Thai Journal of Mathematics : (2016) 61-76
Special Issue (ACFPTO2016) on : Advances in fixed point theory towards real world optimization problems
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Some coincidence points theorems for multi-valued F-weak contractions on complete metric space endowed with a graph

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Abstract : In this paper, we introduce the concepts of weak g-graph-preserving for multi-valued mappings and weak F-G-contractions in a metric space endowed with a directed graph. We establish some coincidence point theorems for this type of mappings in a complete metric space endowed with a directed graph. Examples illustrating our main results are also presented. Our results extend and generalize various known results in the literature.

Keywords : coincidence point; nonlinear integral equation; Closed multi-valued F-contractions; F-contractive condition of Hardy-Rogers-type mappings
2000 Mathematics Subject Classification : 47H10; 47H09.

1 Introduction

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Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers. Similarly, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of real numbers, positive real numbers and the set of nonnegative real numbers, respectively.

Recently many results of the fixed point problems for maps on metric spaces have been proved; see, for instance [1, 2, 3, 4, 5], Wardowski [6] has introduced the concept of an F-contraction as follows:

Definition 1.1. *[6] Let F be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that*

(F1) *F is strictly increasing, i.e. for all $x, y \in \mathbb{R}^+$ such that $x < y, F(x) < F(y)$;*

(F2) *For each sequence $\{\alpha n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha n = 0$ if and only if $\lim_{n\to\infty} F(\alpha n) = \infty$;*

(F3) *There exist $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Definition 1.2. *[6] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \text{ for all } x, y \in X \tag{1.1}$$

Theorem 1.3. *[6] Let (X,d) be a complete metric space and $T : X \to X$ be an F- contraction. Then T has a unique fixed point $x^* \in X$.*

Remark 1.4. *From (F1) and* (1.1) *it is easy to conclude that every Fcontraction is necessarily continuous.*

Very recently, Piri and Kumam [8] extended the result of Wardowski [6] by replacing the condition (F3) in Definition1.1 with the following one:

$(F3')$ F is continuous on $(0, \infty)$.

Let F denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions (F1), (F2) and $(F3')$. Under this new set-up, they proved a fixed point result that generalized the result of Wardowski [6].

The study of fixed point for multi-valued contraction mappings using the Pompeiu- Hausdorff metric was first performed by Nadler [7].

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $D(x, A) = \inf d(x, y) : y \in A$. The class of all nonempty bounded and closed subsets of X is denoted by $CB(X)$. Let H be a Pompeiu-Hausdorff metric induced by the metric d on X, that is,

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$$

for every $A, B \in CB(X)$.

We let $\mathrm{Comp}(X)$ be the set of all nonempty compact subsets of X. It is clear that $\mathrm{Comp}(X)$ is included in $CB(X)$.

Let $T : X \to 2^X$ (collection of all nonempty subsets of X) and $g : X \to X$. An element $x \in X$ is called

(i) fixed point of T if $x \in T(x)$,

(ii) coincidence point of a hybrid pair $\{T, g\}$ if $g(x) \in T(x)$.

In 1969, Nadler [7] introduced the concept of Banach contraction principle for a multi- valued mapping and proved the existence of fixed point for multi-valued version of the Banach contraction principle. The following theorem is the first well-known theorem of multi-valued contractions studied by Nadler.

Theorem 1.5. *Let (X, d) be a complete metric space and $T : XCB(X)$ be a multi-valued mapping. If there exists $k[0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X,$$

then T has a fixed point in X.

In 2007, Berinde and Berinde [9] provided the new type of contraction which is a generalization of the contraction principle considered by Nadler.

Definition 1.6. *[9] Let (X, d) be a metric space and $T : X \to CB(X)$ be a multi-valued mapping. T is said to be a multi-valued weak contraction or a multi-valued (θ, L)-weak contraction if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X.$$

We now recall some notions concerning a directed graph. Let (X, d) be a metric space and Δ denote the diagonal of $X \times X$ .Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges is a subset of $X \times X$. We assume that the graph G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$. We denote by G^{-1} the conversion of a graph G, i.e.,

$$G^{-1} = (x, y) \in X \times X : (y, x) \in E(G).$$

The next definition, G-contraction, was introduced by Jachymski [10] in 2008.

