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The Laplace Transform Dual Reciprocity Method for Linear Wave Equations

P. Satravaha and S. Rujivan

Abstract : The Laplace Transform Dual Reciprocity Method (LTDRM) is extended to solve linear wave equations. The time dependence of the problem is removed temporarily from the equations by the Laplace transform. The transformed equation which is now of an elliptic type can be solved in the Laplace space using the dual reciprocity method. Stehfest's algorithm is then used to retrieve numerical solutions to time domain. The efficiency of the LTDRM is obvious, especially when the solutions at large time are required, due to an allowance of unlimited time-step size to be used. Several examples are presented to demonstrate the acuracy of the method by comparing the results with those obtained from the coupled finite difference - dual reciprocity method and exact solutions.

Keywords : Wave equation; Laplace transform; Dual reciprocity method.
2000 Mathematics Subject Classification : 65R10, 65R20.

1 Introduction

The linear wave equations (LWEs) play a significant role in engineering and applied science. Many problems that occur in engineering pratice and applied science such as vibrations of a membrane [7], propagation of acoustic wave [11], and propagation of electromagnetic wave [4], etc., can be modeled by this type of equations. Efficiently and accurately solving LWEs is a usual task faced by scientists and engineers.

From the starting point of the dual reciprocity method (DRM) in 1983 when Brebbia and Nardini [6] proposed the way of changing domain integral in the BEM (boundary element method) analysis into boundary integrals, DRM now becomes a well-established method for numerically solving many kinds of problems, e.g., solid mechanics [8], elastodynamic problems [1], biharmonic problems [5], etc. On the other hand, the Laplace transform is one of the most classical tool used for solving problems governed by ordinary differential equations or partial differential equations (PDEs), especially for solving transient problems such as heat and wave problems which involve at least one temporal derivative term. In 1994, Zhu, Satravaha and Lu [12] combined DRM with the Laplace transform into the so-called Laplace transform dual reciprocity method (LTDRM) and proposed to solve transient diffusion problems. Since then, the LTDRM has been successfully extended and applied to solve various problems [10, 2].

In this paper, the LTDRM is extended and applied to solve one class of hyperbolic PDEs, i.e. the linear wave equations. This method employs the advantage of the Laplace transform to temporarily remove time dependence of the problem, thus transforming the LWE into a Poisson equation which is of an elliptic type PDE. The DRM is then used to solve the transformed equation, and the solutions in the original time domain are obtained through the utilisation of the Stehfest's numerical inversion of the Laplace transform. In this way, the required solutions can be calculated in one jump no matter how small or large the observation times are. The detail formulation of the LTDRM is given in the next section. Numerical examples are provided in section 3, and discussion and concluding remarks are given in the last section.

2 Method Formulation

The LTDRM will be outlined in this section for the linear wave equation

$$\nabla^2 u(\mathbf{x},t) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}u(\mathbf{x},t), \quad \mathbf{x}\in\Omega,\, t>0 \tag{2.1}$$

where $u(\mathbf{x},t)$ is the unknown function of spatial point $\mathbf{x}=(x,y)$ in a bounded domain Ω with an enclosing boundary Γ at time t, ∇^2 is the two-dimensional Laplace operator, and a non-zero constant c represents the velocity of wave propagation.

By the theory for linear partial differential equations, the LWE is well-posed if it equips with two types of conditions. The first type is initial conditions, also known as Cauchy conditions, which specify values of the unknown function u and its first time-derivative at the initial point, i.e.,

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{2.2}$$

and

$$\frac{\partial u}{\partial t}(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{2.3}$$

where u_0 and v_0 are known functions.

The second type is boundary conditions which fall into the following three categories:

- Dirichlet conditions : values of the unknown function u are prescribed at each point on the boundary Γ_1 as

$$u(\mathbf{x},t) = \overline{u}(\mathbf{x},t), \quad \mathbf{x}\in\Gamma_1,\, t>0 \tag{2.4}$$

- Neumann conditions : values of the normal derivative of the unknown function u are prescribed at each point on the boundary Γ_2 as

$$\frac{\partial u}{\partial n}(\mathbf{x},t) = \overline{q}(\mathbf{x},t), \quad \mathbf{x}\in\Gamma_2,\, t>0 \tag{2.5}$$

- Robin conditions : values of a linear combination of the unknown function u and its normal derivative are prescribed at each point on the boundary Γ_3 as

$$u(\mathbf{x},t) + \lambda \frac{\partial u}{\partial n}(\mathbf{x},t) = \overline{r}(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_3,\ t > 0 \tag{2.6}$$

where Γ_i, $i = 1,2,3$ are complementary segments of Γ, n is the unit outward normal vector on the boundary Γ, λ is a non-zero constant, and $\overline{u}$, $\overline{q}$, $\overline{r}$ are known functions.

Usually in the vibrations of a membrane problem, the edge of a membrane is fixed. Thus the only boundary condition imposed is the Dirichlet boundary condition and it reads $u \equiv 0$ on the boundary for all t. However, sometimes the boundary (or part of it) is left "free" meaning that it can move in the vertical direction and there is no external transverse force acting on it. This is equivalent to the boundary condition $\frac{\partial u}{\partial n} \equiv 0$ on the boundary for all t. Moreover, an intermediate case is also possible; the boundary may be elastically supported and capable of producing a transverse force proportional to the displacement. This situation is equivalent to the boundary condition $u + \lambda \frac{\partial u}{\partial n} \equiv 0$ on the boundary for all t.

