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Convergence theorem for solving the combination of equilibrium problems and fixed point problems in Hilbert spaces

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Abstract : In this article, we propose an iterative algorithm for approximating a common element of solution sets of equilibrium problems, the fixed point sets of nonspreading mappings and the fixed point sets of κ_i-strictly pseudo contractive mappings in Hilbert spaces. Furthermore, we prove that the proposed iterative scheme converges strongly to a common element of those three sets. Finally, to support our main results, the numerical examples are given.

Keywords : strictly-pseudo contractive mapping; nonspreading mapping; equilibrium problem; fixed point; Hilbert space.
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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Throughout this article, the notations $\backslash\backslash \rightharpoonup''$ and $\backslash\backslash \to''$ denote weak convergence and strong convergence, respectively. The fixed point problem for the mapping $T : C \to H$ is to find $x \in C$ such that

$$x = Tx.$$

We denote the fixed point set of a mapping T by $Fix(T)$.

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Definition 1.1. *Let $T : C \to C$ be a mapping. Then*

(i) a mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C;$$

(ii) T is said to be κ-strictly pseudo-contractive *if there exists a constant $\kappa \in [0, 1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in C. \quad (1.1)$$

Note that the class of κ-strictly pseudo-contractions strictly includes the class of nonexpansive mappings.
In 2008, Kohsaka and Takahashi [6] introduced *the nonspreading mapping* T in Hilbert space H as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C. \quad (1.2)$$

In 2009, it is shown by Iemoto and Takahashi [9] that (1.2) is equivalent to the following equation.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \text{ for all } x, y \in C.$$

Many researcher proved the strong convergence theorem for nonspreading mapping and some related mappings in Hilbert space, see for example [13, 14].

Let $F : C \times C \to \mathbb{R}$ be bifunction. *The classical equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C, \quad (1.3)$$

which was first considered and investigated by Blum and Oettli [2] in 1994. The set of solutions of (1.3) is denoted by $EP(F)$.

The equilibrium problem provides a general framework to study a wide class of problems arising in economics, finance, network analysis, transportation, elasticity and optimization. The theory of equilibrium problems has become an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences, see [2, 5, 7, 8, 10, 11, 12].

In 2013, Suwannaut and Kangtunyakarn [16] introduced *the combination of equilibrium problem* which is to find $x \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x, y) \geq 0, \forall y \in C, \quad (1.4)$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \ldots, N$. The set of solution (1.4) is denoted by

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(x, y) \geq 0, \forall y \in C\right\}.$$

If $F_i = F, \forall i = 1, 2, \ldots, N$, then the combination of equilibrium problem (1.4) becomes the classical equilibrium problem (1.3).

Motivated by the above research, we prove a strong convergence theorem for finding a common element of a finite family of solution sets of equilibrium problems, the set of common fixed points of a finite family of nonspreading mappings and the set of common fixed points of a finite family of strictly pseudo contractive mappings in Hilbert spaces. Finally, to support our main results and compare the numerical results between the combination of equilibrium problem and the classical equilibrium problem, a numerical example are given and illustrated.

2 Preliminaries

In this section, some well-known definitions and Lemmas are recalled. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 2.1 ([3]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.2 ([3]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Then, for $\lambda > 0$,*

$$Fix(P_C(I - \lambda A)) = VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.3 ([1]). *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.4 ([4]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F and C satisfy the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) For each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\left(tz + (1-t)x, y\right) \leq F(x, y);$$

(A4) For each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.5 ([16]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0, 1)$ for every $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.6 ([2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2.7 ([5]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$. For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(iii) $Fix(T_r) = EP(F)$;

(iv) $EP(F)$ *is closed and convex.*

Remark 2.8 ([16]). *From Lemma 2.5, it is easy to see that* $\sum_{i=1}^{N} a_i F_i$ *satisfies (A1)-(A4). By using Lemma 2.7, we obtain*

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$, *for each* $i = 1, 2, \ldots, N$, *and* $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.9 ([17]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $T : C \to C$ *be a nonspreading mapping with* $Fix(T) \neq \emptyset$. *Then there hold the following statement:*

(i) $Fix(T) = VI(C, I - T)$;

(ii) For every $u \in C$ *and* $v \in Fix(T)$,

$$\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|, \text{ where } \lambda \in (0, 1).$$

Using the concept of properties of a strictly pseudo-contractive mapping in Banach space, see [15], we can obtain the following results.

Lemma 2.10. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $T : C \to C$ *be a* κ*-strictly pseudo-contractive mapping with* $Fix(T) \neq \emptyset$. *Then there hold the following statement:*

(i) $Fix(T) = VI(C, I - T)$;

(ii) For every $u \in C$ *and* $v \in Fix(T)$,

$$\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|, \text{ where } \lambda \in (0, 1 - \kappa).$$

Proof. To show (i), we let $x^* \in Fix(T)$. Then $x^* = Tx^*$.
Since $\langle y - x^*, (I - T)x^* \rangle = 0, \forall y \in C$, then we have $x^* \in VI(C, I - T)$. This follows that $Fix(T) \subseteq VI(C, I - T)$.
Next, claim that $VI(C, I - T) \subseteq Fix(T)$. Let $\tilde{x} \in VI(C, I - T)$. Then we get

$$\langle y - \tilde{x}, (I - T)\tilde{x} \rangle \geq 0, \forall y \in C. \tag{2.1}$$

Let $x^* \in Fix(T)$. Since T is a strictly pseudo-contraction, we derive that

$$\begin{aligned} \|T\tilde{x} - Tx^*\|^2 &\leq \|\tilde{x} - x^*\|^2 + \kappa \|(I - T)\tilde{x} - (I - T)x^*\|^2 \\ &= \|\tilde{x} - x^*\|^2 + \kappa \|(I - T)\tilde{x}\|^2. \end{aligned} \tag{2.2}$$

Observe that

$$\begin{aligned}\|T\tilde{x} - x^*\|^2 &= \|\tilde{x} - x^* - (I-T)\tilde{x}\|^2 \\ &= \|\tilde{x} - x^*\|^2 - 2\langle \tilde{x} - x^*, (I-T)\tilde{x}\rangle + \|(I-T)\tilde{x}\|^2 . \end{aligned} \tag{2.3}$$

By (2.1), (2.2) and (3.3), we deduce that

$$(1-\kappa)\|(I-T)\tilde{x}\|^2 \leq 2\langle \tilde{x} - x^*, (I-T)\tilde{x}\rangle \leq 0.$$