Definition 1.7. *[10] Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\theta \subseteq E(G)$. We say that a mapping $T : X \to X$ is a G-contraction if T preserves edges of G, i.e., for every $x, y \in X$,*

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G) \tag{1.2}$$

and there exists $\alpha \in (0, 1)$ such that $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y).$$

The mapping $T : X \to X$ satisfying condition (1.2) is called a graph-preserving mapping. Under some additional properties of a metric space X endowed with a directed graph, Jachymski showed that a G-contraction $T : X \to X$ has a fixed point if and only if there exists $x \in X$ such that $(x, T(x)) \in E(G)$.

Subsequently, Beg and Butt [11] tried to introduce the concept of G-contraction for multi-valued mappings, but their extension was not carried correctly (see [12, 13].

In 2011, Nicolae et al. [14] extended the notion of multivalued contraction on a metric space with a graph.

Recently, Dinevari and Frigon [15] introduced a new concept of G-contraction multi- valued mappings.

Definition 1.8. *[15] Let $T : X \to 2^x$ be a map with nonempty values. We say that T is a G-contraction (in the sense of Dinevari and Frigon) if there exists $\alpha \in (0, 1)$ such that for all $(x, y) \in E(G)$ and $u \in Tx$, there exists $v \in Ty$ such that $(u, v) \in E(G)$ and $d(u, v) \leq \alpha d(x, y)$.*

They showed that under some properties, weaker than Property (A), a multi-valued G- contraction with the closed value has a fixed point. Recently, Tiammee and Suantai [16] introduced the concept of graph-preserving for multi-valued mappings and proved their fixed point theorem in a complete metric space endowed with a graph.

Definition 1.9. *[16] Let X be a nonempty set, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and $T : X \to CB(X)$. Then T is said to be graph-preserving if $(x, y) \in E(G) \Rightarrow (u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$.*

Definition 1.10. *[16] Let X be a nonempty set, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X, g : X \to X$, and $T : X \to CB(X)$. Then T is said to be g-graph-preserving if for any $x, y \in X$ such that*

$$g(x), g(y) \in E(G) \Rightarrow (u, v) \in E(G)$$

for all $u \in Tx$ and $v \in Ty$.

Recently, Phon-on et al. [17] introduced a new type of weak G-contraction which is weaker than that of Tiammee and Suantai [16], and they proved some fixed point theorems for this type of mappings with compact values which is a generalization of several known results in a complete metric space endowed with a graph.

Definition 1.11. *[17] Let X be a nonempty set and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and $T : X \to Comp(X)$. Then T is said to be weak graph-preserving if it satisfies the following: for each $x, y \in X$, if $(x, y) \in E(G)$, then for each $u \in Tx$ there is $v \in P_{Ty}(u)$ such that $(u, v) \in E(G)$, where $P_{Ty}(u) = a \in Ty \mid d(u, a) = D(u, Ty)$.*

It is our main aim in this work to establish the main result of Altun et al., [2] without the assumptions F2 and F3 used in their proof and also, we give the main result of Acar et. al. [18] for a mapping $T : X \to CB(X)$ instead of a mapping $T : X \to K(X)$ and we give their result without the continuity of T and assumptions F2 and F3 used in their proof.

2 Preliminaries

2.1 Instructions to Authors

Throughout the article $\mathbb{N}, \mathbb{R}^+$ and $\mathbb{R}$ will denote the set of natural numbers, positive real numbers and real numbers, respectively.

Definition 2.1. *[19] Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be an F-contraction on X if there exists $\tau > 0$ such that*

$$d(fx, fy) > 0 \Rightarrow \tau + F\left(d(fx, fy)\right) \le F(d(x, y))$$

for all $x, y \in X$ and $F \in \mathcal{F}$.

Let (X, d) be a metric space and denote $\mathcal{G} := \{G : G$ is a directed graph with $V(G) = X$ and $\Delta \subseteq E(G)\}$.