Let the Laplace transform of $u(\mathbf{x},t)$ be symbolised as $U(\mathbf{x},p)$ and defined by

$$U(\mathbf{x},p) = \int_0^\infty e^{-pt} u(\mathbf{x},t)\, dt \tag{2.7}$$

Applying Laplace transform to Equations (2.1) and (2.4) - (2.6) gives

$$\nabla^2 U(\mathbf{x},p) = \frac{1}{c^2}\{p^2 U(\mathbf{x},p) - p u_0(\mathbf{x}) - v_0(\mathbf{x})\} \tag{2.8}$$

with transformed boundary conditons

$$U(\mathbf{x},p) = \overline{U}(\mathbf{x},p), \quad \mathbf{x} \in \Gamma_1 \tag{2.9}$$

$$\frac{\partial U}{\partial n}(\mathbf{x},p) = \overline{Q}(\mathbf{x},p), \quad \mathbf{x} \in \Gamma_2 \tag{2.10}$$

$$U(\mathbf{x},p) + \lambda \frac{\partial U}{\partial n}(\mathbf{x},p) = \overline{R}(\mathbf{x},p), \quad \mathbf{x} \in \Gamma_3 \tag{2.11}$$

Equation (2.8) is a Poisson equation and solving it with the traditional BEM [3] leads to an integral equation that contains domain itegral involving initial conditions. Such an obstacle can be overcome by utilising the DRM to convert this domain integral term into equivalent boundary integrals. Zhu et al. [12] proposed using the DRM based on the known fundamental solution to the Laplace operator with the Laplace transformed diffusion equation. Then, considering Equation (2.8), this means that the DRM will be used to convert the right-hand side to equivalent boundary integrals. Thus the required DRM approximation is

$$\frac{1}{c^2}\left\{p^2 U(\mathbf{x},p) - p u_0(\mathbf{x}) - v_0(\mathbf{x})\right\} = \sum_{j=1}^{N+L} f_j(\mathbf{x}) \alpha_j \tag{2.12}$$

where f_j's are interpolation functions and α_j's are the coefficients to be determined by the collocation method with N boundary collocation points and L internal collocation points. After applying Equation (2.12) to all collocation points, the matrix form of this equation is obtained as

$$\vec{\alpha} = \frac{1}{c^2}\mathbf{F}^{-1}\left\{p^2\mathbf{U} - p\mathbf{u}_0 - \mathbf{v}_0\right\} \tag{2.13}$$

The standard DRM can now be applied to Equation (2.8), giving

$$c_l U_l - \sum_{k=1}^{N} Q_k g_{lk} + \sum_{k=1}^{N} U_k h_{lk} = \sum_{j=1}^{N+L} \alpha_j \left\{ c_l(\hat{u}_j)_l - \sum_{k=1}^{N} (\hat{q}_j)_k g_{lk} + \sum_{k=1}^{N} (\hat{u}_j)_k h_{lk} \right\}, \tag{2.14}$$

which is valid at all the point $\mathbf{x}_l$, acting as the source point of the fundamental solution of the Laplace operator, that can be any point inside or on the boundary of the domain Ω. After applying this equation to all collocation points, one has a matrix system of the form

$$\mathbf{HU} - \mathbf{GQ} = \left(\mathbf{H}\widehat{\mathbf{U}} - \mathbf{G}\widehat{\mathbf{Q}}\right)\vec{\alpha} \tag{2.15}$$

It should be noted that the entries in matrices $\mathbf{H}$ and $\mathbf{G}$ depend on boundary elements being used. Substituting the expression from Equation (2.13) into Equation (2.15) gives the LTDRM formulation for linear wave problems in the Laplace space as

$$\left(\mathbf{H} - \frac{p^2}{c^2}\mathbf{S}\right)\mathbf{U} - \mathbf{GQ} = -\frac{1}{c^2}\mathbf{S}(p\mathbf{u}_0 - \mathbf{v}_0) \tag{2.16}$$

which can be reduced to a square system by applying transformed boundary conditions.

Once the values of the unknown functions are found in the Laplace space, the Stehfest's method is used to retrieve the values of the unknown functions in the original time domain. According to Stehfest [9], if $F(p)$ is the known Laplace transform of $f(t)$, $f(t)$ is approximated by

$$f(t) \approx \frac{\ln 2}{t}\sum_{\nu=1}^{N_p} W_\nu F(p_\nu) \tag{2.17}$$

where N_p is an even integer and

$$p_\nu = \frac{\ln 2}{t}\,\nu \tag{2.18}$$

and the weight W_ν is defined as

$$W_\nu = (-1)^{\nu+\sigma} \sum_{k=[(\nu+1)/2]}^{\min\{\nu,\sigma\}} \frac{k^\sigma\,(2k)!}{(\sigma-k)!\,k!\,(k-1)!\,(\nu-k)!\,(2k-\nu)!} \tag{2.19}$$

for $\sigma = N_p/2$ and $[r]$ denoting the integral part of the real number r. For the value of N_p, its optimal value depends on the arithmetic precision using in the calculation and the accuracy in the evaluation of $F(p)$. Stehfest suggested N_p to be 10 for single precision variables, and 18 for double precision variables. However, Zhu et al. [12] reported that accurate solutions can be obtained for N_p as small as 6 and this number will then be used herein.

