *if and only if* $x = y = z$.

(ii) $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$.

*The pair (X, S) is called an S-*metric space.

Immediate examples of such S-metric spaces are:

Example 2.2. [13]

(i) *Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.*

(ii) *Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.*

Lemma 2.3. [13] *Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 2.4. [13] *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \le 2S(x, x, y) + S(y, y, z) \quad \text{and} \quad S(x, x, z) \le 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Definition 2.5. [13] *Let (X, S) be an S-metric space.*

(i) *A sequence $\{x_n\} \subset X$ is said to* converge *to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.*

(ii) *A sequence $\{x_n\} \subset X$ is called a* Cauchy sequence *if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.*

(iii) *The S-metric space (X, S) is said to be* complete *if every Cauchy sequence is a convergent sequence*

Lemma 2.6. [13] *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

In [15], Wardowski introduced a new concept of F-contraction on a complete metric spaces as follows.

Definition 2.7. [15] *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*

(F1) *F is strictly increasing. That is, $\alpha < \beta \implies F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}^+$.*

(F2) *For every sequence $\{\alpha_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

(F3) *There exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Definition 2.8. *Let (X, S) be a S-metric space. A mapping $T : X \to X$ is said to be a F-*contraction *if there exists a number $\tau > 0$ such that*

$$S(Tx, Ty, Tz) > 0 \implies \tau + F(S(Tx, Ty, Tz)) \leq F(S(x, y, z)) \quad \text{for all } x, y, z \in X.$$

Remark 2.9. *Clearly Definition 2.8 and (F1) implies that $S(Tx, Ty, Tz) < S(x, y, z)$ for all $x, y, z \in X$ with $Tx \neq Ty \neq Tz$. Hence every F-contraction mapping is continuous.*

In [7], Kada, Suzuki and Takahashi introduced the concept of a weak distance in a metric space. Analogously we define w-distance in a S-metric space as follows.

Definition 2.10. *Let (X, S) be a S-metric space. A function $\rho : X^3 \to [0, \infty)$ is called a w-*distance *on X if the following conditions hold:*

(w1) *$\rho(x, y, z) \leq \rho(a, a, x) + \rho(a, a, y) + \rho(a, a, z)$ for all $x, y, z, a \in X$;*

(w2) *for any $x, y \in X, \rho(x, x, .) : X \to [0, \infty)$ are lower semicontinuous;*

(w3) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\rho(a, a, x) \leq \delta, \rho(a, a, y) \leq \delta \text{ and } \rho(a, a, z) \leq \delta \implies S(x, y, z) \leq \varepsilon.$$

Example 2.11. (i) *Let (X, S) is a S-metric space and $\rho : X^3 \to [0, \infty]$ is defined by $\rho(x, y, z) = S(x, y, z)$ for $x, y, z \in X$ then ρ is a w-distance on X. In fact, (w1) holds in view of Definition 2.1(ii) and Lemma 2.3; (w2) holds in view of Lemma 2.6 and finally, for a given $\varepsilon > 0$, taking $\delta = \frac{\varepsilon}{3}$ it is easy to verify (w3) in view of Definition 2.1(ii). That is every S-metric on a set X is a w-distance on X.*