Definition 2.2. *[20] Let $F \in \mathcal{F}$, $G \in \mathcal{G}$ be given and (X, d) be a metric space. A mapping $f : X \to X$ is said to be an F-G-contraction if*

- $(c1)$ *$(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$ for all $x, y \in X$,*
- $(c2)$ *there exists a number $\tau > 0$ such that*

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \le F(d(x, y))$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

Remark 2.3. *Let $G \in \mathcal{G}$ be arbitrary and F be given by $F(\alpha) = \ln \alpha$ for all $\alpha > 0$. Then F-G-contraction reduces to G-contraction given in [21].*

Definition 2.4. *[21] A graph G is said to satisfy the property (A) if for any sequence $\{x_n\}$ in $V(G)$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies that $(x_n, x) \in G$.*

Finally, let $C(f, g) := \{x \in X : fx = gx\}$ denote the set of all coincidence points of two self-mappings f and g, defined on X.

Definition 2.5. *[22] Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. A mapping $f : X \to X$ is said to be an F-G-contraction with respect to $g : X \to X$ if*

- (e_1) *$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G)$ for all $x, y \in X$, i.e. f preserves edges w.r.t. g,*
- (e_2) *there exists a number $\tau > 0$ such that*

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \le F(d(gx, gy)) \tag{2.1}$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$.

Remark 2.6. *If $g = I_X$ (identity map on X), then the above definition reduces to Definition 2.2 given in [20] Further, if F is given by $F(\alpha) = \ln \alpha$ for all $\alpha > 0$, then condition* (2.1) *will become Jungck G-contraction*

$$d(fx, fy) \leq e^{-\tau} d(gx, gy)$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$.

Definition 2.7. *Let (X, d) be a metric space. A mapping $T : X \to CB(X)$ is said to be a generalized multivalued F-weak contraction on (X, d), if there exist $F \in F_G$ and > 0 such that*

$$H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty))F(MT(x, y)), \text{ for all } x, y \in X,$$

where

$$M_T(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty) + D(y, Tx)\}.$$

3 Main Results

Definition 3.1. *Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. A mapping $f : X \to X$ is said to be an F-G-contraction with respect to $g : X \to X$ if*

- (e_1) *$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G)$ for all $x, y \in X$, i.e. f preserves edges w.r.t. g,*
- (e_2) *there exists a number $\tau > 0$ such that*

$$H(fx, fy) > 0 \Rightarrow \tau + F(H(fx, fy)) \leq F(M_T(gx, gy)) \tag{3.1}$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$.
where

$$M_T(gx, gy) = max\{d(gx, gy), D(gx, fx), D(gy, fy), \frac{D(gx, fy) + D(gy, fx)}{2}\}. \tag{3.2}$$

Remark 3.2. *If $g = I_X$ (identity map on X), then the above definition reduces to Definition 2.5 given in [20]. Further, if F is given by $F(\alpha) = \ln \alpha$ for all $\alpha > 0$, then condition* (3.1) *will become Jungck G-contraction*

$$d(fx, fy) \leq e^{-\tau} d(gx, gy)$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$.

We say that f is an order F-weak contraction with respect to g if f and g satisfy condition (e_2) above.

Remark 3.3. *Conditions* (e_1) *and* (e_2) *in Definition 3.1 are independent. For example, consider* $f = g = I_X$*, then the identity map always preserves edges but condition* (3.1) *is not satisfied for any* $\alpha > 0$*. Further, consider* $f : \mathbb{R} \to \mathbb{R}$ *given by* $fx = \frac{-x}{2}, x \in \mathbb{R}$ *and* $g = I_{\mathbb{R}}$ *(identity map on* $\mathbb{R}$*), then* f *is an order* $\ln(\cdot)$*-contraction w.r.t.* g *for* $\tau = \ln(2)$ *but* f *is not non-decreasing.*

Property(A) [10] For any sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in N$, then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in N$.

Now we state and prove our first theorem.

Theorem 3.4. *Let* (X, d) *be a complete metric space and* $G = (V(G), E(G))$ *be a directed graph such that* $V(G) = X$ *and let* $f, g : X \to X$ *be two mappings.If* $f : X \to CB(X)$ *is a multi-valued mapping satisfying the following properties:*

(i) f *is a weak g-graph-preserving mapping;*

(ii) there exists $x_0 \in X$ *such that* $(gx_0, fx_0) \in E(G)$*;*

(iii) X *has the property (A);*

(iv) f *is an* F*-*G*-weak contraction with respct to* g*;*

(v) $f(X) \subseteq g(X)$.

