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Fixed Point of $\alpha-\psi-$Geraghty Contraction Type Mappings

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Abstract : In this paper, we improve the notion of $\alpha-\psi-$Geraghty contraction type mappings and establish some common fixed point theorems for the mappings satisfying this conditions. We illustrate an example for support our results.

Keywords : fixed point; $\alpha-\psi-$Geraghty contraction
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1 Introduction

Fixed point theory has gained very large impetus due its wide range of applications in several fields such as Computer Science, Economics, Engineering, Chemistry, Biology, Physics and many others. It is well known that the contractive type conditions are very indispensable in the study of fixed point theory and Banachs fixed point theorem [1] for contraction mappings is one of the pivotal results in analysis. This theorem that has been extended and generalized by several authors which defining new contractive conditions and replacing complete metric spaces with some convenient abstract space.

In 1973, Geraghty [3] studied a generalization of Banach contraction mapping principle in complete metric space. In 2012, Samet et al.[4] introduced the concepts

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of a contractive and $\alpha-$admissible mappings and established several fixed point theorems for such class of mappings defined on complete metric spaces. Thereafter, the existence of fixed points of $\alpha-$admissible contractive type mappings in complete metric spaces. In 2013, Cho et al.[2] defined the concept of $\alpha-$Geraghty contraction type maps in a metric space and proved the existence and uniqueness of a fixed point for the mappings satisfying this conditions. Recently, Karapnar [6] defined the concept of $\alpha-\psi-$Geraghty contraction type mappings. For other results related to Geraghty contractions, see [7, 9, 10].

In this paper, we generalize the results obtained in [6] and give other conditions to prove common fixed point for a pair of $\alpha-\psi-$Geraghty contraction type maps in a complete metric space.

2 Preliminaries

We remind some basic definitions and remarkable results on the topic in the literature.

Definition 2.1. *[4] Let $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that f is $\alpha-$admissible if $x, y \in X$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$.*

Definition 2.2. *[5] Let $f, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that a pair (f, g) is triangular $\alpha-$admissible if*

- *(f1) $\alpha(x,y) \geq 1 \Rightarrow \alpha(fx, gy) \geq 1$ and $\alpha(fx, gy) \geq 1$, $x, y \in X$;*
- *(f2) $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1 \Rightarrow \alpha(x,y) \geq 1$, $x, y, z \in X$.*

Definition 2.3. *[8] Let $f : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. We say that f is $\alpha-$admissible mapping with respect to η if $x, y \in X$,*

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow \alpha(fx, fy) \geq \eta(fx, fy).$$

Lemma 2.4. *[7] Let $f, g : X \to X$ be a pair of triangular $\alpha-$admissible. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $x_{2i+1} = fx_{2i}$ and $x_{2i+2} = gx_{2i+1}$, for $i = 0, 1, 2, \ldots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

For this purpose, we remind the class of Γ all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \ \Rightarrow \ \lim_{n\to\infty} t_n = 0.$$

Theorem 2.5. *[3] Let (X, d) be a complete metric space and $f : X \to X$ be an operator. If f satisfies the following inequality:*

$$d(fx, fy) \leq \beta(d(x,y))d(x,y), \quad \text{for any } x, y \in X, \tag{2.1}$$

where $\beta \in \Gamma$, then f has a unique fixed point.

3 Main Results

We say Ψ be a family of functions $\psi : [0, \infty) \to [0, \infty)$ such that ψ is continuous, strictly increasing and $\psi(0) = 0$.

Definition 3.1. *Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Two mappings $f, g : X \to X$ is called a pair of generalized $\alpha - \psi-$Geraghty contraction type mapping if there exists $\beta \in \Gamma$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi(d(fx, gy)) \leq \beta(\psi(M(f,g,x,y)))\psi(M(f,g,x,y)), \tag{3.1}$$

where

$$M(f,g,x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(y,fx) + d(x,gy)}{2}\right\}$$

and $\psi \in \Psi$.

If $f = g$ then g is called generalized $\alpha - \psi-$Geraghty contraction type mapping if there exists $\beta \in \Gamma$ such that for all $x, y \in X$,

$$\alpha(x,y)\psi(d(fx, gy)) \leq \beta(\psi(M(g,g,x,y)))\psi(M(g,g,x,y)), \tag{3.2}$$

where

$$M(g,g,x,y) = \max\left\{d(x,y), d(x,gx), d(y,gy), \frac{d(x,gy) + d(y,gx)}{2}\right\}$$

and $\psi \in \Psi$.