(ii) *Suppose* $X_0 = \{0, \infty\} \cup \{\frac{1}{n} : n \geq 1\}$ *and define* $S : X_0^3 \to [0, \infty)$ *by* $S(x, y, z) = |x - z| + |y - z|$ *for* $x, y, z \in X_0$*. Then* (X_0, S) *is a* S*-metric space (as a special case of Example 2.2(ii)). Define* $\rho : X_0^3 \to [0, \infty)$ *by* $\rho(x, y, z) = y + 2z$ *for* $x, y, z \in X_0$ *Then* ρ *is a* w*-distance on* X_0*. To verify this, for* $x, y, z, a \in X$ *note that* $\rho(a, a, x) + \rho(a, a, y) + \rho(a, a, z) = 2(x+y+z)+3a \geq y+2z = \rho(x, y, z)$ *which gives (w1); if* $\{y_n\} \in X_0$ *converges to* y *in* (X_0, S) *then (w2) holds, since* $\lim_{n\to\infty} \rho(x, x, y_n) = \lim_{n\to\infty}(x + 2y_n) = x + 2y = \rho(x, x, y)$ *and finally for* $\varepsilon > 0$ *taking* $\delta = \frac{\varepsilon}{3}$*, we find that* $\rho(a, a, x) \leq \delta, \rho(a, a, y) \leq \delta$ *and* $\rho(a, a, z) \leq \delta$ *imply* $2(x+y+z)+3a \leq 3\delta \leq \varepsilon$ *so that* $S(x, y, z) = |x + y - 2z| < 2(x + y + z) + 3a \leq \varepsilon$*, proving (w3).*

Remark 2.12. *For a* w*-distance* ρ *on a* S*-metric space* (X, S) *observe that* $\rho(x, y, z) = 0$ *need not imply* $x = y = z$*. Therefore* $\rho(x, x, y)$ *and* $\rho(y, y, x)$ *need not be equal for* $x, y \in X$*. For instance, in Example 2.11(ii), note that* $\rho(a, 0, 0) = 0$ *for all* $a \in X$*.*

Lemma 2.13. *Let* (X, S) *is a* S*-metric space and* ρ *is a* w*-distance on* X*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, \infty)$ *such that* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ *and let* $x, y, z \in X$*. Then we have the following:*

(i) $\rho(x_n, x_n, y) \leq \alpha_n$ *and* $\rho(x_n, x_n, z) \leq \beta_n$ *for every* $n \geq 1$ *imply* $S(y, y, z) < \varepsilon$ *and hence* $y = z$*.*

(ii) $\rho_n(x_n, x_n, y_n) \leq \alpha_n$ *and* $\rho(x_n, x_n, z) \leq \beta_n$ *for every* $n \geq 1$ *imply that* $S(y_n, y_n, z) \to 0$ *and hence* $y_n \to z$ *as* $n \to \infty$ *in* (X, S)

(iii) $\rho(x_m, x_m, x_n) \leq \alpha_n$ *for all* $m > n \geq 1$ *implies* $\{x_n\}$ *is a Cauchy sequence in* (X, S)

(iv) $\rho(y, y, x_n) \leq \alpha_n$ *for every* $n \geq 1$ *implies* $\{x_n\}$ *is a Cauchy sequence in* (X, S)

Proof. We first prove (ii). Let $\varepsilon > 0$ be given. From the definition of w-distance, there exists a $\delta > 0$ such that $\rho(a, a, u) \leq \delta$ and $\rho(a, a, v) \leq \delta$ imply $S(u, u, v) \leq \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every $n \geq n_0$.Then we have, for any $n \geq n_0$, $\rho(x_n, x_n, y_n) \leq \alpha_n \leq \delta$, $\rho(x_n, x_n, z) \leq \beta_n \leq \delta$, and hence $S(y_n, y_n, z) \leq \varepsilon$, so that $\{y_n\}$ converges to z. It follows from (ii) that (i) holds. Let us now prove (iii). Let $\varepsilon > 0$ be given. As in the proof of (ii), choose $\delta > 0$ such that $\rho(x_n, x_n, x_m) \leq \delta$ for $n, m \geq n_0$. In particular, $\rho(x_{n_0}, x_{n_0}, x_m) \leq \delta$ and $\rho(x_{n_0}, x_{n_0}, x_k) \leq \delta$ for $m > k \geq n_0$ which imply by (w3) that $S(x_m, x_m, x_k) \leq \varepsilon$. whenever $m > k \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence in (X, S). Now prove (iv). For $n \geq n_0$, we have $\rho(y, y, x_n) \leq \delta$ so that for $m > n \geq n_0$, $\rho(y, y, x_m) < \delta$ and $\rho(y, y, x_n) < \delta$ and hence by (w3), $S(x_m, x_m, x_n) \leq \varepsilon$ for $m > n \geq n_0$ giving $\{x_n\}$ is a Cauchy sequence in (X, S). □

We now define the notion of the F_w-contraction in a S-metric space and give some examples.