At this stage one can see that in order to obtain values of the unknowns on the boundary at specific time one needs to solve Equation (2.16) 6 times to get 6 values for each of the unknowns, which takes up the main bulk of the whole computation time. Fortunately, the matrices $\mathbf{H}$, $\mathbf{G}$, $\mathbf{S}$ and vectors $\mathbf{u}_0$, $\mathbf{v}_0$ are independent of the Laplace parameter. Therefore, they have to be constructed only once and stored for subsequent uses. After 6 values of the unknowns are determined, Stefest's method is then used to numerically invert these values to obtain a solution at each nodal point in the time domain. If the value of the unknown at an internal point is required, all the 6 values of the unknowns on the boundary in the Laplace space can be used in conjunction with Equation (2.14) to get 6 values of the unknown at an internal point in the Laplace space and these values are inverted numerically to obtain the required value.

The LTDRM can be easily extended to equations of the form

$$\nabla^2 u(\mathbf{x},t) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} u(\mathbf{x},t) + b\left(\mathbf{x}, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad \mathbf{x} \in \Omega,\ t > 0 \tag{2.20}$$

and the function b is in the form

$$b\left(\mathbf{x}, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = z(\mathbf{x},t) + \beta_1 u + \beta_2 \frac{\partial u}{\partial t} + \beta_3 \frac{\partial u}{\partial x} + \beta_4 \frac{\partial u}{\partial y} \tag{2.21}$$

where z is a known function and $\beta_i, i = 1,2,3,4$ are constants.

For the case $b = z + \beta_1 u$, we obtain a Poisson equation of Equation (2.20) in the Laplace space as

$$\nabla^2 U(\mathbf{x},p) = \frac{1}{c^2}\left\{p^2 U(\mathbf{x},p) - p u_0(\mathbf{x}) - v_0(\mathbf{x})\right\} + Z(\mathbf{x},p) + \beta_1 U(\mathbf{x},p) \tag{2.22}$$

and the LTDRM formulation in this case is

$$\left[\mathbf{H} - \left(\frac{p^2}{c^2} + \beta_1\right)\mathbf{S}\right]\mathbf{U} - \mathbf{GQ} = -\frac{1}{c^2}\mathbf{S}(p\mathbf{u}_0 + \mathbf{v}_0 - \mathbf{Z}) \tag{2.23}$$

For the case $b = \beta_2 \partial u/\partial t$, we get a Poisson equation of Equation (2.20) in the Laplace space as

$$\nabla^2 U(\mathbf{x},p) = \frac{1}{c^2}\{p^2 U(\mathbf{x},p) - p u_0(\mathbf{x}) - v_0(\mathbf{x})\} + \beta_2\{pU(\mathbf{x},p) - u_0(\mathbf{x})\} \tag{2.24}$$

and the LTDRM formulation in this case is

$$\left[\mathbf{H} - \left(\frac{p^2}{c^2} + \beta_2 p\right)\mathbf{S}\right]\mathbf{U} - \mathbf{GQ} = -\mathbf{S}\left[\left(\frac{p}{c^2} + \beta_2\right)\mathbf{u}_0 + \frac{1}{c^2}\mathbf{v}_0\right] \tag{2.25}$$

For the case $b = \beta_3 \partial u/\partial x$ or $b = \beta_4 \partial u/\partial y$, we have a Poisson equation of Equation (2.20) in the Laplace space as

$$\nabla^2 U(\mathbf{x}, p) = \frac{1}{c^2}\left\{p^2 U(\mathbf{x}, p) - p u_0(\mathbf{x}) - v_0(\mathbf{x})\right\} + \beta_3 \frac{\partial U}{\partial x}(\mathbf{x}, p) \tag{2.26}$$

and the LTDRM formulation in this case is

$$\left[\mathbf{H} - \frac{p^2}{c^2}\mathbf{S}\right]\mathbf{U} - \mathbf{G}\mathbf{Q} = -\frac{1}{c^2}\mathbf{S}(p\mathbf{u}_0 + \mathbf{v}_0) + \beta_3 \mathbf{S}\frac{\partial \mathbf{U}}{\partial x} \tag{2.27}$$

To make Equation (2.27) in the form we can solve, we have to relate $\partial \mathbf{U}/\partial x$ to $\mathbf{U}$ via

$$\frac{\partial \mathbf{U}}{\partial x} = \frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1}\mathbf{U} \tag{2.28}$$

Therefore, Equation (2.27) becomes

$$\left[\mathbf{H} - \frac{p^2}{c^2}\mathbf{S} - \beta_3 \mathbf{R}\right]\mathbf{U} - \mathbf{G}\mathbf{Q} = -\frac{1}{c^2}\mathbf{S}(p\mathbf{u}_0 + \mathbf{v}_0) \tag{2.29}$$

where

$$\mathbf{R} = \mathbf{S}\frac{\partial \mathbf{F}}{\partial x}\mathbf{F}^{-1} \tag{2.30}$$

For comparison, the couple finite difference - dual reciprocity method (FDDRM) is considered. This method discretises the time domain in a finite difference manner. At the particular time t_m, the time derivative can be approximated using the centered finite difference scheme as

$$\frac{\partial^2 u}{\partial t^2}(\mathbf{x}, t_m) \approx \frac{u(\mathbf{x}, t_{m+1}) - 2u(\mathbf{x}, t_m) + u(\mathbf{x}, t_{m-1})}{(\Delta t)^2} \tag{2.31}$$

where Δt is a time-step size. This allows the LWE at this particular time to be approximated as

$$\nabla^2 u(\mathbf{x}, t_m) \approx \frac{u(\mathbf{x}, t_{m+1}) - 2u(\mathbf{x}, t_m) + u(\mathbf{x}, t_{m-1})}{(\Delta t)^2} \tag{2.32}$$