Remark 3.2. *Notice that since $\beta : [0, \infty) \to [0, 1)$, we have*

$$\alpha(x,y)\psi(d(fx, gy)) \leq \beta(\psi(M(f,g,x,y)))\psi(M(f,g,x,y)) < \psi(M(f,g,x,y)), \tag{3.3}$$

for any $x, y \in X$, with $x \neq y$.

Theorem 3.3. *Let (X, d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Let $f, g : X \to X$ be two mappings. Suppose that the following conditions are satisfied:*

(i) (f, g) is a pair of generalized $\alpha - \psi-$Geraghty type mappings;

(ii) (f, g) is triangular $\alpha-$admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iv) f and g are continuous.

Then (f, g) have common fixed point.

Proof. Let x_1 in X be such that $x_1 = fx_0$ and $x_2 = gx_1$. Continuing this process, we have

$$x_{2i+1} = fx_{2i} \text{ and } x_{2i+2} = gx_{2i+1}, \text{ for } i = 0, 1, 2, \ldots.$$

By assumption $\alpha(x_0, x_1) \geq 1$ and a pair (f, g) is triangular $\alpha-$admissible, by Lemma 2.4, we have

$$\alpha(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Then, we obtain that

$$\begin{aligned} \psi\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) &= \psi\left(d\left(fx_{2i}, gx_{2i+1}\right)\right) \\ &\leq \alpha\left(x_{2i}, x_{2i+1}\right)\psi\left(d\left(fx_{2i}, gx_{2i+1}\right)\right) \\ &\leq \beta\left(\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right)\right)\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right), \end{aligned} \tag{3.4}$$

for all $i \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned} M\left(f, g, x_{2i}, x_{2i+1}\right) &= \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i}, fx_{2i}\right), d\left(x_{2i+1}, gx_{2i+1}\right), \\ &\qquad \frac{d\left(x_{2i}, fx_{2i}\right) + d\left(x_{2i}, gx_{2i+1}\right)}{2}\Big\} \\ &= \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i+1}, x_{2i+2}\right), \\ &\qquad \frac{d\left(x_{2i+1}, x_{2i+1}\right) + d\left(x_{2i}, x_{2i+2}\right)}{2}\Big\} \\ &= \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i+1}, x_{2i+2}\right), \\ &\qquad \frac{d\left(x_{2i}, x_{2i+2}\right)}{2}\Big\} \\ &\leq \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i+1}, x_{2i+2}\right), \\ &\qquad \frac{d\left(x_{2i}, x_{2i+1}\right) + d\left(x_{2i+1}, x_{2i+2}\right)}{2}\Big\}. \end{aligned}$$

Hence,

$$\begin{aligned} \psi\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) &\leq \beta\left(\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right)\right)\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right) \\ &\leq \beta\left(\psi\left(d\left(x_{2i}, x_{2i+1}\right)\right)\right)\psi\left(d\left(x_{2i}, x_{2i+1}\right)\right) \\ &< \psi\left(d\left(x_{2i}, x_{2i+1}\right)\right). \end{aligned}$$

That is,

$$\psi\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) < \psi\left(d\left(x_{2i}, x_{2i+1}\right)\right).$$

This is the implies that

$$\psi\left(d\left(x_{n+1}, x_{n+2}\right)\right) < \psi\left(d\left(x_n, x_{n+1}\right)\right),$$

for all $n \in \mathbb{N} \cup \{0\}$. By the properties of ψ, we conclude that

$$d\left(x_{n+1}, x_{n+2}\right) < d\left(x_n, x_{n+1}\right) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

So, the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing sequence. Consequently, there exists some positive number γ such that $\lim_{n\to\infty}(x_n, x_{n+1}) = \gamma$. From (3.4), we have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(M(f, g, x_n, x_{n+1}))} \leq \beta(\psi(M(f, g, x_n, x_{n+1}))) < 1.$$

So,

$$1 \leq \beta(\psi(d(x_n, x_{n+1}))) < 1.$$

Now, by taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1.$$

Since $\beta \in \Gamma$, we have

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0,$$

which yields that

$$\gamma = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.5}$$

Next, we will show that the sequence $\{x_n\}$ is a Cauchy by using contradiction, we suppose that $\{x_n\}$ is not a Cauchy sequence. Therefore, there exists $\varepsilon > 0$ and sequence $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k, we have $m_k > n_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon$$

and

$$d\left(x_{m_k}, x_{n_{k-1}}\right) < \varepsilon.$$

By the triangle inequality, we obtain

$$\begin{aligned}\varepsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d\left(x_{m_k}, x_{n_{k-1}}\right) + d\left(x_{n_{k-1}}, x_{n_k}\right) \\ &< \varepsilon + d\left(x_{n_{k-1}}, x_{n_k}\right).\end{aligned}$$