This yields that $\tilde{x} \in Fix(T)$. Hence we get $VI(C,T) \subseteq Fix(T)$.
Therefore $Fix(T) = VI(C, I-T)$.
To prove (ii), let $x \in C, y \in Fix(T)$ and $\lambda \in (0, 1-\kappa)$. Since T is a κ-strictly pseudo-contractive mapping, we obtain

$$\begin{aligned}\|Tx - Ty\|^2 &\leq \|x-y\|^2 + \kappa\|(I-T)x - (I-T)y\|^2 \\ &= \|x-y\|^2 + \kappa\|(I-T)x\|^2 . \end{aligned} \tag{2.4}$$

Since

$$\begin{aligned}\|Tx - y\|^2 &= \|x - y - (I-T)x\|^2 \\ &= \|x-y\|^2 - 2\langle x-y, (I-T)x\rangle + \|(I-T)x\|^2 , \end{aligned}$$

by (3.4), we have

$$(1-\kappa)\|(I-T)x\|^2 \leq 2\langle x-y, (I-T)x\rangle \leq 0. \tag{2.5}$$

From (i) and Lemma 2.2, we get

$$y \in Fix(T) = VI(C, I-T) = Fix\left(P_C(I - \lambda(I-T))\right). \tag{2.6}$$

By nonexpansiveness of P_C, (3.5) and (3.6), we derive that

$$\begin{aligned}\|P_C(I-\lambda(I-T))x - y\|^2 &= \|P_C(I-\lambda(I-T))x - P_C(I-\lambda(I-T))y\|^2 \\ &\leq \|(I-\lambda(I-T))x - (I-\lambda(I-T))y\|^2 \\ &= \|x - y - \lambda((I-T)x - (I-T)y)\|^2 \\ &= \|x - y - \lambda(I-T)x\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, (I-T)x\rangle + \lambda^2\|(I-T)x\|^2 \\ &\leq \|x-y\|^2 - \lambda(1-\kappa)\|(I-T)x\|^2 + \lambda^2\|(I-T)x\|^2 \\ &= \|x-y\|^2 - \lambda((1-\kappa)-\lambda)\|(I-T)x\|^2 \\ &\leq \|x-y\|^2 . \end{aligned}$$

That is, $\|P_C(I-\lambda(I-T))x - y\| \leq \|x-y\|$. Hence $P_C(I-\lambda(I-T))$ is a quasi-nonexpansive mapping. □

3 Strong convergence theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $\{T_i\}_{i=1}^{\bar{N}}$ be a finite family of κ_i-strictly pseudo-contractive mappings of C into itself and let $\{S_i\}_{i=1}^{\bar{N}} : C \to C$ be a finite family of nonspreading mappings. Suppose that $\Theta = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{\bar{N}} Fix(T_i) \cap \bigcap_{i=1}^{\bar{N}} Fix(S_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1 \in C$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n^i = \beta_n P_C (I - \mu_n (I - S_i)) x_n + (1 - \beta_n) P_C (I - \lambda_n (I - T_i)) u_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \sum_{i=1}^{\bar{N}} \rho_n^i y_n^i, \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $f : C \to C$ is a contractive mapping with coefficient τ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\rho_n^i\} \subseteq (0, 1)$, for every $i = 1, 2, \ldots, \bar{N}$, $\{\lambda_n\} \in (0, 1 - \kappa)$, where $\kappa = \max_{i=1,2,\ldots,\bar{N}} \kappa_i$, and $0 < a_i < 1$, for all $i = 1, 2, \ldots, N$ with $\sum_{i=1}^{N} a_i = 1$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < \eta \leq \beta_n \leq \mu < 1, \forall n \in \mathbb{N}$,

(iii) $0 < \omega \leq r_n \leq \varrho < 1, \forall n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$,

(v) $0 < \nu \leq \rho_n^i \leq \xi < 1, \forall n \in \mathbb{N}$ *and* $i = 1, 2, \ldots, \bar{N}$ *with* $\sum_{i=1}^{\bar{N}} \rho_n^i = 1$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\rho_{n+1}^i - \rho_n^i| < \infty$, *for each* $i = 1, 2, \ldots, \bar{N}$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta f(q)$.

Proof. The proof will be divided into five steps.
Step 1. We show that $\{x_n\}$ is bounded.
Since $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C,$$

by Lemma 2.7 and Remark 2.8, we have $u_n = T_{r_n} x_n$ and $Fix(T_{r_n}) = \bigcap_{i=1}^{N} EP(F_i)$.
From Lemma 2.2 and Lemma 2.9(i), we obtain

$$Fix(S_i) = VI(C, I - S_i) = Fix(P_C (I - \mu_n (I - S_i))), \text{ for every } i = 1, 2, \ldots, N.$$

Similarly, by Lemma 2.2 and Lemma 2.10(i), we also have

$$Fix\left(T_i\right) = VI(C, I - T_i) = Fix\left(P_C\left(I - \lambda_n\left(I - T_i\right)\right)\right), \text{ for all } i = 1, 2, \ldots, N.$$

Let $z \in \Theta$. By nonexpansiveness of P_C and T_{r_n}, for each $i = 1, 2, \ldots, \bar{N}$, we derive that

$$\begin{aligned}
&\left\|y_n^i - z\right\| \\
=&\left\|\beta_n\left(P_C\left(I - \mu_n\left(I - S_i\right)\right)x_n - z\right) + \left(1 - \beta_n\right)\left(P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n - z\right)\right\| \\
\leq&\beta_n\left\|P_C\left(I - \mu_n\left(I - S_i\right)\right)x_n - z\right\| \left(1 - \beta_n\right)\left\|P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n - z\right\| \\
\leq&\beta_n\left\|x_n - z\right\| \left(1 - \beta_n\right)\left\|T_{r_n}x_n - z\right\| \\
\leq&\left\|x_n - z\right\|. \qquad (3.2)
\end{aligned}$$