Definition 2.14. *Let (X,S) be a S-metric space and ρ be a w-distance on X. Let F be a mapping as defined in Definition 2.7. A mapping $T : X \to X$ is said to be a F_w-*contraction *if*

(i) $\rho(x,y,z) = 0 \implies \rho(Tx,Ty,Tz) = 0$;

(ii) *There exists a number $\tau > 0$ such that*

$$\tau + F(\rho(Tx,Ty,Tz)) \leq F(\rho(x,y,z)) \qquad \text{for all } x,y,z \in X$$

with $\rho(Tx,Ty,Tz) > 0$.

If condition (ii) replace by

(ii′) *There exists a number $\tau > 0$ such that*

$$\tau + F(\rho(Tx,Tx,Ty)) \leq F\Big(\alpha\rho(x,x,y) + \beta\rho(x,x,Tx) + \gamma\rho(y,y,Ty) + \delta\rho(x,x,Ty) + L\rho(y,y,Tx)\Big), \tag{2.1}$$

for all $x,y \in X$ with $\rho(Tx,Tx,Ty) > 0$, where $\alpha+\beta+\gamma+3\delta = 1, \gamma \neq 1$ and $L \geq 0$.

*Then T is called an F-*contraction of Hardy-Rogers-type.

Remark 2.15. *Clearly, (ii) of Definition 2.14 implies that*

$$\rho(Tx,Ty,Tz) < p(x,y,z) \qquad \text{for all } x,y,z \in X \quad \text{with} \quad \rho(Tx,Ty,Tz) > 0.$$

Example 2.16. *Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = \ln\alpha$. Then F satisfies* (F1), (F2) *and* (F3) *(for all $k \in (0,1)$) of Definition 2.7. A mapping $T : X \to X$ satisfies*

$$\rho(Tx,Ty,Tz) \leq \lambda\rho(x,y,z), \tag{2.2}$$

for all $x,y,z \in X$ and some $\lambda \in [0,1)$ if and only if T is a F_w-contraction.

Proof. Let us start with a mapping $T : X \to X$ satisfying (2.2). If $\lambda = 0$ then (i) and (ii) in Definition 2.14 are satisfied. For $0 < \lambda < 1$, (i) is obvious and (ii) is satisfied for $\tau = \ln\frac{1}{\lambda}$. Thus T is a F_w-contraction. Conversely, if $T : X \to X$ is a F_w-contraction then (ii) of Definition 2.14 implies that $\rho(Tx,Ty,Tz) \leq e^{-\tau}\rho(x,y,z)$ for all $x,y,z \in X$ with $\rho(Tx,Ty,Tz) > 0$. Clearly it is satisfied even for $\rho(Tx,Ty,Tz) = 0$. Thus $\rho(Tx,Ty,Tz) \leq \lambda\rho(x,y,z)$ for all $x,y,z \in X$, where $\lambda = e^{-\tau} \in [0,1)$. □

Example 2.17. (i) *Consider $G(\alpha) = \ln\alpha + \alpha$ for all $\alpha > 0$. Then G satisfies* (F1), (F2) *and* (F3) *of Definition 2.7. A mapping $T : X \to X$ is an G_w-contraction if and only if*

$$\rho(Tx,Ty,Tz)e^{\rho(Tx,Ty,Tz)-\rho(x,y,z)} \leq \lambda\rho(x,y,z) \tag{2.3}$$

for all $x,y,z \in X$ and $\lambda = e^{-\tau} \in [0,1)$. Reason is similar to above example.