Then f,g has a coincidence point .

Proof. Let $x_0 \in X$ be as in (ii). Since $f(X) \subseteq g(X)$, then there exists a point $x \in X, Tx \neq \emptyset$ such that $gx_1 = fx_0$. From (ii), we have $(gx_0, fx_0) \in E(G)$, i.e., $(gx_0, gx_1) \in E(G)$ and, since f preserves edges with respect to g, we get $(fx_0, fx_1) \in E(G)$.
Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that

$$(gx_n, gx_{n+1}) = (fx_{n-1}, fx_n) \in E(G) \text{ for every } n \in \mathbb{N}.$$

For sake of simplicity take

$$H(fx_{n-1}, fx) \geq d(gx_n, gx_{n+1}) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $gx_{n_0} = gx_{n_0+1}$, then $fx_{n_0} = gx_{n_0+1}$ implies that $fx_{n_0+1} = gx_{n_0+1}$ that is $x_{n_0+1} \in C(f, g)$. Now we assume $gx_n \neq gx_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$. Since f is an F-G-contraction w.r.t. g on $E(G)$, then we write

$$
\begin{aligned}
\tau + F(d(gx_n, gx_{n+1})) &\leq \tau + F(H(fx_{n1}, fx_n)) \\
&\leq F(M_T(gx_{n1}, gx_n)) \\
&= F\left(\max \left\{ \begin{array}{c} d(gx_{n-1}, gx_n), D(gx_{n-1}, fx_{n-1}), \\ D(gx_n, fx_n), \frac{D(gx_{n-1}, fx_n) + D(gx_n, fx_{n-1})}{2} \end{array} \right\} \right) \\
&\leq F\left(\max \left\{ \begin{array}{c} d(gx_{n-1}, gx_n), \\ d(gx_{n-1}, gx_n) + D(gx_n, fx_{n-1}), \\ d(gx_n, gx_{n+1}) + D(gx_{n+1}, fx_n), \\ \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2}, \\ \frac{D(gx_{n-1}, fx_n) + D(gx_n, fx_{n-1})}{2} \end{array} \right\} \right) \\
&= F\left(\max \left\{ \begin{array}{c} d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n) + 0, \\ d(gx_n, gx_{n+1}) + 0, \\ \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + 0 + 0}{2} \end{array} \right\} \right)
\end{aligned}
$$

$$
\leq F(\max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}) \tag{3.4}
$$

If there exists $n \in \mathbb{N}$ such that $\max d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}) = d(gx_n, gx_{n+1})$, then (3.4) becomes

$$
\tau + F(d(gx_n, gx_{n+1})) \leq F(d(x_n, gx_{n+1})),
$$

which is a contradiction. Thus, we conclude that $\max d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}) = d(gx_{n-1}, gx_n)$, for all $n \in \mathbb{N}$. Hence, the inequality (3.4) turns into

$$
F(d(gx_n, gx_{n+1})) \leq F(d(gx_{n-1}, gx_n))\tau, \text{ for all } n \in \mathbb{N}. \tag{3.5}
$$

Since $\tau > 0$, so we get

$$
F(d(gx_n, gx_{n+1})) < F(d(gx_{n-1}, gx_n)), \text{ for all } n \in \mathbb{N}. \tag{3.6}
$$