From (3.2), it deduces that

$$\begin{aligned}
\left\|x_{n+1} - z\right\| &\leq \left\|\alpha_n\left(f\left(x_n\right) - z\right) + \left(1 - \alpha_n\right)\left(\sum_{i=1}^{\bar{N}}\rho_n^i y_n^i - z\right)\right\| \\
&\leq \alpha_n\left\|f\left(x_n\right) - z\right\| + \left(1 - \alpha_n\right)\left\|\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i - z\right)\right\| \\
&\leq \alpha_n\left\|f\left(x_n\right) - f(z)\right\| + \alpha_n\left\|f(z) - z\right\| + \left(1 - \alpha_n\right)\sum_{i=1}^{\bar{N}}\rho_n^i\left\|y_n^i - z\right\| \\
&\leq \alpha_n\tau\left\|x_n - z\right\| + \alpha_n\left\|f(z) - z\right\| + \left(1 - \alpha_n\right)\left\|x_n - z\right\| \\
&= \left(1 - \alpha_n(1 - \tau)\right)\left\|x_n - z\right\| + \alpha_n\left\|f(z) - z\right\| \\
&\leq \max\left\{\left\|x_1 - z\right\|, \frac{\left\|f(z) - z\right\|}{1 - \tau}\right\}.
\end{aligned}$$

Using mathematical induction, this implies that $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\left\{y_n^i\right\}$, for any $i = 1, 2, \ldots, \bar{N}$.

Step 2. Prove that $\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = 0$.

Using the same argument as [16], we get

$$\left\|u_n - u_{n-1}\right\| \leq \left\|x_n - x_{n-1}\right\| + \frac{1}{\omega}\left|r_n - r_{n-1}\right|\left\|u_n - x_n\right\|. \qquad (3.3)$$

By (3.3), for each $i = 1, 2, \ldots, \bar{N}$, we derive

$$\begin{aligned}
&\left\|y_n^i - y_{n-1}^i\right\| \\
\leq&\beta_n\left\|P_C\left(I - \mu_n\left(I - S_i\right)\right)x_n - P_C\left(I - \mu_{n-1}\left(I - S_i\right)\right)x_{n-1}\right\| \\
&+ \left|\beta_n - \beta_{n-1}\right|\left\|P_C\left(I - \mu_{n-1}\left(I - S_i\right)\right)x_{n-1}\right\| \\
&+ \left(1 - \beta_n\right)\left\|P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n - P_C\left(I - \lambda_{n-1}\left(I - T_i\right)\right)u_{n-1}\right\| \\
&+ \left|\beta_n - \beta_{n-1}\right|\left\|P_C\left(I - \lambda_{n-1}\left(I - T_i\right)\right)u_{n-1}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \|x_n - x_{n-1}\| + \beta_n \Big[\mu_n \|(I - S_i) x_n - (I - S_i) x_{n-1}\| \Big] \\
&\quad + |\mu_n - \mu_{n-1}| \|(I - S_i) x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C (I - \mu_{n-1} (I - S_i)) x_{n-1}\| \\
&\quad + (1 - \beta_n) \|u_n - u_{n-1}\| + (1 - \beta_n) \Big[\lambda_n \|(I - T_i) u_n - (I - T_i) u_{n-1}\| \Big] \\
&\quad + |\lambda_n - \lambda_{n-1}| \|(I - T_i) u_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C (I - \lambda_{n-1} (I - T_i)) u_{n-1}\| \\
&\leq \beta_n \|x_n - x_{n-1}\| + \mu_n \|(I - S_i) x_n - (I - S_i) x_{n-1}\| \\
&\quad + |\mu_n - \mu_{n-1}| \|(I - S_i) x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C (I - \mu_{n-1} (I - S_i)) x_{n-1}\| \\
&\quad + (1 - \beta_n) \Big[\|x_n - x_{n-1}\| + \frac{1}{\omega} |r_n - r_{n-1}| \|u_n - x_n\| \Big] \\
&\quad + \lambda_n \|(I - T_i) u_n - (I - T_i) u_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|(I - T_i) u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C (I - \lambda_{n-1} (I - T_i)) u_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \mu_n \|(I - S_i) x_n - (I - S_i) x_{n-1}\| + |\mu_n - \mu_{n-1}| \|(I - S_i) x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C (I - \mu_{n-1} (I - S_i)) x_{n-1}\| + \frac{1}{\omega} |r_n - r_{n-1}| \|u_n - x_n\| \\
&\quad + \lambda_n \|(I - T_i) u_n - (I - T_i) u_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|(I - T_i) u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|P_C (I - \lambda_{n-1} (I - T_i)) u_{n-1}\| . \qquad (3.4)
\end{aligned}
$$