(ii) *Consider* $K(\alpha) = \ln(\alpha^2 + \alpha)$ *for all* $\alpha > 0$. *Then* K *satisfies (F1), (F2) and (F3) of Definition 2.7. A mapping* $T : X \to X$ *is a* K_w*-contraction if and only if*

$$\frac{\rho(Tx,Ty,Tz)\big(\rho(Tx,Ty,Tz)+1\big)}{\rho(x,y,z)\big(\rho(x,y,z)+1\big)} \;\le\; \lambda, \tag{2.4}$$

for all $x, y, z \in X$ *and* $\lambda = e^{-\tau} \in [0, 1)$.

Remark 2.18. *Let* $F, G : \mathbb{R}^+ \to \mathbb{R}$ *be mappings satisfying (F1), (F2) and (F3) of Definition 2.7 together with* $F(\alpha) \le G(\alpha)$ *for all* $\alpha > 0$. *Let* $H = G - F$ *be nondecreasing. Then every* F_w*-contraction* $T : X \to X$ *is a* G_w*-contraction. Indeed for any* $x, y, z \in X$ *with* $\rho(Tx, Ty, Tz) > 0$, *we have,*

$$\begin{aligned}\tau + G\big(\rho(Tx,Ty,Tz)\big) &= \tau + F\big(\rho(Tx,Ty,Tz)\big) + H\big(\rho(Tx,Ty,Tz)\big)\\ &\le F\big(\rho(x,y,z)\big) + H\big(\rho(x,y,z)\big) = G\big(\rho(x,y,z)\big).\end{aligned}$$

Example 2.19. *Let* $X = [0, \infty)$ *and* $S(x, y, z) = |x-z|+|y-z|$ *for all* $x, y, z \in X$. *Then* (X, S) *is a complete* S*-metric space. Define* $\rho : X^3 \to \mathbb{R}^+$ *by* $\rho(x, y, z) = \max\{y, z\}$ *for* $x, y, z \in X$ *Then* ρ *is a* w*-distance on* X. *Define a mapping* $T : X \to X$ *by*

$$Tx \;=\; \begin{cases} \frac{x^2}{2} & \text{if } 0 \le x \le 1,\\ 0 & \text{if } x > 1.\end{cases}$$

Since T *is not continuous, therefore it is not a* F*-contraction for any mapping* F *as described in Definition 2.7. Now consider the mapping* F *as described in Example 2.16. We note that* $\rho(Tx, Ty, Tz) = \max\{Ty, Tz\} > 0$ *if and only if* $0 \le y \le 1$ *or* $0 \le z \le 1$.

Now we have the following cases:

For $x, y, z \in X$ *with* $0 < y \le 1 < z$, *we have* $\dfrac{\rho(Tx,Ty,Tz)}{\rho(x,y,z)} = \dfrac{y^2/2}{z} \le \dfrac{1}{2}$.

For $x, y, z \in X$ *with* $0 < z \le 1 < y$, *we have* $\dfrac{\rho(Tx,Ty,Tz)}{\rho(x,y,z)} = \dfrac{z^2/2}{y} \le \dfrac{1}{2}$.

For $x, y, z \in X$ *with* $0 < y < z \le 1$, *we have* $\dfrac{\rho(Tx,Ty,Tz)}{\rho(x,y,z)} = \dfrac{z^2/2}{z} \le \dfrac{1}{2}$.

For $x, y, z \in X$ *with* $0 < z < y \le 1$, *we have* $\dfrac{\rho(Tx,Ty,Tz)}{\rho(x,y,z)} = \dfrac{y^2/2}{y} \le \dfrac{1}{2}$.

So ρ *satisfies* (2.2) *for all* $x, y, z \in X$ *and for* $\lambda = \frac{1}{2}$. *Thus* T *is a* F_w*-contraction which is not a* F*-contraction for any* F.