This is,

$$\varepsilon < \varepsilon + d\left(x_{n_{k-1}}, x_{n_k}\right), \tag{3.6}$$

for all $k \in \mathbb{N}$. From (3.6) and (3.5), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.7}$$

By using triangle inequality, we have

$$\psi\left(\left(d\left(x_{m_k}, x_{n_k}\right)\right) \leq \psi\left(d\left(x_{m_k}, x_{m_{k+1}}\right) + d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) + d\left(x_{n_{k+1}}, x_{n_k}\right)\right)\right)$$

and

$$\psi\left(\left(d\left(x_{m_{k+1}}, x_{n_{k+1}}\right)\right) \leq \psi\left(d\left(x_{m_{k+1}}, x_{m_k}\right) + d\left(x_{m_k}, x_{n_k}\right) + d\left(x_{n_k}, x_{n_{k+1}}\right)\right)\right).$$

Taking limit as $k \to \infty$ and using (3.5) and (3.7), we get

$$\lim_{k\to\infty} d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) = \varepsilon.$$

By Lemma 2.4, $d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \geq 1$, we obtain that

$$\begin{aligned}\psi\left(d\left(x_{n_{k+1}}, x_{n_{k+2}}\right)\right) &= \psi\left(d\left(fx_{n_k}, gx_{m_{k+1}}\right)\right)\\ &\leq \alpha\left(x_{n_k}, x_{m_{k+1}}\right)\psi\left(d(fx_{n_k}, gx_{m_{k+1}})\right)\\ &\leq \beta\left(\psi\left(M\left(f,g,x_{n_k},x_{m_{k+1}}\right)\right)\right) M\left(f,g,x_{n_k},x_{n_{k+1}}\right).\end{aligned}$$

Finally, we conclude that

$$\frac{\psi\left(d\left(x_{n_{k+1}}, x_{m_{k+2}}\right)\right)}{\psi\left(M\left(f,g,x_{n_k},x_{m_{k+1}}\right)\right)} \leq \beta\left(\psi\left(M\left(f,g,x_{n_k},x_{m_{k+1}}\right)\right)\right). \tag{3.8}$$

Keeping (3.4) and taking limit as $k \to \infty$ in (3.8), we get

$$\lim_{k\to\infty} \beta\left(\psi\left(d\left(x_{n_k}, x_{m_{k+1}}\right)\right)\right) = 1.$$

Since $\beta \in \Gamma$, we have

$$\lim_{k\to\infty} \psi\left(d\left(x_{n_k}, x_{m_{k+1}}\right)\right) = 0.$$

So, $\lim_{k\to\infty} d\left(x_{n_k}, x_{m_{k+1}}\right) = 0 < \varepsilon$, which is a contraction. Similarly, for other cases, we can show that $\{x_n\}$ is a Cauchy sequence. Since X is a complete, so there exists $x^* \in X$ such that $x_n \to x^*$ implies that $x_{2i+1} \to x^*$ and $x_{2i+2} \to x^*$. By f and g are continuous, we get $gx_{2i+1} \to gx^*$ and $fx_{2i+2} \to fx^*$. Hence, $x^* = fx^*$ similarly, $x^* = gx^* = x^*$. Then (f,g) have common fixed point. □

In the following Theorem, we dropped the continuity.

Theorem 3.4. *Let (X,d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Let $f,g : X \to X$ be two mapping. Suppose that the following conditions are satisfied:*

(i) (f,g) is a pair of generalized $\alpha-\psi-$Geraghty contraction type mappings;

(ii) (f,g) is triangular $\alpha-$admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x^* \in X$ *an* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x^*) \geq 1$ *for all* k*.*