By the definition of x_n and (3.4), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\quad + (1 - \alpha_n) \left\| \sum_{i=1}^{\bar{N}} \rho_n^i y_n^i - \sum_{i=1}^{\bar{N}} \rho_n^i y_{n-1}^i \right\| + (1 - \alpha_n) \left\| \sum_{i=1}^{\bar{N}} \rho_n^i y_{n-1}^i - \sum_{i=1}^{\bar{N}} \rho_{n-1}^i y_{n-1}^i \right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\| \sum_{i=1}^{\bar{N}} \rho_{n-1}^i y_{n-1}^i \right\| \\
&\leq \alpha_n \tau \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left(\|f(x_{n-1})\| + \left\| \sum_{i=1}^{\bar{N}} \rho_{n-1}^i y_{n-1}^i \right\| \right) \\
&\quad + (1 - \alpha_n) \sum_{i=1}^{\bar{N}} \rho_n^i \left\| y_n^i - y_{n-1}^i \right\| + (1 - \alpha_n) \sum_{i=1}^{\bar{N}} \left| \rho_n^i - \rho_{n-1}^i \right| \left\| y_{n-1}^i \right\| \\
&\leq \alpha_n \tau \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left(\|f(x_{n-1})\| + \left\| \sum_{i=1}^{\bar{N}} \rho_{n-1}^i y_{n-1}^i \right\| \right) \\
&\quad + (1 - \alpha_n) \sum_{i=1}^{\bar{N}} \rho_n^i \Big[\|x_n - x_{n-1}\| + \mu_n \|(I - S_i) x_n - (I - S_i) x_{n-1}\| \\
&\quad + |\mu_n - \mu_{n-1}| \|(I - S_i) x_{n-1}\| + |\beta_n - \beta_{n-1}| \|P_C (I - \mu_{n-1} (I - S_i)) x_{n-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\omega}\left|r_n-r_{n-1}\right|\left\|u_n-x_n\right\|+\lambda_n\left\|\left(I-T_i\right)u_n-\left(I-T_i\right)u_{n-1}\right\| \\
&+\left|\lambda_n-\lambda_{n-1}\right|\left\|\left(I-T_i\right)u_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|P_C\left(I-\lambda_{n-1}\left(I-T_i\right)\right)u_{n-1}\right\|\Big] \\
&+\left(1-\alpha_n\right)\sum_{i=1}^{\bar{N}}\left|\rho_n^i-\rho_{n-1}^i\right|\left\|y_{n-1}^i\right\| \\
\leq&\left(1-\alpha_n(1-\tau)\right)\left\|x_n-x_{n-1}\right\|+\left|\alpha_n-\alpha_{n-1}\right|\left(\left\|f\left(x_{n-1}\right)\right\|+\left\|\sum_{i=1}^{\bar{N}}\rho_{n-1}^i y_{n-1}^i\right\|\right) \\
&+\mu_n\sum_{i=1}^{\bar{N}}\rho_n^i\left\|\left(I-S_i\right)x_n-\left(I-S_i\right)x_{n-1}\right\|+\left|\mu_n-\mu_{n-1}\right|\sum_{i=1}^{\bar{N}}\rho_n^i\left\|\left(I-S_i\right)x_{n-1}\right\| \\
&+\left|\beta_n-\beta_{n-1}\right|\sum_{i=1}^{\bar{N}}\rho_n^i\left\|P_C\left(I-\mu_{n-1}\left(I-S_i\right)\right)x_{n-1}\right\|+\frac{1}{\omega}\left|r_n-r_{n-1}\right|\left\|u_n-x_n\right\| \\
&+\lambda_n\sum_{i=1}^{\bar{N}}\rho_n^i\left\|\left(I-T_i\right)u_n-\left(I-T_i\right)u_{n-1}\right\|+\left|\lambda_n-\lambda_{n-1}\right|\sum_{i=1}^{\bar{N}}\rho_n^i\left\|\left(I-T_i\right)u_{n-1}\right\| \\
&+\left|\beta_n-\beta_{n-1}\right|\sum_{i=1}^{\bar{N}}\rho_n^i\left\|P_C\left(I-\lambda_{n-1}\left(I-T_i\right)\right)u_{n-1}\right\|+\sum_{i=1}^{\bar{N}}\left|\rho_n^i-\rho_{n-1}^i\right|\left\|y_{n-1}^i\right\| \\
\leq&\left(1-\alpha_n(1-\tau)\right)\left\|x_n-x_{n-1}\right\|+2M\left|\alpha_n-\alpha_{n-1}\right|+2M\mu_n+M\left|\mu_n-\mu_{n-1}\right| \\
&+2M\left|\beta_n-\beta_{n-1}\right|+\frac{M}{\omega}\left|r_n-r_{n-1}\right|+M\lambda_n+M\left|\lambda_n-\lambda_{n-1}\right|+M\sum_{i=1}^{\bar{N}}\left|\rho_n^i-\rho_{n-1}^i\right|,
\end{aligned}
\tag{3.5}
$$

where $M=\max_{n\in\mathbb{N}}\left\{\left\|f\left(x_n\right)\right\|,\left\|\sum_{i=1}^{\bar{N}}\rho_n^i y_n^i\right\|,\left\|\left(I-S_i\right)x_n\right\|,\left\|P_C\left(I-\mu_n\left(I-S_i\right)\right)x_n\right\|,\right.$
$\left.\left\|u_n-x_n\right\|,\left\|\left(I-T_i\right)u_n\right\|,\left\|P_C\left(I-\lambda_n\left(I-T_i\right)\right)u_n\right\|,\left\|y_n^i\right\|\right\}$, for $i=1,2,\ldots,\bar{N}$.
By (3.5), applying Lemma 2.4 and the condition (i), we obtain

$$
\left\|x_{n+1}-x_n\right\|\to 0 \text{ as } n\to\infty. \tag{3.6}
$$

Step 3. Claim that $\lim_{n\to\infty}\left\|P_C\left(I-\mu_n(I-S_i)\right)x_n-x_n\right\|=0$, $\lim_{n\to\infty}\left\|P_C\left(I-\lambda_n(I-T_i)\right)x_n-x_n\right\|=0$ and $\lim_{n\to\infty}\left\|u_n-x_n\right\|=0$, for all $i=1,2,\ldots,\bar{N}$.
To prove this, let $z\in\Theta$. Since $u_n=T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive, we have

$$
\begin{aligned}
\left\|z-u_n\right\|^2&=\left\|T_{r_n}z-T_{r_n}x_n\right\|^2 \\
&\leq\left\langle T_{r_n}z-T_{r_n}x_n,z-x_n\right\rangle \\
&=\frac{1}{2}\left[\left\|T_{r_n}x_n-z\right\|^2+\left\|x_n-z\right\|^2-\left\|T_{r_n}x_n-x_n\right\|^2\right],
\end{aligned}
$$

which follows that

$$
\left\|u_n-z\right\|^2\leq\left\|x_n-z\right\|^2-\left\|u_n-x_n\right\|^2. \tag{3.7}
$$

Next, from (3.7), we derive

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
\leq&\alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\left\|\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i-z\right)\right\|^2\\
\leq&\alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\sum_{i=1}^{\bar{N}}\rho_n^i\left\|y_n^i-z\right\|^2\\
\leq&\alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\Big[\sum_{i=1}^{\bar{N}}\rho_n^i\Big(\beta_n\|P_C(I-\mu_n(I-S_i))x_n-z\|^2\\
&+(1-\beta_n)\|P_C(I-\lambda_n(I-T_i))u_n-z\|^2\Big)\Big]\\
\leq&\alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\left(\beta_n\|x_n-z\|^2+(1-\beta_n)\|u_n-z\|^2\right)\\
\leq&\alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\Big(\beta_n\|x_n-z\|^2+(1-\beta_n)\big(\|x_n-z\|^2\\
&-\|u_n-x_n\|^2\big)\Big)\\
\leq&\alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2\\
\leq&\|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|f(x_n)-z\|^2\\
\leq&\|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)+\alpha_n\|f(x_n)-z\|^2. \qquad (3.8)
\end{aligned}$$