3 Main results

Theorem 3.1. *Let* (X, S) *be a complete* S*-metric space and* ρ *be a* w*-distance on* X. *Let* $T : X \to X$ *be a* F_w*-contraction. Then* T *has a unique fixed point* x^* *in* X *and for every* $x_0 \in X$, *there is a sequence* $\{T^n x_0\}$ *in* X *that converges to* x^*. *Further* $\rho(x^*, x^*, x^*) = 0$.

Proof. For any two fixed points x^* and y^* of T in X with $\rho(Tx^*, Tx^*, Ty^*) > 0$, we have

$$\tau \ \leq \ F(\rho(x^*, x^*, y^*)) - F(\rho(Tx^*, Tx^*, Ty^*)) = 0.$$

Thus $\rho(Tx^*, Tx^*, Ty^*) = \rho(x^*, x^*, y^*) = 0$ for any two fixed points x^* and y^* of T in X. In particular, $\rho(Tx^*, Tx^*, Tx^*) = \rho(x^*, x^*, x^*) = 0$. By Lemma 2.13 (i), we obtain $x^* = y^*$ for any two fixed points x^* and y^* of T in X. Hence fixed point x^* of T if exists is unique and satisfies $\rho(x^*, x^*, x^*) = 0$.

Now we show the existence of a fixed point of T. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ with $\rho(x_{k-1}, x_{k-1}, x_k) = 0$ then, by Definition 2.14(i), $\rho(Tx_{k-1}, Tx_{k-1}, Tx_k) = 0$, that is, $\rho(x_k, x_k, x_{k+1}) = 0$. Therefore

$$\rho(x_{k-1}, x_{k-1}, x_{k+1}) \ \leq \ 2\rho(x_{k-1}, x_{k-1}, x_k) + \rho(x_k, x_k, x_{k+1}) = 0.$$

By Lemma 2.13(i) we have $x_k = x_{k+1}$. Inductively, we have $x_k = x_{k+i}$ for all $i \in \mathbb{N}$. This implies $T^i(x_k) = x_k$ for all $i \in \mathbb{N}$ and in particular, for $i = 1, T(x_k) = x_k$. Also $\lim\limits_{n\to\infty} T^n(x_0) = \lim\limits_{i\to\infty} T^{k+i}(x_0) = \lim\limits_{i\to\infty} T^i(x_k) = x_k$. Thus we can take $x^* = x_k$ in this case and settle the proof.

Now assume that $\rho_n := \rho(x_n, x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Then by Definition 2.14(ii) we get

$$F(\rho_n) \ \leq \ F(\rho_{n-1}) - \tau \ \leq \ F(\rho_{n-2}) - 2\tau \ \leq \cdots \leq \ F(\rho_0) - n\tau. \tag{3.1}$$

From (3.1), we get $\lim\limits_{n\to\infty} F(\rho_n) = -\infty$. By (F2) of Definition 2.7, we have

$$\lim_{n\to\infty} \rho_n \ = \ 0. \tag{3.2}$$

Now, by (F3) of Definition 2.7, we find that there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \rho_n^k F(\rho_n) = 0. \tag{3.3}$$

By (3.1), we find that following holds for all $n \in \mathbb{N}$.

$$\rho_n^k F(\rho_n) - \rho_n^k F(\rho_0) \leq \rho_n^k \big(F(\rho_0) - n\tau\big) - \rho_n^k F(\rho_0) = -\rho_n^k n\tau \leq 0. \tag{3.4}$$

Letting $n \to \infty$ in (3.4) and using (3.2) and (3.3), we have

$$\lim_{n\to\infty} n\rho_n^k \ = \ 0. \tag{3.5}$$

By (3.5), there exists a positive integer n_0 such that $np_n^k < 1$ for all $n \geq n_0$. Consequently, we have

$$\rho_n \ < \ \frac{1}{n^{1/k}} \ ; \qquad \forall n \geq n_0. \tag{3.6}$$

Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/k}}$ is convergent, therefore, by (3.6), the series $\sum\limits_{n=1}^{\infty} \rho_n$ is also convergent. Now for any $m > n$ we have