Then (f, g) *have common fixed point.*

Proof. Setting a sequence $x_{2i+1} = fx_{2i+1}$ and $x_{2i+2} = gx_{2i+1}$, for $i = 0, 1, 2, \ldots$ converges to $x^* \in X$. By the assumption (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, x^*) \geq$ for all k. By using (3.12) for all k, we get

$$\begin{aligned}\psi\left(d\left(x_{2n_{k+1}}, gx^*\right)\right) &= \psi\left(d\left(fx_{2n_k}, gx^*\right)\right) \\ &\leq \alpha\left(x_{2n_k}, x^*\right)\psi\left(d\left(fx_{2n_k}, gx^*\right)\right) \\ &\leq \beta\left(\psi\left(M\left(f, g, x_{2n_k}, x^*\right)\right)\right)\psi\left(M\left(f, g, x_{2n_k}, x^*\right)\right).\end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}\psi\left(M\left(f, g, x_{2n}, x^*\right)\right) = \psi\Big(&\max\Big\{d\left(x_{2n_k}, x^*\right), d\left(x_{2n_k}, x_{2n_k}\right), d\left(x^*, gx^*\right), \\ &\frac{d\left(x^*, fx_{2n_k}\right) + d\left(x_{2n_k}, gx^*\right)}{2}\Big\}\Big).\end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \psi\left(M\left(f, g, x_{2n_k}, x^*\right)\right) = \psi\left(d\left(x^*, gx^*\right)\right). \tag{3.10}$$

Suppose that $\psi\left(d\left(x^*, gx^*\right)\right) > 0$. From (3.10), for an enough large k, we get $\psi\left(M\left(f, g, x_{2n_k}, x^*\right)\right)$, which implies that

$$\beta\left(\psi\left(M\left(f, g, x_{2n_k}, x^*\right)\right)\right) < \psi\left(M\left(f, g, x_{2n_k}, x^*\right)\right).$$

Then, we have

$$d\left(x_{2n_k}, gx^*\right) < M\left(f, g, x_{2n_k}, x^*\right). \tag{3.11}$$

Taking limit as $k \to \infty$, we get that $d\left(x^*, gx^*\right) < d\left(x^*, gx^*\right)$, which is a contradiction. Hence, $d\left(x^*, gx^*\right) = 0$ implies that $x^* = gx^*$. Similarly, $x^* = fx^*$. Thus, $x^* = gx^* = fx^*$. □

If $M(f, g, x, y) = \max\left\{d(x, y), d(x, gx), d(y, gy), \dfrac{d(x, gy) + d(y, gx)}{2}\right\}$ and $f = g$ in Theorem 3.3 and Theorem 3.4 , we have the following corollaries.

Corollary 3.5. *Let* (X, d) *be a complete metric space and* $\alpha : X \times X \to \mathbb{R}$ *be a function. Let* $g : X \to X$ *is a mapping. Suppose that the following conditions are satisfied:*

(i) g *is a generalized* $\alpha-\psi-$*Geraghty type mappings;*

(ii) g *is triangular* $\alpha-$*admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$*;*

(iv) g *is continuous.*

Then g *has a fixed point.*

Corollary 3.6. *Let* (X, d) *be a complete metric space and* $\alpha : X \times X \to \mathbb{R}$ *be a function. Let* $g : X \to X$ *is a mapping. Suppose that the following conditions are satisfied:*

(i) g *is a generalized* $\alpha - \psi -$*Geraghty type mappings;*

(ii) g *is triangular* $\alpha -$*admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$*;*

(iv) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x^* \in X$ *an* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x^*) \geq 1$ *for all* k*.*

Then g *has a fixed point.*

Let $f, g : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. We say that (f, g) is $\alpha-$admissible mapping with respect to η if $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(fx, gy) \geq \eta(fx, gy).$$

Let (X, d) be a metric space and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Two mappings $f, g : X \to X$ is called a pair of generalized $\alpha - \eta - \psi-$Geraghty contraction type mapping if there exists $\beta \in \Gamma$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow \psi(d(fx, gy)) \leq \beta(\psi(M(f, g, x, y)))\psi(M(f, g, x, y)), \quad (3.12)$$

where

$$M(f, g, x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(y, fx) + d(x, gy)}{2}\right\}.$$

Theorem 3.7. *Let* (X, d) *be a complete metric space and* $\alpha : X \times X \to \mathbb{R}$ *be a function. Let* $f, g : X \to X$ *are* $\alpha-$*admissible mappings with respect to* η*. Suppose that the following conditions are satisfied:*

(i) (f, g) *is a pair of generalized* $\alpha - \eta - \psi-$*Geraghty type mappings;*

(ii) (f, g) *is triangular* $\alpha-$*admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$*;*

