



The resolvent operator techniques with perturbations for finding zeros of maximal monotone operator and fixed point problems in Hilbert spaces

Pongsakorn Sunthrayuth[†] and Poom Kumam^{‡1}

[†] Division of mathematics, Department of Mathematics and Computer Science,
Faculty of Science and Technology,
Rajamangala University of Technology Thanyaburi, Pathumthani, Thailand
e-mail : pongsakorn_su@rmutt.ac.th

[‡] Department of Mathematics, Faculty of Science,
King Mongkut's University of Technology Thonburi, Bangkok, Thailand
e-mail : poom.kum@kmutt.ac.th

Abstract : In this paper, we introduce iterative schemes with perturbations for finding zeros of the sum of two monotone operators and a fixed point problem of a nonexpansive mapping in Hilbert spaces. We prove a strong convergence theorem of the proposed iterative schemes under some certain conditions. Furthermore, we also apply our results to solving the variational inequality and equilibrium problems. The results obtained in this paper are improve and generalize many known recent results in this field.

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1 Introduction

¹Corresponding author email: poom.kum@kmutt.ac.th

Let C be a nonempty, closed and convex subset of a real Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$ respectively. Let $\{x_n\}$ be a sequence in C , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$) denotes strong (weak) convergence of the sequence $\{x_n\}$ to x . A mapping $S : C \rightarrow C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If $0 < L < 1$, then S is a contraction and if $L = 1$, then S is nonexpansive. We denote the fixed points set of the mapping S by $F(S)$, i.e., $F(S) = \{x \in C : x = Sx\}$.

Let $A : H \rightarrow 2^H$ be a set-valued mapping. We denote $D(A)$ by domain of A , that is, $D(A) = \{x \in H : Ax \neq \emptyset\}$.

Definition 1.1. A set-valued mapping $A : D(A) \subset H \rightarrow 2^H$ is said to be monotone if for all $x, y \in D(A)$ such that

$$\langle u - v, x - y \rangle \geq 0 \quad \text{for } u \in Ax \text{ and } v \in Ay.$$

Definition 1.2. A monotone operator $A : D(A) \subset H \rightarrow 2^H$ is said to be maximal if its graph is not strictly contained in the graph of any other monotone operator on H .

Definition 1.3. ([28]) Let $A : D(A) \subset H \rightarrow 2^H$ be a maximal monotone. The resolvent operator of A , denoted by $J_\lambda^A : H \rightarrow D(A)$ which is defined by

$$J_\lambda^A = (I + \lambda A)^{-1},$$

where λ is any positive number and also denote $A^{-1}0$ by the set of zeros of A , that is, $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$.

For the resolvent J_λ^A , $\lambda > 0$ the following facts are well known (see [25]).

- (1) J_λ^A is a single-valued nonexpansive mapping;
- (2) $F(J_\lambda^A) = A^{-1}0$.

In this paper, we consider the following so-called *variational inclusion problem*: Find $x \in H$ of the sum of two monotone operators A and B such that

$$0 \in Ax + Bx, \tag{1.1}$$

where $A : C \rightarrow H$ is a single-valued mapping, $B : H \rightarrow 2^H$ is a set-valued mapping and 0 is a zero vector in H . The set of solutions of (1.1) is denoted by $(A + B)^{-1}0$. It is well known that the problem (1.1) has wide applications in the fields of economics, structural analysis, mechanics, optimization problems, signal processing, image recovery and applied sciences (see, e.g., [17, 18, 19, 20, 21, 22, 23, 24] and the references therein).

Now, we consider two special cases of the problem (1.1).

- (1) If $H = \mathbb{R}^n$ then the problem (1.1) reduces to the *generalized equation*, which is introduced by Robinson [1].
- (2) If $A = 0$ then the problem (1.1) reduces to the *inclusion problem*, which is introduced by Rockafella [2, 3].

In recent years, many authors have constructed the several iterative methods for solving variational inclusion in several settings (see, e.g., [4, 5, 15, 26, 33, 8, 9, 10, 11, 12, 34, 14, 15, 16] and the references therein).

In 2011, Manaka and Takahashi [26] introduced modified Mann's iteration for solving the problem (1.1) in a real Hilbert space H as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S J_{\lambda_n}^B (x_n - \lambda_n A x_n), \quad \forall n \geq 1, \quad (1.2)$$

where S is a nonexpansive mapping on C , $A : C \rightarrow H$ is an α -inverse strongly monotone mapping, B is a maximal monotone operator on H such that the domain of B is included in C and $J_{\lambda}^B = (I + \lambda B)^{-1}$ is a resolvent of B for all $\lambda > 0$. They proved that the sequence $\{x_n\}$ defined by (1.2) converges weakly to point in $F(S) \cap (A + B)^{-1}0$ under suitable conditions on the parameters $\{\alpha_n\}$ and $\{\lambda_n\}$.