After applying the standard DRM to Equation (2.32) we arrive at the matrix equation

$$\mathbf{H}\mathbf{u}_m - \mathbf{G}\mathbf{q}_m = \left(\mathbf{H}\widehat{\mathbf{U}} - \mathbf{G}\widehat{\mathbf{Q}}\right)\vec{\alpha}_m \tag{2.33}$$

where

$$\vec{\alpha}_m = \mathbf{F}^{-1}\frac{1}{(c\Delta t)^2}\left(\mathbf{u}_{m+1} - 2\mathbf{u}_m + \mathbf{u}_{m-1}\right) \tag{2.34}$$

Rearranging terms we finally have the FDDRM formulation for the LWP as

$$\omega \mathbf{S}\mathbf{u}_{m+1} + \mathbf{G}\mathbf{q}_m = (\mathbf{H} + 2\omega \mathbf{S})\,\mathbf{u}_m - \omega \mathbf{S}\mathbf{u}_{m-1} \tag{2.35}$$

where $\omega = 1/(c\Delta t)^2$ and initial conditions $\mathbf{u}_{-1}$ and $\mathbf{u}_0$ can be found from Equations (2.2) and (2.3).

The FDDRM can also be easily extended to a more general Equation (2.20) in a similar fashion as the LTDRM.

3 Numerical Examples

In this section, several examples representing various kinds of vibrations of membrane problems are presented to illustrate the efficiency and the accuracy of the LTDRM. Constant boundary elements together with linear radial basis functions as interpolation functions are used in both the LTDRM and the FDDRM. In order to measure the accuracy of the obtained numerical solutions, an average relative error at time t denoted by $E_{av}(t)$ will be used and is defined by

$$E_{av}(t) = \frac{1}{N_s} \sum_{k=1}^{N_s} \frac{|u_e(\mathbf{x}_k, t) - u_a(\mathbf{x}_k, t)|}{|u_e(\mathbf{x}_k, t)|} \times 100 \tag{3.1}$$

where $\mathbf{x}_k$'s are sample points, N_s is the number of sample points, and u_e and u_a are exact and approximated values of u, respectively. For all the examples presented, $N_s = N + L$.

Example 1. In this example, the boundary of a membrane is rectangular such as $\Omega = [0, a] \times [0, b]$ and it is fixed, i.e. $u \equiv 0$ on Γ. The analytical solution can be obtained using the method of separation of variables as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t)\} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{3.2}$$

where

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b u_0(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dy\, dx, \tag{3.3}$$

$$B_{mn} = \frac{4}{ab\omega_{mn}} \int_0^a \int_0^b v_0(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dy\, dx, \tag{3.4}$$

and

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{3.5}$$

For this example, we let $a = b = 1$ and two initial conditions are

$$u_0(x, y) = x(1 - x)y(1 - y) \tag{3.6}$$

and

$$v_0(x, y) = 0 \tag{3.7}$$

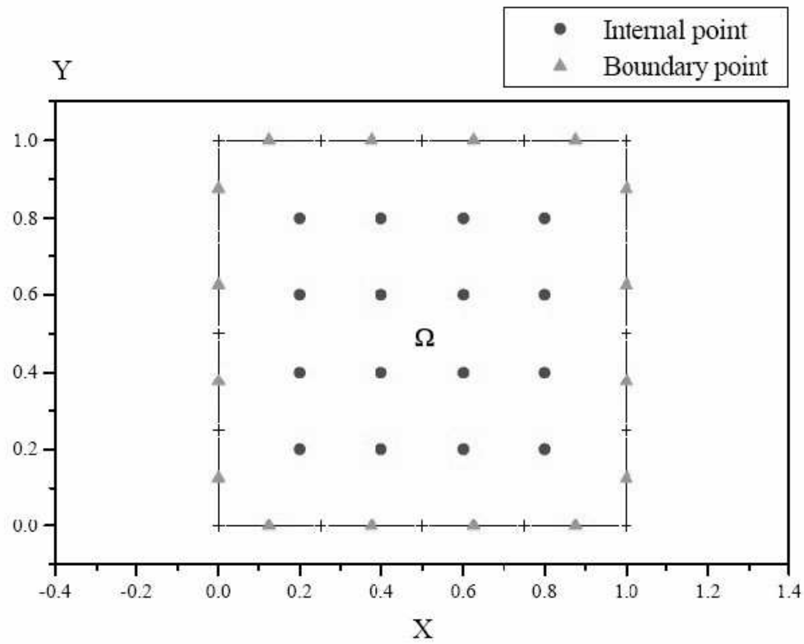

Figure 1: The domain Ω and all collocation points in Example 1.

In applying the LTDRM and the FDDRM to this problem, the boundary Γ is discretised into 20 equal-size constant elements and 16 internal points are used as shown in Figure 1. Average relative errors $E_{av}(t)$ obtained using LTDRM and FDDRM to solve this problem for $c = 10^{-4}$ and $c = 10^{-5}$ are illustrated in Figures 2 and 3. It can be seen that numerical solutions obtained from both methods when $c = 10^{-5}$ are more accurate than the ones obtained when $c = 10^{-4}$. In fact, our experiments have shown that they are in very good agreement with analytical solution for small observation times as well as large observation times when $c \leq 10^{-5}$. On the other hand, they are often accurate only at small observation times when $c > 10^{-5}$.