From (3.6) and the condition (i), (ii), taking $n\to\infty$ in (3.8), we get

$$\|u_n-x_n\|\to 0 \text{ as } n\to\infty. \qquad (3.9)$$

Consider

$$\begin{aligned}
\|x_{n+1}-z\|^2&=\left\|\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i-z\right)+\alpha_n\left(f(x_n)-\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i\right)\right)\right\|^2\\
&\leq\left\|\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i-z\right)\right\|^2+2\alpha\left\langle f(x_n)-\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i\right),x_{n+1}-z\right\rangle\\
&\leq\sum_{i=1}^{\bar{N}}\rho_n^i\left\|y_n^i-z\right\|^2+2\alpha\left\|f(x_n)-\sum_{i=1}^{\bar{N}}\rho_n^i\left(y_n^i\right)\right\|\|x_{n+1}-z\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{\bar{N}} \rho_n^i \Big[\beta_n \left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - z \right\|^2 + \left(1 - \beta_n \right) \left\| P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n - z \right\|^2 \\
&\quad - \beta_n \left(1 - \beta_n \right) \left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n \right\|^2 \Big] \\
&\quad + 2\alpha \left\| f\left(x_n \right) - \sum_{i=1}^{\bar{N}} \rho_n^i \left(y_n^i \right) \right\| \left\| x_{n+1} - z \right\| \\
&\leq \beta_n \left\| x_n - z \right\|^2 + \left(1 - \beta_n \right) \left\| u_n - z \right\|^2 \\
&\quad - \beta_n \left(1 - \beta_n \right) \left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n \right\|^2 \\
&\quad + 2\alpha \left\| f\left(x_n \right) - \sum_{i=1}^{\bar{N}} \rho_n^i \left(y_n^i \right) \right\| \left\| x_{n+1} - z \right\| \\
&\leq \left\| x_n - z \right\|^2 - \beta_n \left(1 - \beta_n \right) \left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n \right\|^2 \\
&\quad + 2\alpha \left\| f\left(x_n \right) - \sum_{i=1}^{\bar{N}} \rho_n^i \left(y_n^i \right) \right\| \left\| x_{n+1} - z \right\|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
&\beta_n \left(1 - \beta_n \right) \left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n \right\|^2 \\
&\leq \left\| x_n - x_{n+1} \right\| \left(\left\| x_n - z \right\| + \left\| x_{n+1} - z \right\| \right) + 2\alpha \left\| f\left(x_n \right) - \sum_{i=1}^{\bar{N}} \rho_n^i \left(y_n^i \right) \right\| \left\| x_{n+1} - z \right\|.
\end{aligned}
$$

From (3.6), the condition (i), we obtain

$$
\left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n \right\| \to 0 \text{ as } n \to \infty. \tag{3.10}
$$

Since

$$
\begin{aligned}
&\left\| x_n - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\| \\
&\leq \left\| x_n - x_{n+1} \right\| + \left\| x_{n+1} - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\| \\
&\leq \left\| x_n - x_{n+1} \right\| + \alpha_n \left\| f\left(x_n \right) - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\| \\
&\quad + \left(1 - \alpha_n \right) \sum_{i=1}^{\bar{N}} \rho_n^i \left\| y_n^i - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\| \\
&= \left\| x_n - x_{n+1} \right\| + \alpha_n \left\| f\left(x_n \right) - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\| \\
&\quad + \left(1 - \alpha_n \right) \left(1 - \beta_n \right) \sum_{i=1}^{\bar{N}} \rho_n^i \left\| P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\|,
\end{aligned}
$$

by (3.6), (3.10) and the condition (i), we have

$$
\left\| x_n - P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n \right\| \to 0 \text{ as } n \to \infty, \tag{3.11}
$$

for each $i = 1, 2, \ldots, \bar{N}$. Similarly, we observe that

$$\begin{aligned}
&\|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|f\left(x_n\right) - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| \\
&\quad + \left(1 - \alpha_n\right)\sum_{i=1}^{\bar{N}}\rho_n^i\left\|y_n^i - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\right\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|f\left(x_n\right) - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| \\
&\quad + \left(1 - \alpha_n\right)\beta_n\sum_{i=1}^{\bar{N}}\rho_n^i\|P_C\left(I - \mu_n\left(I - S_i\right)\right)x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\|.
\end{aligned}$$

From (3.6), (3.10) and the condition (i), we obtain

$$\|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| \to 0 \text{ as } n \to \infty, \tag{3.12}$$

for any $i = 1, 2, \ldots, \bar{N}$. Since

$$\begin{aligned}
&\|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)x_n\| \\
&\leq \|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| + \|P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)x_n\| \\
&\leq \|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| + \|\left(I - \lambda_n\left(I - T_i\right)\right)u_n - \left(I - \lambda_n\left(I - T_i\right)\right)x_n\| \\
&\leq \|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)u_n\| + \|u_n - x_n\| + \lambda_n\|\left(I - T_i\right)u_n - \left(I - T_i\right)x_n\|,
\end{aligned}$$

by (3.9), (3.12), the condition (iii), we deduce that

$$\|x_n - P_C\left(I - \lambda_n\left(I - T_i\right)\right)x_n\| \to 0 \text{ as } n \to \infty, \tag{3.13}$$

for any $i = 1, 2, \ldots, \bar{N}$.

Step 4. Show that $\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle \leq 0$, where $q = P_\Theta f(q)$.
To show this, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \lim_{k\to\infty}\langle f(q) - q, x_{n_k} - q\rangle.$$

Since $\{x_n\}$ is bounded, then we can assume that $x_{n_k} \rightharpoonup \theta$ as $k \to \infty$. From (3.9), we also obtain $u_{n_k} \rightharpoonup \theta$ as $k \to \infty$.
Since $u_{n_k} \rightharpoonup \theta$ as $k \to \infty$ and (3.9), applying the same method as in [16], we get

$$\theta \in \bigcap_{i=1}^{N} EP\left(F_i\right). \tag{3.14}$$

Assume $\theta \notin \bigcap_{i=1}^{\bar{N}} Fix\left(S_i\right)$, that is, $\theta \notin Fix\left(S_i\right), \forall i = 1, 2, \ldots, \bar{N}$. Since $Fix\left(S_i\right) = Fix\left(P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)\right)$, for every $i = 1, 2, \ldots, \bar{N}$, then we get $\theta \neq P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)\theta$,

for any $i = 1, 2, \ldots, \bar{N}$. By nonexpansiveness of P_C, (3.11), the condition (iv) and the Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \theta\| &< \liminf_{k\to\infty} \|x_{n_k} - P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)\theta\| \\
&\leq \liminf_{k\to\infty} \Big[\|x_{n_k} - P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)x_{n_k}\| \\
&\quad + \|P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)x_{n_k} - P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)\theta\| \Big] \\
&\leq \liminf_{k\to\infty} \Big[\|x_{n_k} - P_C\left(I - \mu_{n_k}\left(I - S_i\right)\right)x_{n_k}\| \\
&\quad + \|x_{n_k} - \theta\| + \mu_{n_k}\|(I - S_i)\,x_{n_k} - (I - S_i)\,\theta\| \Big] \\
&= \|x_{n_k} - \theta\|.
\end{aligned}$$