(iv) f *and* g *are continuous.*

Then (f, g) *have common fixed point.*

Proof. Let x_1 in X be such that $x_1 = fx_0$ and $x_2 = gx_1$. Continuing this process, we have

$$x_{2i+1} = fx_{2i} \text{ and } x_{2i+2} = gx_{2i+1}, \text{ for } i = 0, 1, 2, \ldots.$$

By assumption $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and a pair (f, g) is $\alpha-$admissible with respect to η, we have, $\alpha(fx_0, gx_1) \geq \eta(fx_0, gx_1)$, which we deduce that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ which implies that $\alpha(gx_1, fx_2) \geq (gx_1, fx_2)$. Continuing in this way, we get $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Then, we have that

$$\begin{aligned}
\psi\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) &= \psi\left(d\left(fx_{2i}, gx_{2i+1}\right)\right) \\
&\leq \alpha\left(x_{2i}, x_{2i+1}\right)\psi\left(d\left(fx_{2i}, gx_{2i+1}\right)\right) \\
&\leq \beta\left(\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right)\right)\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right),
\end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}
M\left(f, g, x_{2i}, x_{2i+1}\right) &= \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i}, fx_{2i}\right), d\left(x_{2i+1}, gx_{2i+1}\right), \\
&\qquad \frac{d\left(x_{2i}, fx_{2i}\right) + d\left(x_{2i}, gx_{2i+1}\right)}{2}\Big\} \\
&= \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i+1}, x_{2i+2}\right), \\
&\qquad \frac{d\left(x_{2i+1}, x_{2i+1}\right) + d\left(x_{2i}, x_{2i+2}\right)}{2}\Big\} \\
&= \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i+1}, x_{2i+2}\right), \\
&\qquad \frac{d\left(x_{2i}, x_{2i+2}\right)}{2}\Big\} \\
&\leq \max\Big\{d\left(x_{2i}, x_{2i+1}\right), d\left(x_{2i+1}, x_{2i+2}\right), \\
&\qquad \frac{d\left(x_{2i}, x_{2i+1}\right) + d\left(x_{2i+1}, x_{2i+2}\right)}{2}\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\psi\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) &\leq \beta\left(\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right)\right)\psi\left(M\left(f, g, x_{2i}, x_{2i+1}\right)\right) \\
&\leq \beta\left(\psi\left(d\left(x_{2i}, x_{2i+1}\right)\right)\right)\psi\left(d\left(x_{2i}, x_{2i+1}\right)\right) \\
&< \psi\left(d\left(x_{2i}, x_{2i+1}\right)\right).
\end{aligned}$$

That is,

$$\psi\left(d\left(x_{2i+1}, x_{2i+2}\right)\right) < \psi\left(d\left(x_{2i}, x_{2i+1}\right)\right).$$

This is the implies that

$$\psi\left(d\left(x_{n+1}, x_{n+2}\right)\right) < \psi\left(d\left(x_n, x_{n+1}\right)\right),$$

for all $n \in \mathbb{N} \cup \{0\}$. Follows the proof in the Theorem 3.3. Thus, f and g have common fixed point.

□

Theorem 3.8. *Let (X,d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Let $f,g : X \to X$ are $\alpha-$admissible mappings with respect to η. Suppose that the following conditions are satisfied:*

(i) *(f,g) is a pair of generalized $\alpha - \eta - \psi-$Geraghty type mappings;*

(ii) *(f,g) is triangular $\alpha-$admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$ an $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all k.*

Then (f,g) have common fixed point.

Proof. Follows the proof in Theorem 3.4. □

Corollary 3.9. *Let (X,d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Let $g : X \to X$ is $\alpha-$admissible mappings with respect to η. Suppose that the following conditions are satisfied:*

(i) *g is a generalized $\alpha - \psi-$Geraghty type mappings;*

(ii) *g is triangular $\alpha-$admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;*

(iv) *g is continuous.*

Then g has a fixed point.

Corollary 3.10. *Let (X,d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Let $g : X \to X$ is $\alpha-$admissible mappings with respect to η. Suppose that the following conditions are satisfied:*

(i) *g is a generalized $\alpha - \psi-$Geraghty type mappings;*

(ii) *g is triangular $\alpha-$admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$ an $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all k.*

Then g has a fixed point.