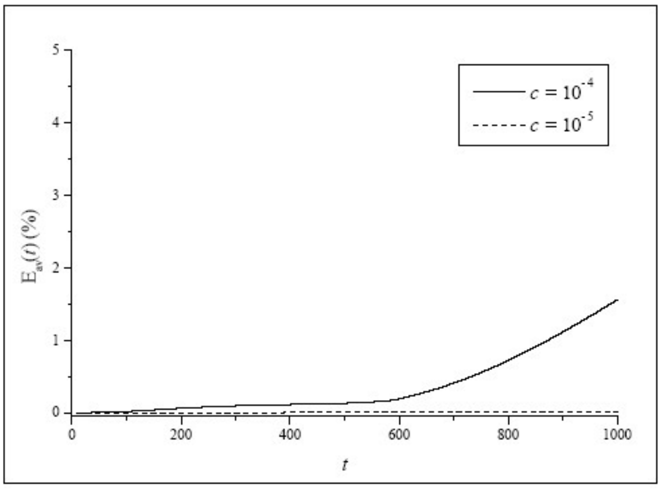

Figure 2: Average relative errors of the LTDRM.

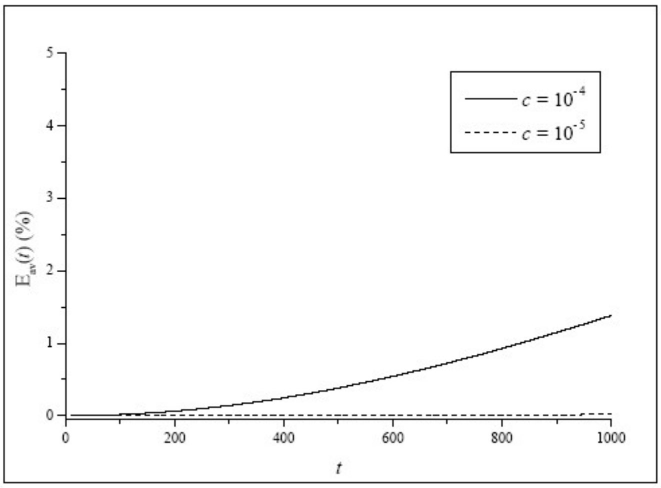

Figure 3: Average relative errors of the FDDRM.

Example 2. The vibration of a membrane problem without source term with $\Omega = [0,1]\times[0,1]$ is solved in this example. Unlike Example 1, two initial conditions are

$$u_0(x,y) = 0 \tag{3.8}$$

and

$$v_0(x,y) = x \tag{3.9}$$

and the free boundary condition $\partial u/\partial n \equiv 0$ on Γ is imposed. To solve this problem, all collocation points in Example 1 are used. The analytical solution, which is obtained via the method of separation of variables, is of the form

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\cos(n\pi x)\cos(m\pi y)h_{mn}(t) \tag{3.10}$$

where

$$h_{mn}(t) = \begin{cases} t, & m=0, n=0 \\ \sin(\omega_{mn}t), & \text{otherwise} \end{cases} \tag{3.11}$$

in which ω_{mn} is defined in Equation (3.5) and

$$A_{mn}h'_{mn}(0) = \frac{\int_0^1\int_0^1 x\cos(n\pi x)\cos(m\pi y)\,dy\,dx}{\int_0^1\int_0^1 \cos^2(n\pi x)\cos^2(m\pi y)\,dy\,dx} \tag{3.12}$$

Figures 4 and 5 show average relative errors of the LTDRM and the FDDRM for the case $c = 10^{-4}$. It can be seen that results obtained from the LTDRM agree well with those obtained from the FDDRM for the case $c = 10^{-4}$. Our experiments have also shown that both methods are very accurate when $c \leq 10^{-4}$ and start to deteriorate when $c > 10^{-4}$.

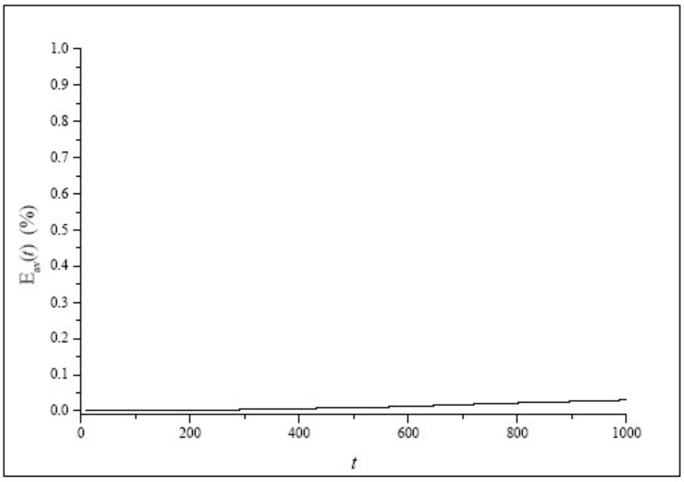

Figure 4: Average relative errors of the LTDRM with $c = 10^{-4}$.