This is a contradiction. Thus we obtain

$$\theta \in \bigcap_{i=1}^{\bar{N}} Fix\left(S_i\right). \tag{3.15}$$

Similarly, let $\theta \notin \bigcap_{i=1}^{\bar{N}} Fix\left(T_i\right)$, that is, $\theta \notin Fix\left(T_i\right), \forall i = 1, 2, \ldots, \bar{N}$. Because $Fix\left(T_i\right) = Fix\left(P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)\right)$, for each $i = 1, 2, \ldots, \bar{N}$, we have $\theta \neq P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)\theta$, for all $i = 1, 2, \ldots, \bar{N}$. Using nonexpansiveness of P_C, (3.13), the condition (iv) and the Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \theta\| &< \liminf_{k\to\infty} \|x_{n_k} - P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)\theta\| \\
&\leq \liminf_{k\to\infty} \Big[\|x_{n_k} - P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)x_{n_k}\| \\
&\quad + \|P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)x_{n_k} - P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)\theta\| \Big] \\
&\leq \liminf_{k\to\infty} \Big[\|x_{n_k} - P_C\left(I - \lambda_{n_k}\left(I - T_i\right)\right)x_{n_k}\| \\
&\quad + \|x_{n_k} - \theta\| + \lambda_{n_k}\|(I - T_i)\,x_{n_k} - (I - T_i)\,\theta\| \Big] \\
&= \|x_{n_k} - \theta\|.
\end{aligned}$$

This is a contradiction. Thus we obtain

$$\theta \in \bigcap_{i=1}^{\bar{N}} Fix\left(T_i\right). \tag{3.16}$$

From (3.14), (3.15) and (3.16), it yields that

$$\theta \in \Theta. \tag{3.17}$$

Since $x_{n_k} \rightharpoonup \theta$ as $k \to \infty$, (3.17) and Lemma 2.1, thus we get

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle = \langle f(q) - q, \theta - q\rangle \leq 0. \tag{3.18}$$

Step 5. Finally, Prove that the sequence $\{x_n\}$ converges strongly to $q = P_\Theta f(q)$. Observe that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \left\| \alpha_n \left(f\left(x_n\right) - q \right) + \left(1 - \alpha_n\right) \sum_{i=1}^{\bar{N}} \rho_n^i \left(y_n^i - q \right) \right\|^2 \\
&\leq \left(1 - \alpha_n\right)^2 \left\| \sum_{i=1}^{\bar{N}} \rho_n^i \left(y_n^i - q \right) \right\|^2 + 2\alpha_n \left\langle f\left(x_n\right) - q, x_{n+1} - q \right\rangle \\
&\leq \left(1 - \alpha_n\right)^2 \sum_{i=1}^{\bar{N}} \rho_n^i \left\| y_n^i - q \right\|^2 + 2\alpha_n \left\langle f\left(x_n\right) - f(q), x_{n+1} - q \right\rangle \\
&\quad + 2\alpha_n \left\langle f(q) - q, x_{n+1} - q \right\rangle \\
&\leq \left(1 - \alpha_n\right)^2 \sum_{i=1}^{\bar{N}} \rho_n^i \Big[\beta_n \left\| P_C \left(I - \mu_n \left(I - S_i \right) \right) x_n - q \right\|^2 \\
&\quad + \left(1 - \beta_n\right) \left\| P_C \left(I - \lambda_n \left(I - T_i \right) \right) u_n - q \right\|^2 \Big] + 2\alpha_n \left\| f\left(x_n\right) - f(q) \right\| \left\| x_{n+1} - q \right\| \\
&\quad + 2\alpha_n \left\langle f(q) - q, x_{n+1} - q \right\rangle \\
&\leq \left(1 - \alpha_n\right)^2 \left[\beta_n \left\| x_n - q \right\|^2 + \left(1 - \beta_n\right) \left\| u_n - q \right\|^2 \right] + 2\alpha_n \tau \left\| x_n - q \right\| \left\| x_{n+1} - q \right\| \\
&\quad + 2\alpha_n \left\langle f(q) - q, x_{n+1} - q \right\rangle \\
&\leq \left(1 - \alpha_n\right)^2 \left\| x_n - q \right\|^2 + \alpha_n \tau \Big[\left\| x_n - q \right\|^2 + \left\| x_{n+1} - q \right\|^2 \Big] \\
&\quad + 2\alpha_n \left\langle f(q) - q, x_{n+1} - q \right\rangle .
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\leq \frac{\left(1 - \alpha_n\right)^2 + \alpha_n \tau}{1 - \alpha_n \tau} \left\| x_n - q \right\|^2 + \frac{2\alpha_n}{1 - \alpha_n \tau} \left\langle f(q) - q, x_{n+1} - q \right\rangle \\
&= \left(1 - \frac{2\alpha_n (1 - \tau)}{1 - \alpha_n \tau} \right) \left\| x_n - q \right\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \tau} \left\| x_n - q \right\|^2 + \frac{2\alpha_n}{1 - \alpha_n \tau} \left\langle f(q) - q, x_{n+1} - q \right\rangle \\
&= \left(1 - \frac{2\alpha_n (1 - \tau)}{1 - \alpha_n \tau} \right) \left\| x_n - q \right\|^2 + \frac{2\alpha_n (1 - \tau)}{1 - \alpha_n \tau} \Big[\frac{\alpha_n}{2(1 - \tau)} \left\| x_n - q \right\|^2 \\
&\quad + \frac{1}{1 - \tau} \left\langle f(q) - q, x_{n+1} - q \right\rangle \Big].
\end{aligned}$$