To obtain strong convergence, Zhang et al. [15] introduced the following iterative method base on Halpern's iteration for finding a common element of the set of solutions to the problem (1.1) and the set of fixed points of a nonexpansive mapping S :

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S J_{\lambda_n}^B (x_n - \lambda_n A x_n), \quad \forall n \geq 1, \quad (1.3)$$

where $A : C \rightarrow H$ is an α -inverse strongly monotone mapping, B is a maximal monotone operator on H such that the domain of B is included in C and $J_{\lambda}^B = (I + \lambda B)^{-1}$ is a resolvent of B for all $\lambda > 0$. Under some mild conditions, they proved that the sequence $\{x_n\}$ defined by (1.3) converges strongly to point in $F(S) \cap (A + B)^{-1}0$.

Very, recently, Takahashi et al. [33] introduced an iterative method for finding a common element of the set of solutions to the problem (1.1) and the set of fixed points of nonexpansive mappings. They obtained the following convergence theorem:

Theorem TTT Let C be a closed and convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping and let B be a maximal monotone operator on H such that the domain of B is included in C . Let $J_{\lambda}^B = (I + \lambda B)^{-1}$ be a resolvent of B for all $\lambda > 0$ and $S : C \rightarrow C$ be a nonexpansive mapping such that $F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S (\alpha_n x + (1 - \alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n)), \quad \forall n \geq 1,$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 1, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0, \quad \lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point in $F(S) \cap (A + B)^{-1}0$.

Motivated by the above works, we construct iterative methods with perturbations for finding a common element $x^* \in \Omega$, where Ω is the set of common element of the set of solutions of a variational inclusion problem (1.1) and the set of fixed points of a nonexpansive mapping in Hilbert spaces. We prove a strong convergence theorem of the proposed iterative schemes under some certain conditions. As special cases, we can obtain x^* is the minimum-norm common element of Ω . Furthermore, we also apply our results to solving the variational inequality and equilibrium problems. The main results obtained in this paper are improve and generalize many known recent results in this field.

2 Preliminaries

Definition 2.1. *A mapping $A : C \rightarrow H$ is said to be α -inverse strongly monotone if there exists a constant $\alpha > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Remark 2.1. *If A is α -inverse strongly monotone, then A is $\frac{1}{\alpha}$ -Lipschitzian.*

Lemma 2.2. ([30]) *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|l_n - x_n\| = 0$.*

Lemma 2.3. ([27]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an α -inverse strongly monotone operator. Then, we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \quad (2.1)$$

where $\lambda > 0$. In particular, if $0 < \lambda \leq 2\alpha$ then $I - \lambda A$ is nonexpansive.

Lemma 2.4. (The Resolvent Identity [31]) *For $\lambda > 0$, $\mu > 0$ and $x \in H$, then*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda x \right).$$

Lemma 2.5. *For each $r, s > 0$ then*

$$\|J_r x - J_s x\| \leq \left| 1 - \frac{s}{r} \right| \|J_r x - x\| \quad \text{for all } x \in H.$$

Proof. Follows from the resolvent identity, we can conclude the desired result easily. \square

Lemma 2.6. ([15]) *Let $A : C \rightarrow H$ be a mapping, B be a maximal monotone operator in H such that the domain of B is included in C and let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent operator of B for all $\lambda > 0$. Then $F(J_\lambda^B(I - \lambda B)) = (A + B)^{-1}0$.*

Lemma 2.7. ([32]) (*Demiclosed principle*) *Let C be a nonempty, closed and convex subset of a Hilbert space H and $T : C \rightarrow C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Tx_n \rightarrow 0$ implies $x = Tx$.*

Lemma 2.8. ([29]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that

- (i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.9. *Let H be a real Hilbert space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