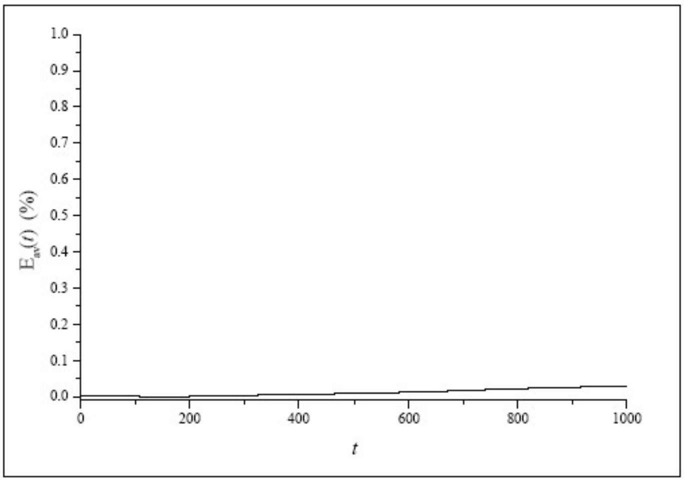

Figure 5: Average relative errors of the FDDRM with $c = 10^{-4}$.

Example 3. In this example, we investigate the vibration of a circular membrane. The governing equation of this problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{3.13}$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplace operator in polar coordinates. The domain in this case is a circular disk $\Omega = \{(r,\theta)\,|\,0 < r \leq r_0, -\pi < \theta \leq \pi\}$ for some $r_0 > 0$. If the boundary of the membrane is fixed, i.e. $u \equiv 0$ on Γ, and initial conditions $u_0(r,\theta) = (r_0^2 - r^2)\sin(\theta)$ and $v_0(r,\theta) \equiv 0$ are prescribed, the analytical solution is expressed as

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left\{A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right\}\cos(\lambda_{mn} ct) \tag{3.14}$$

where J_m is the Bessel function of the first kind of order m, $\lambda_{mn} r_0$ is the nth root of J_m and

$$A_{mn} = \frac{2}{\pi r_0^2 J_{m+1}^2(\lambda_{mn} r_0)} \int_0^{r_0}\int_{-\pi}^{\pi} r J_m(\lambda_{mn} r) u_0(r,\theta)\cos(m\theta)\, d\theta\, dr \tag{3.15}$$

and

$$B_{mn} = \frac{2}{\pi r_0^2 J_{m+1}^2(\lambda_{mn} r_0)} \int_0^{r_0}\int_{-\pi}^{\pi} r J_m(\lambda_{mn} r) u_0(r,\theta)\sin(m\theta)\, d\theta\, dr \tag{3.16}$$

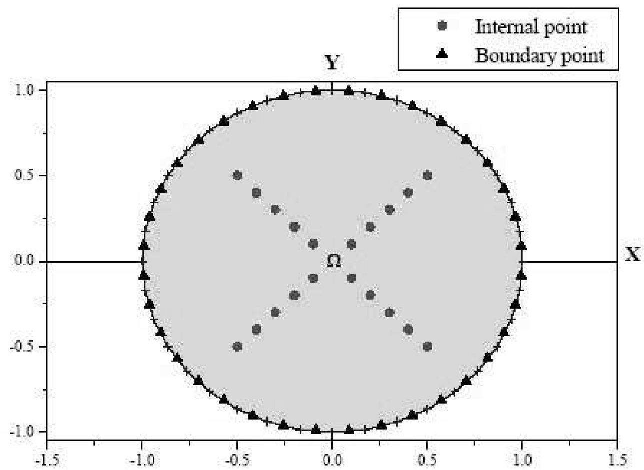

Figure 6: The domain Ω and 56 collocation points in Example 3.

To solve this problem, 56 collocation points as shown in Figure 6 are chosen. Average relative errors of the LTDRM and the FDDRM for the case $c = 10^{-4}$ and $c = 10^{-5}$ with $r_0 = 1$ are illustrated in Figures 7 and 8. It can be noticed for both methods that E_{av} varies with time when $c = 10^{-4}$ while it is small (less than 1 %) and stable over a long time period when $c = 10^{-5}$. In fact, E_{av} of both method are small when $c \leq 10^{-5}$. Again, the results from both methods agree well with each other.

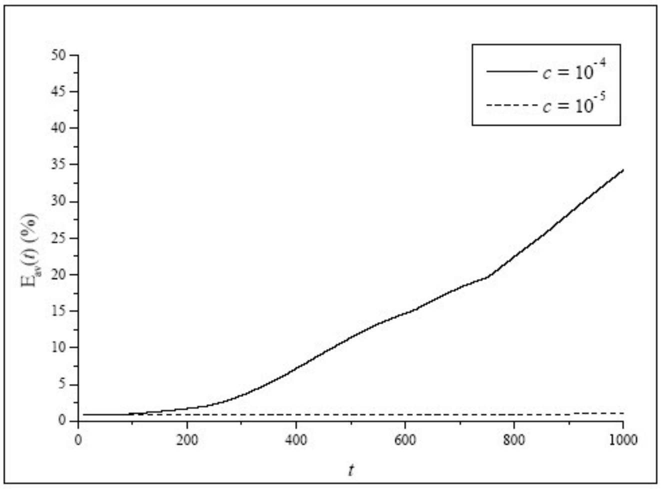

Figure 7: Average relative errors of the LTDRM.

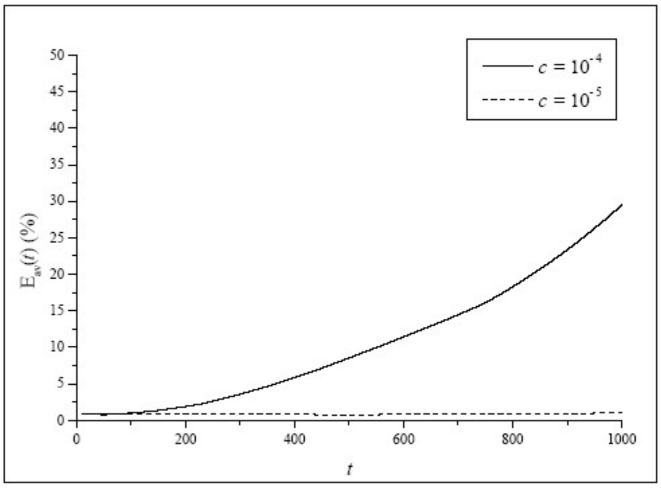

Figure 8: Average relative errors of the FDDRM.