It follows from (3.6) and (F1) that

$$
d(gx_n, gx_{n+1}) < d(gx_{n-1}, gx_n), \text{ for all } n \in \mathbb{N}.
$$

Therefore $d(gx_n, gx_{n+1})_{n\in\mathbb{N}}$ is a nonnegative nonincreasing sequence, and hence

$$
\lim d(gx_n, gx_{n+1}) = \gamma \leq 0.n \to \infty
$$

Now, we claim that $\gamma = 0$. Arguing by contradiction, we assume that $\gamma > 0$. Since $d(gx_n, gx_{n+1})_{n\in\mathbb{N}}$ is a nonnegative nonincreasing sequence, so there exists $n_1 \in \mathbb{N}$ such that

$$
d(gx_n, gx_{n+1}) > \gamma, \text{ for all } n > n_1. \tag{3.7}
$$

So from (3.5) and (3.7) and (F1), we get

$$\begin{aligned} F(\gamma) < F(d(gx_n, gx_{n+1})) &\leq F(d(gx_{n-1}, gx_n)) - \tau \\ &\leq F(d(gx_{n-2}, gx_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(d(gx_{n_1}, gx_{n_1} + 1)) - (n - n1)\tau, \text{ for all } n > n_1. \end{aligned} \tag{3.8}$$

Since $F(\gamma) \in \mathbb{R}$ and $\lim_{n\to\infty}(F(d(gx_{n_1}, gx_{n_1} + 1)) - (n - n_1)\tau) = -\infty$. So there exists $n_2 \in \mathbb{N}$ such that

$$F(d(gx_{n_1}, gx_{n_1} + 1)) - (n - n_1)\tau < F(\gamma), \text{ for all } n > n_2. \tag{3.9}$$

Setting $n_3 = \max n_1, n_2$. From (3.8) and (3.9), we get

$$F(\gamma) < F(d(gx_{n_1}, gx_{n_1} + 1)) - (n - n_1)\tau < F(\gamma), \text{ for all } n > n_3.$$

Which is a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{3.10}$$

Now, we claim that

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 0. \tag{3.11}$$

Arguing by contradiction, we assume that there exist $\epsilon > 0$ and subsequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for all $n \in \mathbb{N}$,

$$p(n) > q(n) > n, d(gx_{p(n)}, gx_{q(n)}) \leq \epsilon, d(gx_{p(n)-1}, gx_{q(n)}) < \epsilon. \tag{3.12}$$

Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned} \epsilon \leq d(gx_{p(n)}, gx_{q(n)}) &\leq d(gx_{p(n)}, gx_{p(n)-1}) + d(gx_{p(n)-1}, gx_{q(n)}) \\ &\leq d(gx_{p(n)}, gx_{p(n)-1}) + \epsilon. \end{aligned} \tag{3.13}$$

It follows from (3.10), (3.13) and the Sandwich theorem that

$$\lim_{n\to\infty} d(gx_{p(n)}, gx_{q(n)}) = \epsilon. \tag{3.14}$$

From (3.14), there exists $n_4 \in \mathbb{N}$ such that

$$0 < \epsilon < d(gx_{p(n)}, gx_{q(n)}) \leq H(fx_{p(n)}, fx_{q(n)}), \text{ for all } n > n_4.$$

Since $gx_{p(n)+1} \in fx_{p(n)}, gx_{q(n)+1} \in fx_{q(n)}$ and f is a multivalued F -weak contraction, we have

$$\begin{aligned}
\tau &+ F(d(gx_{p(n)}, gx_{q(n)})) \\
&\leq \tau + F(H(fx_{p(n)}, fx_{q(n)})) \\
&\leq F(M_f(gx_{p(n)}, gx_{q(n)})) \\
&= F\left(\max\left\{\begin{array}{c} d(gx_{p(n)}, gx_{q(n)}), D(gx_{p(n)}, fx_{p(n)}), \\ D(gx_{q(n)}, fx_{q(n)}), \\ \frac{D(gx_{p(n)}, fx_{q(n)}) + D(gx_{q(n)}, fx_{p(n)})}{2}\}) \end{array}\right\}\right) \\
&\leq F\left(\max\left\{\begin{array}{c} d(gx_{p(n)}, gx_{q(n)}), \\ d(gx_{p(n)}, gx_{p(n)+1}) + D(gx_{p(n)+1}, fx_{p(n)}), \\ d(gx_{q(n)}, gx_{q(n)+1}) + D(gx_{q(n)+1}, fx_{q(n)}), \\ \frac{1}{2}[d(gx_{p(n)}, gx_{q(n)}) + d(gx_{q(n)}, gx_{q(n)} + 1) \\ +D(gx_{q(n)+1}, fx_{q(n)}) + d(gx_{q(n)}, gx_{p(n)}) \\ +d(gx_{p(n)}, gx_{p(n)+1}) + D(gx_{p(n)+1}, fx_{p(n)})] \end{array}\right\}\right) \\
&= F\left(\max\left\{\begin{array}{c} d(gx_{p(n)}, gx_{q(n)}), d(gx_{p(n)}, gx_{q(n)+1}) + 0, \\ d(gx_{q(n)}, gx_{q(n)+1}) + 0, \\ \frac{d(gx_{p(n)}, gx_{q(n)}) + d(gx_{p(n)}, gx_{p(n)+1}) + 0}{2} \\ \frac{d(gx_{q(n)}, gx_{p(n)}) + d(gx_{p(n)}, gx_{p(n)+1}) + 0}{2}\} \end{array}\right\}\right)
\end{aligned}$$