Using the condition (i), (3.18) and Lemma 2.2, we can conclude that the sequence $\{x_n\}$ converges strongly to $q = P_\Theta f(q)$. By (3.9), we also obtain $\{u_n\}$ converges strongly to $q = P_\Theta f(q)$. This completes the proof. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $\{T_i\}_{i=1}^{\bar{N}}$ be a finite family of κ_i-strictly pseudo-contractive mappings of C into itself and let $\{S_i\}_{i=1}^{\bar{N}} : C \to C$ be a finite family of nonspreading mappings. Suppose that $\Theta = EP(F) \cap \bigcap_{i=1}^{\bar{N}} Fix(T_i) \cap \bigcap_{i=1}^{\bar{N}} Fix(S_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1 \in C$ and*

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ y_n^i = \beta_n P_C \left(I - \mu_n (I - S_i)\right) x_n + (1 - \beta_n) P_C \left(I - \lambda_n (I - T_i)\right) u_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \sum_{i=1}^{\bar{N}} \rho_n^i y_n^i, \forall n \in \mathbb{N}, \end{cases} \tag{3.19}$$

where $f : C \to C$ is a contractive mapping with coefficient τ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\rho_n^i\} \subseteq (0,1)$, for every $i = 1, 2, \ldots, \bar{N}$, $\{\lambda_n\} \in (0, 1 - \kappa)$, where $\kappa = \max_{i=1,2,\ldots,\bar{N}} \kappa_i$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < \eta \leq \beta_n \leq \mu < 1, \forall n \in \mathbb{N}$,

(iii) $0 < \omega \leq r_n \leq \varrho < 1, \forall n \in \mathbb{N}$,

(iv) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$,

(v) $0 < \nu \leq \rho_n^i \leq \xi < 1, \forall n \in \mathbb{N}$ *and* $i = 1, 2, \ldots, \bar{N}$ *with* $\sum_{i=1}^{\bar{N}} \rho_n^i = 1$,

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\rho_{n+1}^i - \rho_n^i| < \infty$, *for each* $i = 1, 2, \ldots, \bar{N}$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta f(q)$.

Proof. Put $F_i = F, \forall i = 1, 2, \ldots, N$. Then we can obtain the desired result. □

4 A Numerical Example

In this section, we give a numerical example to support our main theorem. According to numerical results, we have that the iteration algorithm for the combination of equilibrium problem (3.1) converges faster than the iterative algorithm for the classical equilibrium problem (3.19).

Example 4.1. *Let $\mathbb{R}$ be the space of real numbers. For every $i = 1, 2, \ldots, N$, let $f : \mathbb{R} \to \mathbb{R}$ and $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$fx = 1 + \frac{x}{4},$$
$$F_i(x, y) = i\left(-5x^2 + xy + 4y^2\right), \text{ for all } x, y \in \mathbb{R}.$$

For each $i = 1, 2, \ldots, \bar{N}$, *let* $S_i : \mathbb{R} \to \mathbb{R}$ *and* $T_i : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$S_i x = -\frac{(i+3)x}{i+1}, \text{ for all } x \in \mathbb{R}$$
$$T_i x = \begin{cases} x, & \text{if } x \in (-\infty, 0), \\ \frac{x}{7i}, & \text{if } x \in [0, \infty). \end{cases}$$

Let $\alpha_n = \frac{1}{8n}$, $\beta_n = \frac{5n}{7n+6}$, $r_n = \frac{3n}{7n+3}$, $\mu_n = \frac{1}{3n^3}$, $\lambda_n = \frac{1}{n^2}$ *and* $\rho_n^1 = \frac{2n+1}{5n+8}$, $\rho_n^2 = \frac{2n+3}{5n+8}$, $\rho_n^3 = \frac{n+4}{5n+8}$, *for every* $n \in \mathbb{N}$. *Put* $a_i = \frac{5}{6^i} + \frac{1}{N6^N}$, *for every* $i = 1, 2, \ldots, N$.
Solution. *For every* $i = 1, 2, \ldots, \bar{N}$, *it is obvious to see that* T_i *is a nonspreading mapping and*

$$Fix(T_i) = \begin{cases} \{x\}, & \text{if } x \in (-\infty, 0), \\ \{0\}, & \text{if } x \in [0, \infty). \end{cases}$$

Moreover, S_i *is a* $\frac{1}{i+2}$*-strictly pseudo-contractive mapping, for each* $1, 2, \ldots, \bar{N}$ *with* $\bigcap_{i=1}^{\bar{N}} Fix(S_i) = \{0\}$. *Since* $a_i = \frac{5}{6^i} + \frac{1}{N6^N}$, *for* $i = 1, 2, \ldots, N$, *we obtain*

$$\begin{aligned} \sum_{i=1}^{N} a_i F_i(x, y) &= \sum_{i=1}^{N} \left(\frac{5}{6^i} + \frac{1}{N6^N} \right) i \left(-5x^2 + xy + 4y^2 \right) \\ &= \sigma \left(-5x^2 + xy + 4y^2 \right), \end{aligned}$$

where $\sigma = \sum_{i=1}^{N} \left(\frac{5}{6^i} + \frac{1}{N6^N} \right) i$. *It is easy to see that* $\sum_{i=1}^{N} a_i F_i$ *satisfies all conditions in Theorem 3.1 and* $1 \in EP(\sum_{i=1}^{N} a_i F_i) = \bigcap_{i=1}^{N} EP(F_i)$. *If we choose* $x_1 \in [0, \infty)$, *then we have*

$$\bigcap_{i=1}^{\bar{N}} Fix(T_i) \cap \bigcap_{i=1}^{\bar{N}} Fix(S_i) \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}.$$

By the definition of F, *we have*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \\ &= \sigma \left(-5x^2 + xy + 4y^2 \right) + \frac{1}{r_n} (y - u_n)(u_n - x_n) \\ &\Leftrightarrow \\ 0 &\leq \sigma r_n \left(-5x^2 + xy + 4y^2 \right) + \frac{1}{r_n} (y - u_n)(u_n - x_n) \\ &= 4\sigma r_n y^2 + (\sigma u_n r_n + u_n - x_n) y + u_n x_n - u_n^2 - 5\sigma u_n^2 r_n. \end{aligned}$$

Let $G(y) = 4\sigma r_n y^2 + (\sigma u_n r_n + u_n - x_n) y + u_n x_n - u_n^2 - 5\sigma u_n^2 r_n$. *Then* $G(y)$ *is a quadratic function of* y *with coefficient* $a = 4\sigma r_n$, $b = \sigma u_n r_n + u_n - x_n$, *and*