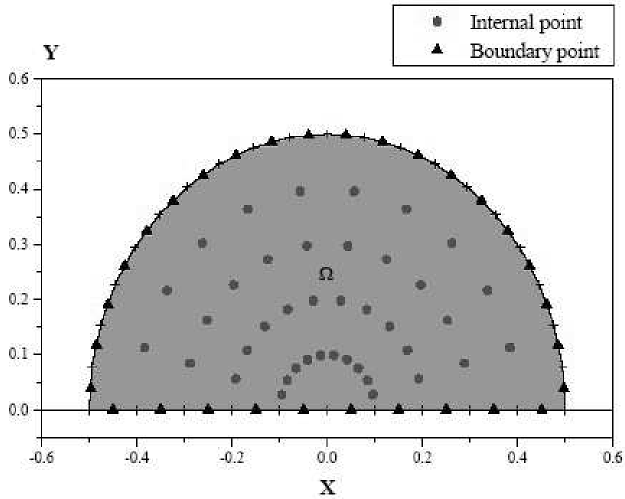

Figure 9: The domain Ω and all collocation points in Example 4.

Example 4. Consider a more general form of the LWE

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} u + (-2 + cx^2)\cos(ct) + (2 + cy^2)\sin(ct) + u - \frac{\partial u}{\partial t} \tag{3.17}$$

defined on a half circular domain Ω (see Figure 9), with two initial conditions

$$u_0(x, y) = y^2 \tag{3.18}$$

and

$$v_0(x, y) = cx^2 \tag{3.19}$$

and the Dirichlet boundary condition

$$u(x, y, t) = x^2 \sin(ct) - y^2 \cos(ct) \tag{3.20}$$

70 collocation points as shown in Figure 9 are used in the LTDRM and the FDDRM procedures. Average relative errors from both methods for the case $c = 10^{-5}$ and $c = 10^{-6}$ are shown in Figures 10 and 11. It can be seen again that E_{av} reduces as c decreases.

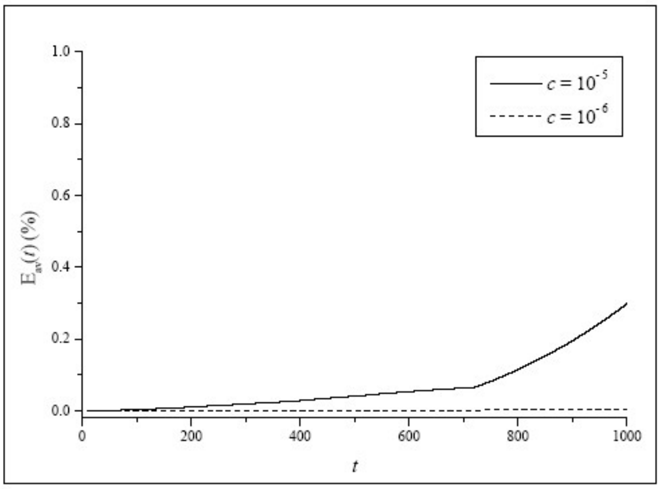

Figure 10: Average relative errors of the LTDRM.

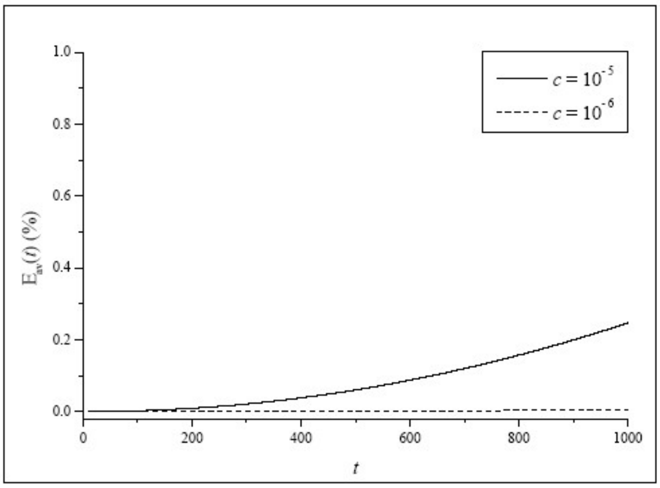

Figure 11: Average relative errors of the FDDRM.

Example 5. As for the last example, we consider the vibration of a membrane problem with a source term of the form

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}u + z - 10u - \frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \tag{3.21}$$

where

$$z = -e^{-\sin(ct)}\left\{\cos(x^2+y^2)\left(-10+4x^2+4y^2+c\cos(ct)+\cos^2(ct)+\sin(ct)\right)\right.$$
$$\left. + 2(2+3x-y)\sin(x^2+y^2)\right\} \tag{3.22}$$

The Neuman boundary conditions are imposed on Γ_1 and Γ_3. On Γ_2 the Dirichlet boundary condition is prescribed. The exact solution to this problem is

$$u(x,y,t) = \cos(x^2+y^2)e^{-\sin(ct)} \tag{3.23}$$

The domain Ω for this problem together with 50 collocation points used in the numerical procedures is shown in Figure 12. Average relative errors for the case $c = 10^{-5}$ and $c = 10^{-6}$ given in Figures 13 and 14 show that the LTDRM gives accurate results which agree well with the FDDRM and is stable for a long time period when $c \leq 10^{-6}$. However, the method deteriorates when c gets bigger.