Letting $n \to \infty$ in the inequality above by taking (3.10), (3.14) and $(F3')$ into account, we get that

$$\tau + F(\epsilon) \leq F(\epsilon).$$

Since $\tau > 0$, this is a contradicts. Hence

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 0.$$

Therefore, we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, so there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(gx_n, gx^*) = 0. \tag{3.15}$$

In this case we claim $x^* \in fx^*$. Suppose contrary that $x^* \notin fx^*$. In this case there exists $n_0 \in \mathbb{N}_0$ and a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that $D(gx_{n_{k+1}}, fx^*) > 0$ for all $n_k \leq n_0$ (otherwise , there exists $n_1 \in \mathbb{N}_0$ such that $gx_n \in fx^*$ for all $n \leq n_1$, which implies that $gx^* \in fx^*$. This is a contradiction, Since $x^* \notin fx^*$). Since $D(gx_{n_{k+1}}, fx^*) > 0$ for all $n_k \leq n_0$, and f is multivalued Fweak contraction

, we obtain

$$\begin{aligned}
\tau \quad &+ \quad F(D(gx_{n_{k+1}}, fx^*)) \\
&\leq \tau + F(H(fx_{n_k}, fx^*)) \\
&\leq F(M(gx_{n_k}, gx^*)) \\
&= F\left(\max\left\{ \begin{array}{c} d(gx_{n_k}, gx^*), D(gx_{n_k}, fx_{n_k}), D(gx^*, fx^*), \\ \frac{D(gx_{n_k}, fx^*)+D(gx^*, fx_{n_k})}{2} \} \end{array} \right\}\right) \\
&\leq F\left(\max\left\{ \begin{array}{c} d(gx_{n_k}, gx^*), D(gx^*, fx^*), \\ d(gx_{n_k}, gx_{n_{k+1}}) + D(gx_{n_k+1}, fx_{n_k}), \\ \frac{D(gx_{n_k}, fx^*)+d(gx^*, gx_{n_k+1})+D(gx_{n_k+1}, fx_{n_k})}{2} \end{array} \right\}\right) \\
&= F\left(\max\left\{ \begin{array}{c} d(gx_{n_k}, gx^*), d(gx_{n_k}, gx_{n_k+1}) + 0, D(gx^*, fx^*), \\ \frac{D(gx_{n_k}, fx^*)+d(gx^*, gxn_k+1)+0}{2} \end{array} \right\}\right)
\end{aligned}$$

Letting $n \to \infty$ in the inequality above by taking (3.10), (3.15) and ($F3'$) into account, we get that

$$\tau + F(D(gx^*, fx^*)) \leq F(D(gx^*, fx^*)).$$

Which is a contradiction. Therefore, we have $gx^* \in fx^*$. Hence x^* is a coincidence point of f and g. □

4 Applications

4.1 Existence of solution for nonlinear integral equation

Let $X = C([0,T], \mathbb{R})$ denote the space of all continuous functions on $[0,T]$ and, for $u \in X$, define the supremum norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$. Clearly, $(X, \|\cdot\|)$ endowed with the metric d defined by

$$d(u,v) = \sup_{t\in[0,T]} \{|u(t) - v(t)|\}, \tag{4.1}$$

for all $u, v \in X$, is a complete metric space.