$$\rho(x_n, x_n, x_m) \ \leq \ \rho_{n+1} + \rho_{n+2} + \cdots + \rho_m < \alpha_n, \tag{3.7}$$

where $\alpha_n = \sum_{i=n+1}^{\infty} \rho_i \to 0$ as $n \to \infty$. By Lemma 2.13 (iii), $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. From (3.7) and (w2) of Definition 2.10, we get

$$\rho(x_n, x_n, x^*) \;\leq\; \alpha_n. \tag{3.8}$$

Now for $\rho(Tx_{n-1}, Tx_{n-1}, Tx^*) > 0$, we find from Remark 2.15 and (3.8) that

$$\rho(x_n, x_n, Tx^*) = \rho(Tx_{n-1}, Tx_{n-1}, Tx^*) < \rho(x_{n-1}, x_{n-1}, x^*) \leq \alpha_{n-1}. \tag{3.9}$$

Clearly (3.9) is satisfied even for $\rho(Tx_{n-1}, Tx_{n-1}, Tx^*) = 0$. Thus

$$\rho(x_n, x_n, Tx^*) \;\leq\; \alpha_{n-1}; \quad \forall n \in \mathbb{N}. \tag{3.10}$$

From (3.8), (3.10) and Lemma 2.13 (i), we get $Tx^* = x^*$. Also we have seen above that $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0)$. □

Example 3.2. *Consider the F_w-contraction T defined in Example 2.19. We note that $x = 0$ is the unique fixed point of T and $\rho(0, 0, 0) = 0$.*

Since every contraction $T : X \to X$ satisfying 2.2 is an F_w-contraction for $F(\alpha) = \ln \alpha$, $\alpha > 0$, $F(\alpha) < \ln \alpha + \alpha = G(\alpha)$ for all $\alpha > 0$ and $G - F$ is non decreasing, therefore, by Remark 2.18, T is an G_w-contraction and hence satisfies (2.3). In the following example we shall present a mapping $T : X \to X$ which is an G_w-contraction but not an F_w-contraction and hence satisfies (2.3) but not (2.2). Thus our theorem deals with the fixed points of a more general class of contractions.

Example 3.3. *Consider the sequence $a_n = \frac{n(n-1)}{2}$ for $n \in \mathbb{N}$. Let $X = \{a_n : n \in \mathbb{N}\}$ and $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. Define $\rho : X^3 \to \mathbb{R}^+$ by $\rho(x, y, z) = \max\{x, y\}$ for all $x, y, z \in X$. Then ρ is a w-distance on X. Define a mapping $T : X \to X$ by $Ta_1 = a_1$, $Ta_n = a_{n-1}$ for $n > 1$. Take F as in Example 2.16 and G as in Example 2.17(i). T is not a F_w-contraction as*

$$\lim_{n\to\infty} \frac{\rho(Ta_n, Ta_n, Ta_1)}{\rho(a_n, a_n, a_1)} \;=\; \lim_{n\to\infty} \frac{a_{n-1}}{a_n} \;=\; 1.$$

But T is a G_w-contraction. We first observe that $\rho(Ta_m, Ta_n, Ta_r) > 0 \Leftrightarrow Ta_m > 0$ or $Ta_n > 0 \Leftrightarrow m > 2$ or $n > 2$. Now we have the following cases:

For $m > 2 > n$ or $m > n > 2$, we have

$$\begin{aligned}\frac{\rho(Ta_m, Ta_n, Ta_r)}{\rho(a_m, a_n, a_r)} e^{\rho(Ta_m, Ta_n, Ta_r) - \rho(a_m, a_n, a_r)} &= \frac{a_{m-1}}{a_m} e^{a_{m-1} - a_m} \\ &= \left(1 - \frac{2}{m}\right) e^{1-m} \;<\; e^{1-m} \;<\; e^{-1}.\end{aligned}$$