$c = u_n x_n - u_n^2 - 5\sigma u_n^2 r_n$. *Determine the discriminant* Δ *of* G *as follows*

$$\begin{aligned}\Delta =& b^2 - 4ac \\ =& \left(\sigma u_n r_n + u_n - x_n\right)^2 - 4\left(4\sigma r_n\right)\left(u_n x_n - u_n^2 - 5\sigma u_n^2 r_n\right) \\ =& u_n^2 - 2u_n x_n + x_n^2 + 18\sigma r_n u_n^2 - 18\sigma r_n u_n x_n + 81\sigma^2 r_n^2 u_n^2 \\ =& \left(u_n - x_n + 9\sigma r_n u_n\right)^2.\end{aligned}$$

We know that $G(y) \geq 0, \forall y \in \mathbb{R}$. *If it has most one solution in* $\mathbb{R}$, *then* $\Delta \leq 0$, *so we obtain*

$$u_n = \frac{x_n}{1 + 9\sigma r_n}, \text{ where } \sigma = \sum_{i=1}^{N}\left(\frac{5}{6^i} + \frac{1}{N6^N}\right) i. \tag{4.1}$$

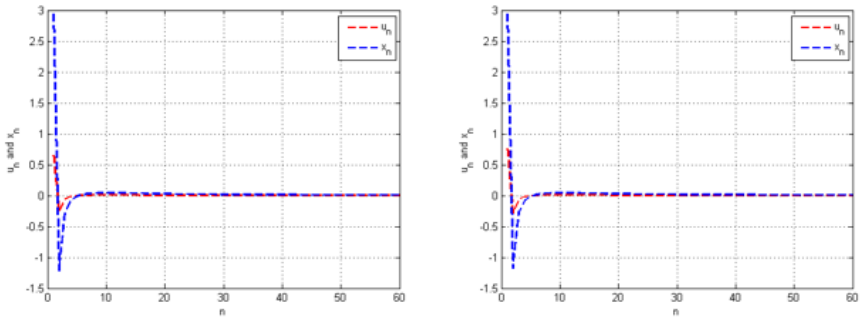
From (4.1), the iterative method (3.1) becomes

$$\begin{cases} y_n^i = \frac{5n}{7n+6}\left(I - \frac{1}{3n^3}\left(I - S_i\right)\right) x_n + \left(1 - \frac{5n}{7n+6}\right)\left(I - \frac{1}{n^2}\left(I - T_i\right)\right)\left(\frac{x_n}{1+9\sigma r_n}\right), \\ x_{n+1} = \frac{1}{8n}\left(1 + \frac{x_n}{4}\right) + \left(1 - \frac{1}{8n}\right)\sum_{i=1}^{\bar{N}} \rho_n^i y_n^i, \forall n \in \mathbb{N}, \end{cases} \tag{4.2}$$

Furthermore, if we put $F_i = F, \forall i = 1, 2, \ldots, N$, *where the bifunction* $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is defined by* $F(x, y) = -5x^2 + xy + 4y^2$, *for all* $x, y \in \mathbb{R}$., *then we obtain the following iterative scheme*

$$\begin{cases} y_n^i = \frac{5n}{7n+6}\left(I - \frac{1}{3n^3}\left(I - S_i\right)\right) x_n + \left(1 - \frac{5n}{7n+6}\right)\left(I - \frac{1}{n^2}\left(I - T_i\right)\right)\left(\frac{x_n}{1+9r_n}\right), \\ x_{n+1} = \frac{1}{8n}\left(1 + \frac{x_n}{4}\right) + \left(1 - \frac{1}{8n}\right)\sum_{i=1}^{\bar{N}} \rho_n^i y_n^i, \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

For the iterative scheme (4.2) and (4.3), Table 1 and Figure 1 show the numerical results of sequences $\{u_n\}$, *and* $\{x_n\}$ *with* $x_1 = 3$ *and* $n = N = 60, \bar{N} = 3$.



(a) the iterative algorithm (3.1) (b) the iterative algorithm (3.15)

Figure 1: The convergence comparison of $\{u_n\}$ and $\{x_n\}$ with initial value $x_1 = 3$ for the iterative algorithm (3.1) and (3.15).

	the iterative algorithm (3.1)		the iterative algorithm (3.19)	
n	u_n	x_n	u_n	x_n
1	0.707547	3.000000	0.810811	3.000000
2	-0.252068	-1.212893	-0.281186	-1.174365
3	-0.051259	-0.258859	-0.060812	-0.266053
4	-0.015262	-0.079069	-0.019155	-0.085886
5	-0.002439	-0.012836	-0.003880	-0.017665
⋮	⋮	⋮	⋮	⋮
30	0.003387	0.018841	0.004084	0.019617
⋮	⋮	⋮	⋮	⋮
56	0.001757	0.009829	0.002114	0.010208
57	0.001725	0.009651	0.002076	0.010023
58	0.001694	0.009479	0.002039	0.009844
59	0.001664	0.009313	0.002003	0.009671
60	0.001636	0.009153	0.001968	0.009504

Table 1: The values of $\{u_n\}$ and $\{x_n\}$ with an initial value $x_1 = 3$ for the iterative algorithm (3.1) and (3.19).

Remark 4.2. *From the above numerical results, we can conclude that*

(i) For the iterative algorithm (3.1), Table 1 shows that the sequences $\{u_n\}$ *and* $\{x_n\}$ *converge to* $\{0\} = \bigcap_{i=1}^{\bar{N}} Fix(T_i) \cap \bigcap_{i=1}^{\bar{N}} Fix(S_i) \cap \bigcap_{i=1}^{N} EP(F_i)$ *and the convergence of* $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ *and* $\{x_n\}$ *can be guaranteed by Theorem 3.1.*

(ii) For the iterative algorithm (3.15), Table 1 shows that the sequences $\{u_n\}$ *and* $\{x_n\}$ *converge to* $\{0\} = \bigcap_{i=1}^{\bar{N}} Fix(T_i) \cap \bigcap_{i=1}^{\bar{N}} Fix(S_i) \cap EP(F)$ *and Corollary 3.2 guarantees the convergence of* $\{u_n\}$ *and* $\{x_n\}$.

(iii) From Table 1, we have that the iterative method for the combination of equilibrium problem (3.1) converges faster than the that the iterative method for the classical equilibrium problem (3.19).

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