Now, following the idea in [23], we discuss the application of fixed point techniques to the solution of the nonlinear integral equation:

$$x(t) = h_1(t) - h_2(t) + \mu \int_0^t m_1(t,s)k_1(s, x(s))\, ds + \rho \int_0^T m_2(t,s)k_2(s, x(s))ds, \tag{4.2}$$

where $t \in [0,T]$, μ, ρ are real numbers, $h_1, h_2 \in C([0,T],\mathbb{R})$ with $h_1 \geq h_2$, and $m_1(t,s)$, $m_2(t,s)$, $k_1(t,s)$, $k_2(t,s)$ are continuous real valued functions in $[0,T]\times\mathbb{R}$. Next, consider the graph G with $V(G) = X$ and

$$E(G) = \{(u,v) \in X \times X \,:\, u(t) \leq v(t), \text{ for all } t \in [0,T]\}.$$

Clearly, $E(G)$ is a partial order. In particular, in [24], the authors showed that $(X, E(G))$ is regular, that is property (A) holds true.
We will prove the following theorem.

Theorem 4.1. *For each $x \in X$, define the operators:*

$$Bx(t) = -h_2(t) + \mu \int_0^t m_1(t,s)k_1(s,x(s))ds$$

and

$$Sx(t) = x(t) - h_1(t) - \rho \int_0^T m_2(t,s)k_2(s,x(s))\,ds,$$

where $t \in [0,T]$, μ, ρ are real numbers, $h_1, h_2 \in C([0,T],\mathbb{R})$ with $h_1 \geq h_2$, and $m_1(t,s)$, $m_2(t,s)$, $k_1(t,s)$, $k_2(t,s)$ are continuous real valued functions in $[0,T]\times\mathbb{R}$. Suppose that the following hypotheses are satisfied:

- *(i)* $\int_0^T \sup_{t\in[0,T]} |m_i(t,s)|ds = M_i < \infty$, $i \in \{1,2\}$,
- *(ii) for each $s \in [0,T]$ and for all $x, y \in X$ with $(Sx, Sy) \in E(G)$, there exist $L_i \geq 0$ such that*

$$|k_i(s,x(s)) - k_i(s,y(s))| \leq L_i|x(s) - y(s)|, \quad i \in \{1,2\},$$

- *(iii)* $\rho \int_0^T m_2(t,s)k_2\left(s, \mu\int_0^s m_1(s,\lambda)k_1(\lambda,x(\lambda))d\lambda + h_1(s) - h_2(s)\right) ds = 0,$
- *(iv) $(Sx, Sy) \in E(G)$ implies $(Bx, By) \in E(G)$ for all $x, y \in X$,*
- *(v) there exists $x_0 \in X$ such that*

$$x_0 \leq h_1(t) - h_2(t) + \mu \int_0^t m_1(t,s)k_1(s,x_0(s))\,ds + \rho \int_0^T m_2(t,s)k_2(s,x_0(s))ds.$$

Then the integral equation (4.2) has a solution, provided that $\frac{|\mu|L_1M_1}{1-|\rho|L_2M_2} < 1$.

Proof. Note that the integral equation (4.2) has a solution if and only if the operators B and S have a coincidence point. Clearly, B and S are self-operators on X. Now, for all $x, y \in X$ with $(Sx, Sy) \in E(G)$, by assumptions (i) and (ii), we have

$$\begin{aligned} |Bx(t)-By(t)| &\leq |\mu|\int_0^t |m_1(t,s)||k_1(s,x(s))-k_1(s,y(s))|ds \\ &\leq |\mu|\int_0^t \sup_{t\in[0,T]} |m_1(t,s)||k_1(s,x(s))-k_1(s,y(s))|ds \\ &= |\mu|\int_0^t \sup_{t\in[0,T]} |m_1(t,s)|L_1|x(s)-y(s)|ds \\ &\leq |\mu|L_1\|x-y\|\int_0^t \sup_{t\in[0,T]} |m_1(t,s)|ds \\ &\leq |\mu|L_1M_1\|x-y\|. \end{aligned}$$

This implies that

$$\|Bx-By\| = \sup_{t\in[0,T]} |Bx(t)-By(t)| \leq |\mu|L_1M_1\|x-y\|, \tag{4.3}$$