3 Main Results

3.1 An implicit iteration scheme

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let B be a maximal monotone operator in H such that the domain of B is included in C and let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent operator of B for all $\lambda > 0$. Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Assume that λ is a positive constant such that $\lambda \in [a, b] \subset (0, 2\alpha)$ and $\{u_t\} \subset H$ is a perturbation satisfy $\lim_{t \rightarrow 0^+} u_t = u' \in H$. For each $t \in (0, 1 - \frac{\lambda}{2\alpha})$, we define a mapping $S_t : C \rightarrow C$ by

$$S_t x := SJ_\lambda^B(tu_t + (1 - t)x - \lambda Ax), \quad \forall x \in C.$$

By the nonexpansiveness of S , J_λ^B and by Lemma 2.3, for all $x, y \in C$, we have

$$\begin{aligned} \|S_t x - S_t y\| &= \|SJ_\lambda^B(tu_t + (1 - t)x - \lambda Ax) - SJ_\lambda^B(tu_t + (1 - t)y - \lambda Ay)\| \\ &\leq \|J_\lambda^B(tu_t + (1 - t)x - \lambda Ax) - J_\lambda^B(tu_t + (1 - t)y - \lambda Ay)\| \\ &= (1 - t) \left\| \left(I - \frac{\lambda}{1 - t} A \right) x - \left(I - \frac{\lambda}{1 - t} A \right) y \right\| \\ &\leq (1 - t) \|x - y\|, \end{aligned}$$

which implies that the mapping S_t is a contraction. Hence, S_t has a unique fixed point, denoted by x_t , which uniquely solves the fixed point equation

$$x_t = SJ_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t). \quad (3.1)$$

Theorem 3.1. *Assume that $\{x_t\}$ is defined by (4.6), then $\{x_t\}$ converges strongly as $t \rightarrow 0^+$ to a point $x^* \in \Omega$, where x^* is the unique solution of the variational inequality*

$$\langle u' - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega. \quad (3.2)$$

As a special case, if we take $u_t = 0$, then the sequence $\{x_t\}$ converges strongly to the minimum-norm common element of Ω .

Proof. First, we show that $\{x_t\}$ is bounded. Set $x_t = Sy_t$, where $y_t = J_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t)$. Take $p \in \Omega$, we observe that

$$p = Sp = SJ_\lambda^B(p - \lambda Ap) = SJ_\lambda^B\left(tp + (1-t)\left(p - \frac{\lambda}{1-t}Ap\right)\right), \quad \forall t \in \left(0, 1 - \frac{\lambda}{2\alpha}\right).$$

Since S, J_λ^B and $I - \frac{\lambda}{1-t}A$ are nonexpansive (see Lemma 2.3), we have

$$\begin{aligned} \|y_t - p\| &= \left\| J_\lambda^B\left(tu_t + (1-t)\left(I - \frac{\lambda}{1-t}A\right)x_t\right) - J_\lambda^B\left(tp + (1-t)\left(I - \frac{\lambda}{1-t}A\right)p\right) \right\| \\ &\leq \left\| t(u_t - p) + (1-t)\left[\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right] \right\| \\ &\leq t\|u_t - p\| + (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right\| \\ &\leq t\|u_t - p\| + (1-t)\|x_t - p\|. \end{aligned} \quad (3.3)$$

Then, it follows that

$$\begin{aligned} \|x_t - p\| &= \|Sy_t - Sp\| \\ &\leq \|y_t - p\| \\ &\leq t\|u_t - p\| + (1-t)\|x_t - p\|, \end{aligned}$$

which implies that

$$\|x_t - p\| \leq \|u_t - p\|.$$

Since $\lim_{t \rightarrow 0^+} u_t = u'$, then there exists a constant $K_1 > 0$ such that $K_1 = \sup_{t > 0} \{\|u_t\|\}$. Hence, $\{x_t\}$ is bounded, so are $\{y_t\}$, $\{Sx_t\}$ and $\{Ax_t\}$.

Next, we show that $\lim_{t \rightarrow 0^+} \|x_t - Sx_t\| = 0$. By the convexity of $\|\cdot\|^2$ and

(3.3), we have

$$\begin{aligned}
 \|x_t - p\|^2 &\leq \|y_t - p\|^2 \\
 &\leq \left\| (1-t) \left[\left(x_t - \frac{\lambda}{1-t} Ax_t \right) - \left(p - \frac{\lambda}{1-t} Ap \right) \right] + t(u_t - p) \right\|^2 \\
 &\leq (1-t) \left\| \left(x_t - \frac{\lambda}{1-t} Ax_t \right) - \left(p - \frac{\lambda}{1-t} Ap \right) \right\|^2 + t\|u_t - p\|^2 \\
 &= (1-t) \left\| (x_t - p) - \frac{\lambda}{1-t} (Ax_t - Ap) \right\|^2 + t\|u_t - p\|^2 \\
 &= (1-t) \left[\|x_t - p\|^2 - \frac{2\lambda}{1-t} \langle Ax_t - Ap, x_t - p \rangle + \frac{\lambda^2}{(1-t)^2} \|Ax_t - Ap\|^2 \right] + t\|u_t - p\|^2 \\
 &\leq (1-t) \left[\|x_t - p\|^2 - \frac{2\lambda\alpha}{1-t} \|Ax_t - Ap\|^2 + \frac{\lambda^2}{(1-t)^2} \|Ax_t - Ap\|^2 \right] + t\|u_t - p\|^2 \\
 &= (1-t) \left[\|x_t - p\|^2 + \frac{\lambda}{(1-t)^2} (\lambda - 2(1-t)\alpha) \|Ax_t - Ap\|^2 \right] + t\|u_t - p\|^2 \\
 &\leq \|x_t - p\|^2 + \frac{\lambda}{1-t} (\lambda - 2(1-t)\alpha) \|Ax_t - Ap\|^2 + t\|u_t - p\|^2,
 \end{aligned}$$