For $n > 2 > m$ or $n > m > 2$, we have

$$\frac{\rho(Ta_m, Ta_n, Ta_r)}{\rho(a_m, a_n, a_r)} e^{\rho(Ta_m,Ta_n,Ta_r)-\rho(a_m,a_n,a_r)} = \frac{a_{n-1}}{a_n} e^{a_{n-1}-a_n}$$
$$= \left(1 - \frac{2}{n}\right) e^{1-n} < e^{1-n} < e^{-1}.$$

Thus T is an G_w-contraction for $\tau = 1$. Clearly $a_1 = 0$ is a fixed point of T, $\rho(a_1, a_1, a_1) = a_1 = 0$ and for any $a_m \in X$,

$$\lim_{n\to\infty} T^n a_m = \lim_{n\to\infty} T^{n+m} a_m = \lim_{n\to\infty} T^n(T^m a_m) = \lim_{n\to\infty} T^n a_1 = a_1$$

Theorem 3.4. *Let (X, S) be a complete S-metric space and ρ be a w-distance on X. Let $T : X \to X$ be a F_w-contraction of Hardy-Rogers-type, that is, there exists a number $\tau > 0$ such that*

$$\tau + F(\rho(Tx, Tx, Ty)) \leq F\Big(\alpha\rho(x, x, y) + \beta\rho(x, x, Tx) + \gamma\rho(y, y, Ty) + \delta\rho(x, x, Ty) + L\rho(y, y, Tx)\Big), \tag{3.11}$$

for all $x, y \in X$ with $\rho(Tx, Tx, Ty) > 0$, where $\alpha + \beta + \gamma + 3\delta = 1, \gamma \neq 1$ and $L \geq 0$. Then T has a fixed point. Moreover, if $\alpha + \delta + L \leq 1$, then the fixed point of T is unique.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now, let $\rho_n := \rho(x_n, x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n \neq x_{n+1}$, that is, $Tx_{n-1} \neq Tx_n$ for all $n \in \mathbb{N}$, using the contractive condition (3.11) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned}
\tau + F(\rho_n) &= \tau + F\Big(\rho(x_n, x_n, x_{n+1})\Big) = \tau + F\Big(\rho(Tx_{n-1}Tx_{n-1}, Tx_n)\Big) \\
&\leq F\Big(\alpha\rho(x_{n-1}, x_{n-1}, x_n) + \beta\rho(x_{n-1}, x_{n-1}, Tx_{n-1}) + \gamma\rho(x_n, x_n, Tx_n) \\
&\qquad + \delta\rho(x_{n-1}, x_{n-1}, Tx_n) + Ld(x_n, x_n, Tx_{n-1})\Big) \\
&= F\Big(\alpha\rho(x_{n-1}, x_{n-1}, x_n) + \beta\rho(x_{n-1}, x_{n-1}, x_n) + \gamma\rho(x_n, x_n, x_{n+1}) \\
&\qquad + \delta\rho(x_{n-1}, x_{n-1}, x_{n+1}) + L\rho(x_n, x_n, x_n)\Big) \\
&= F\Big(\alpha\rho_{n-1} + \beta\rho_{n-1} + \gamma\rho_n + \delta\rho(x_{n-1}, x_{n-1}, x_{n+1})\Big) \\
&\leq F\Big((\alpha + \beta)\rho_{n-1} + \gamma\rho_n + \delta[2\rho_{n-1} + \rho_n]\Big) \\
&= F\Big((\alpha + \beta + 2\delta)\rho_{n-1} + (\gamma + \delta)\rho_n\Big).
\end{aligned}$$

Since F is strictly increasing, we deduce

$$\rho_n < (\alpha + \beta + 2\delta)\rho_{n-1} + (\gamma + \delta)\rho_n$$

and hence

$$(1-\gamma-\delta)\rho_n \; < \; (\alpha+\beta+2\delta)\rho_{n-1}\;, \quad \text{for all } n \in \mathbb{N}.$$