By a similar reasoning we get

$$\begin{aligned} &\left|\rho\int_0^T m_2(t,s)k_2(s,x(s))\,ds - \rho\int_0^T m_2(t,s)k_2(s,y(s))\,ds\right| \\ &\leq |\rho|\int_0^T |m_2(t,s)||k_2(s,x(s))-k_2(s,y(s))|ds \\ &\leq |\rho|\int_0^T \sup_{t\in[0,T]} |m_2(t,s)||k_2(s,x(s))-k_2(s,y(s))|ds \\ &\leq |\rho|\int_0^T \sup_{t\in[0,T]} |m_2(t,s)|L_2|x(s)-y(s)|ds \\ &\leq |\rho|L_2M_2\|x-y\|, \end{aligned}$$

which implies

$$\sup_{t\in[0,T]}\left|\rho\int_0^T m_2(t,s)k_2(s,x(s))\,ds - \rho\int_0^T m_2(t,s)k_2(s,y(s))\,ds\right| \leq |\rho|L_2M_2\|x-y\|.$$

Consequently, we note that

$$\begin{aligned} &\|Sx-Sy\| \\ &\geq \|x-y\| - \sup_{t\in[0,T]}\left|\rho\int_0^T m_2(t,s)k_2(s,x(s))\,ds - \rho\int_0^T m_2(t,s)k_2(s,y(s))\,ds\right| \\ &\geq (1-|\rho|L_2M_2)\|x-y\|, \end{aligned}$$

which implies

$$\|x-y\| \leq \frac{1}{1-|\rho|L_2M_2}\|Sx-Sy\|. \tag{4.4}$$

Finally, by merging (4.3) and (4.4), we obtain

$$\|Bx - By\| \leq \frac{|\mu|L_1 M_1}{1 - |\rho|L_2 M_2}\|Sx - Sy\|.$$

and, since $\frac{|\mu|L_1 M_1}{1-|\rho|L_2 M_2} < 1$, we can choose $\tau > 0$ such that $e^{-\tau} = \frac{|\mu|L_1 M_1}{1-|\rho|L_2 M_2}$ and hence, in view of (4.1), we deduce that

$$d(Bx, By) \leq e^{-\tau} d(Sx, Sy).$$

By passing to logarithms, we can write this as

$$\ln(d(Bx, By)) \leq \ln(e^{-\tau} d(Sx, Sy)),$$

and, after routine calculations, we get

$$\tau + \ln(d(Bx, By)) \leq \ln(d(Sx, Sy)),$$

for all $x, y \in X$ such that $(Sx, Sy) \in E(G)$. Thus, condition (2.5) is trivially satisfied. Next, adopting the same reasoning in [23], we can show that $B(X) \subseteq S(X)$. Indeed, by (iii), for $x(t) \in X$ we have

$$\begin{aligned}
&S(Bx(t) + h_1(t)) \\
&= Bx(t) + h_1(t) - h_1(t) - \rho \int_0^T m_2(t,s)k_2(s, Bx(s) + h_1(s))ds \\
&= Bx(t) - \rho \int_0^T m_2(t,s)k_2\left(s, \mu \int_0^s m_1(s,\lambda)k_1(\lambda, x(\lambda))d\lambda + h_1(s) - h_2(s)\right)ds \\
&= Bx(t).
\end{aligned}$$

Clearly, hypothesis (iv) means that B preserves edges w.r.t. S. Next, by hypothesis (v) we get

$$x_0 - h_1(t) - \rho \int_0^T m_2(t,s)k_2(s, x_0(s))ds \leq h_2(t) + \mu \int_0^t m_1(t,s)k_1(s, x_0(s))\ ds,$$

that is $(Sx_0, Bx_0) \in E(G)$. Thus all the conditions of Theorem 3.4 are satisfied and hence its conclusion holds true, that is B and S have at least a coincidence point. Consequently, the integral equation (4.2) has a solution in X. □

Acknowledgement(s) : The authors was supported by Muban Chombueng Rajabhat University and Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT).

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(Received 21 May 2016)
(Accepted 14 September 2016)