Example 3.11. *Let $X = [0,\infty)$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Let $\beta(t) = \dfrac{1}{1+2t}$, for all $t > 0$ and $\beta(0) = 0$. Then, $\beta \in \Gamma$. Let $\psi(t) = \dfrac{t}{3}$ and two mappings $f,g : X \to X$ be defined by*

$$fx = \begin{cases} \frac{1}{4}x, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases} \quad \text{and} \quad gx = x.$$

Define functions $\alpha, \eta : X \times X \to [0, \infty)$ *by*

$$\alpha(x,y) = \begin{cases} 1, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} \frac{1}{3}, & \text{if } 0 \leq x, y \leq 1 \\ 4, & \text{otherwise.} \end{cases}$$

Condition (iii) of Theorem 3.7 is satisfied with $x_0 = 1$ *since* $\alpha(1, f(1)) = \alpha(1, \frac{1}{4}) = 1 > \frac{1}{3} = \eta(1, \frac{1}{4}) = \eta(1, f(1))$. *Condition* (iv) *of Theorem 3.7 is satisfied with* $fx_n = \frac{1}{4}x_n$ *and* $gx_n = x_n$. *Obviously, condition* (ii) *is satisfied. Let* $x, y \in X$ *be such that* $\alpha(x,y) \geq 1$. *Then,* $x, y \in [0,1]$ *and so* $fx \in [0,1]$, $gy \in [0,1]$ *and* $\alpha(fx, gy) = 1$. *Hence,* (f, g) *are* $\alpha-$*admissible, and hence* (ii) *is satisfied. Finally, we shall prove that* (i) *is satisfied. Let* $\alpha(x,y) \geq \eta(x,y)$. *Thus,* $x, y \in [0,1]$. *It follows that*

$$\begin{aligned} \beta(\psi(d(x,y)))\psi(d(x,y)) - \psi(d(fx,gy)) &= \beta(d\frac{1}{3}d(x,y)) \cdot \frac{1}{3}d(x,y) - \frac{1}{3}d(fx,gy) \\ &= \beta(d\frac{1}{3}|x-y|) \cdot \frac{1}{3}|x-y| - \frac{1}{3}|fx-gy| \\ &= \frac{1}{1+\frac{1}{2}|x-y|} \cdot \frac{1}{3}|x-y| - \frac{1}{3}|\frac{1}{4}x-y| \\ &= \frac{\frac{1}{3}|x-y|}{1+\frac{1}{2}|x-y|} - \frac{1}{3}|\frac{1}{4}x-y| \\ &\geq 0. \end{aligned}$$

Then we have $\psi(d(fx,gy)) \leq \beta(\psi(d(x,y)))\psi(d(x,y))$. *Thus all assumptions of Theorem 3.7 are satisfied. Thus,* f *and* g *have common point.*

Remark 3.12. *More detailed, applications and examples see in [6] and references therein. Our results are more general than those in [2, 6, 8] and improve several results existing in literature.*

4 Conclusions

This paper presents some common fixed point theorems for a pair of $\alpha - \psi-$Geraghty contraction type. The presented theorems extend and generalize classical results in fixed point theory, in particular the very famous Banach contraction principle. The present version of these results make significant and useful contribution in the existing literature.

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References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fund. Math. 3(1922) 133-181.

[2] S. Cho, J. Bae and E. Karapinar, Fixed point theorems for $\alpha-$Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2013, 2013:329

[3] M. Geraghty, On contractive mappings, Proc. Amer. Math. Soc.40 (1973) 604-608.

[4] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi-$contractive type mappings, Nonlinear Analysis. 75(2012) 2154-2165.

[5] T. Abdeljawad, Meir-Keeler, $\alpha-$contractive fixed and common fixed point theorems, Fixed Point Theory Appl. 2013 doi:10.1186/1687-1812-2013-19.

[6] E. Karapinar, $\alpha - \psi-$Geraghty contraction type mappings and some related fixed point results. Filomat. 28(1)(2014) 3748.

[7] M. Arshad, A. Hussain, A. Azam, Fixed point of $\alpha-$Geraghty contraction with applications, U.P.B. Sci. Bull., Series A, 78 Iss. 2 (2016) 1223-7027.

[8] P. Salimi, A. Latif and N. Hussain, Modified $\alpha - \psi-$contractive mappings with applications, Fixed Point Theory Appl.,151 (2013) 2013.

[9] P. Chaipunya, Y. J. Cho, P. Kumam, Geraghty-type theorems in modular metric spaces with an application to partial diferential equation, Adv. Diference Equ., 2012 (2012).

[10] E, Karapinar, P. Kumam, P. Salimi, On $\alpha - \psi-$Meir$-$Keeler contractive mappings, Fixed Point Theory and Applications 2013, 2013:94.

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