From $\alpha+\beta+\gamma+3\delta = 1$ and $\gamma \neq 1$, we deduce that $1-\gamma-\delta > 0$ and so

$$\rho_n \; < \; \frac{\alpha+\beta+2\delta}{1-\gamma-\delta}\rho_{n-1} = \rho_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

Consequently,

$$\tau + F(\rho_n) \; \leq F(\rho_{n-1})\;, \quad \text{for all } n \in \mathbb{N}.$$

This implies

$$F(\rho_n) \; \leq \; F(\rho_{n-1}) - \tau \; \leq \; F(\rho_{n-2}) - 2\tau \; \leq \cdots \leq \; F(\rho_0) - n\tau.$$

In the same way to proof Theorem 3.1, we can conclude that T has a fixed point. Now, we prove the uniqueness of the fixed point. Assume that $z \in X$ is another fixed point of T, different from x^*. This means that $\rho(x^*,x^*,z) > 0$. Taking $x = x^*$ and $y = z$ in the contractive condition (3.11), we have

$$\begin{aligned}
\tau + F(\rho(x^*,x^*,z)) \; &= \; \tau + F(\rho(Tx^*,Tx^*,Tz)) \\
&\leq \; F\Big(\alpha\rho(x^*,x^*,z) + \beta\rho(x^*,x^*,Tx^*) + \gamma\rho(z,z,Tz) \\
&\qquad + \delta\rho(x^*,x^*,Tz) + Ld(z,z,Tx^*)\Big) \\
&= \; F\Big((\alpha+\delta+L)\rho(x^*,x^*,z)\Big),
\end{aligned}$$

which is a contradiction, if $\alpha+\delta+L \leq 1$, and hence $x^* = z$. □

As a first corollary of Theorem 3.4, taking $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, we obtain the Wardowski's result [15] in a complete S-metric space with w-distance. Further, putting $\alpha = \delta = L = 0$ and $\beta+\gamma = 1$ and $\beta \neq 0$, we obtain the following version of Kannan's result [8] in a complete S-metric space with w-distance.

Corollary 3.5. *Let (X,S) be a complete S-metric space and ρ be a w-distance on X. Assume that there exist $T : X \to X$ be a F_w-contraction and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F\Big(\rho(Tx,Tx,Ty)\Big) \; \leq \; F\Big(\beta\rho(x,x,Tx) + \gamma\rho(y,y,Ty)\Big),$$

for all $x, y \in X$, $Tx \neq Ty$, where $\beta+\gamma = 1$, $\gamma \neq 1$. Then T has a unique fixed point in X.

A version of the Chatterjea [3] fixed point theorem in a complete S-metric space with w-distance is obtained from the Theorem 3.4, putting $\alpha = \beta = \gamma = 0$ and $\delta = 1/2$.

Corollary 3.6. *Let (X, S) be a complete S-metric space and ρ be a w-distance on X. Assume that there exist $T : X \to X$ be a F_w-contraction and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F\Big(\rho(Tx, Tx, Ty)\Big) \ \leq \ F\Big(\frac{1}{2}\rho(x, x, Ty) + L\rho(y, y, Tx)\Big),$$

for all $x, y \in X$, $Tx \neq Ty$. Then T has a unique fixed point in X. If $L \leq 1/2$ then the fixed point of T is unique.

Finally, if we choose $\delta = L = 0$, we obtain a version of Reich [11] type theorem. in a complete S-metric space with w-distance.

Corollary 3.7. *Let (X, S) be a complete S-metric space and ρ be a w-distance on X. Assume that there exist $T : X \to X$ be a F_w-contraction and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F\Big(\rho(Tx, Tx, Ty)\Big) \ \leq \ F\Big(\alpha\rho(x, x, y) + \beta\rho(x, x, Tx) + \gamma\rho(y, y, Ty)\Big),$$

for all $x, y \in X$, $Tx \neq Ty$. where $\alpha + \beta + \gamma = 1, \gamma \neq 1$. Then T has a unique fixed point in X.

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