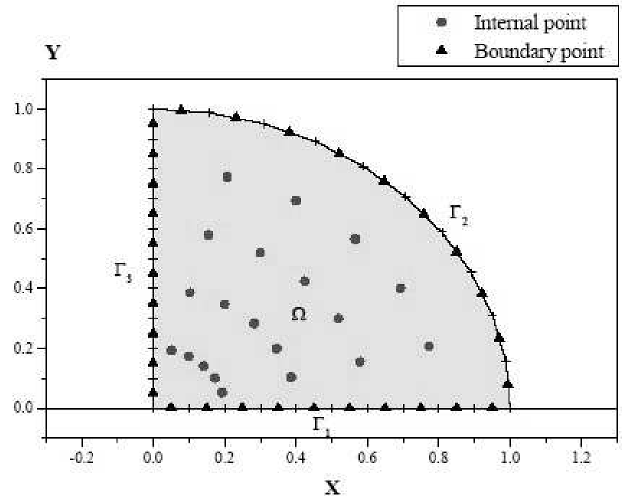

Figure 12: The domain Ω and all collocation points in Example 5.

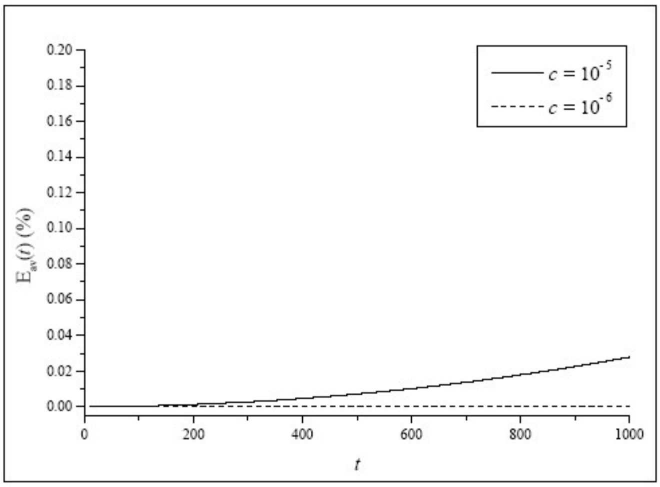

Figure 13: Average relative errors of the LTDRM.

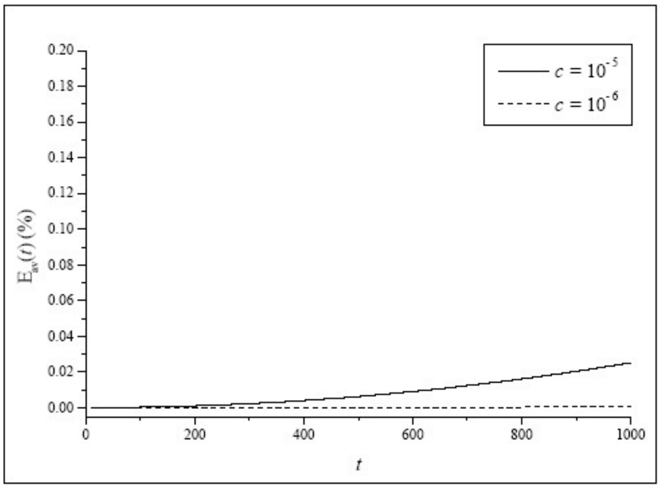

Figure 14: Average relative errors of the FDDRM.

4 Discussions and Concluding Remarks

The Laplace transform dual reciprocity method is devised to numerically solve linear wave equations and is extended to a more general form of LWEs. The highlights of this method are the transformation of an LWE into a Poisson equation using Laplace transform technique and the utilisation of the DRM to solve this transformed equation. This allows a time-free and boundary-only integral equation to be obtained. Consequently, the dimension of the problem is virtually reduced by two. Although a system of linear equations need to be solved several times in the Laplace space when the solutions at a specific time are required, the matrices involved in the calculation have to be constructed only once and stored for later uses. In this way, solutions at any observation time can be calculated swiftly in one jump no matter how small or large the observation times are. The efficiency of the LTDRM is therefore obvious and it becomes prominent when solutions at a large observation time are required, comparing to the FDDRM which uses a step-by-step calculation in the time domain.

Several examples are included to demonstrate the accuracy of the LTDRM. It was shown that numerical solutions obtained from the LTDRM are in very good agreement with the corresponding analytical solutions and numerical solutions obtained from the FDDRM, for small as well as large observation times with the same accuracy when $c \leq c_0$ for some $c_0 > 0$. In most cases, c_0 is approximately 10^{-5} even though there might be some problems that c_0 can be bigger, such as 1.

The numerical procedures described in this paper use only constant boundary elements and linear radial basis functions as interpolation functions. The accuracy of solutions in the Laplace space can be improved through the use of higher order elements and higher order radial basis functions, instead of increasing the number of constant boundary elements. This could stretch the limitation on the value of c_0 to be larger than presently is. Moreover, it is possible to extend the LTDRM to solve nonlinear wave equations which is left for future works.

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P. Satravaha and S. Rujivan
Department of Mathematics
Chulalongkorn University
Bangkok 10330, Thailand.
e-mail : Pornchai.S@chula.ac.th