which implies that

$$\frac{\lambda}{1-t} (2(1-t)\alpha - \lambda) \|Ax_t - Ap\|^2 \leq t\|u_t - p\|^2.$$

Since $t \in (0, 1 - \frac{\lambda}{2\alpha})$, we have $2(1-t)\alpha - \lambda > 0$. Then, we obtain

$$\lim_{t \rightarrow 0^+} \|Ax_t - Ap\| = 0. \tag{3.4}$$

Since J_λ^B is firmly nonexpansive, we have

$$\begin{aligned}
 &\|y_t - p\|^2 \\
 &= \|J_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t) - J_\lambda^B(p - \lambda Ap)\|^2 \\
 &\leq \langle tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap), y_t - p \rangle \\
 &= \frac{1}{2} \left[\|tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap)\|^2 + \|y_t - p\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
& \|y_t - p\|^2 \\
& \leq \|tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap)\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \\
& = \left\| (1-t) \left[\left(I - \frac{\lambda}{1-t} A \right) x_t - \left(I - \frac{\lambda}{1-t} A \right) p \right] + t(u_t - p) \right\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \\
& \leq (1-t) \left\| \left(I - \frac{\lambda}{1-t} A \right) x_t - \left(I - \frac{\lambda}{1-t} A \right) p \right\|^2 + t\|u_t - p\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \\
& \leq (1-t)\|x_t - p\|^2 + t\|u_t - p\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \\
& \leq \|x_t - p\|^2 + t\|u_t - p\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \\
& \leq \|y_t - p\|^2 + t\|u_t - p\|^2 - \|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2.
\end{aligned}$$

Then, we have

$$\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|^2 \leq t\|u_t - p\|^2.$$

From (3.4), we obtain that

$$\lim_{t \rightarrow 0^+} \|x_t - y_t\| = 0,$$

and hence

$$\lim_{t \rightarrow 0^+} \|y_t - Sy_t\| = \lim_{t \rightarrow 0^+} \|y_t - x_t\| = 0. \quad (3.5)$$

Moreover, we get that

$$\begin{aligned}
\|x_t - Sx_t\| & \leq \|x_t - y_t\| + \|y_t - Sy_t\| + \|Sy_t - Sx_t\| \\
& \leq 2\|x_t - y_t\| + \|y_t - Sy_t\| \rightarrow 0 \text{ as } t \rightarrow 0^+. \quad (3.6)
\end{aligned}$$

For $z \in \Omega$, by Lemmas 2.3 and 2.9, we obtain that

$$\begin{aligned}
\|x_t - z\|^2 & \leq \left\| (1-t) \left[\left(I - \frac{\lambda}{1-t} A \right) x_t - \left(I - \frac{\lambda}{1-t} A \right) z \right] + t(u_t - z) \right\|^2 \\
& \leq (1-t)^2 \left\| \left(I - \frac{\lambda}{1-t} A \right) x_t - \left(I - \frac{\lambda}{1-t} A \right) z \right\|^2 + 2t\langle u_t - z, x_t - z \rangle \\
& \leq (1-t)^2 \|x_t - z\|^2 + 2t\langle u' - z, x_t - z \rangle + 2t\langle u_t - u', x_t - z \rangle \\
& = (1-2t)\|x_t - z\|^2 + t^2\|x_t - z\|^2 + 2t\langle u' - z, x_t - z \rangle + 2t\langle u_t - u', x_t - z \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_t - z\|^2 & \leq \langle u' - z, x_t - z \rangle + \langle u_t - u', x_t - z \rangle + \frac{t}{2}\|x_t - z\|^2 \\
& \leq \langle u' - z, x_t - z \rangle + \left(\|u_t - u'\| + \frac{t}{2} \right) K_2, \quad (3.7)
\end{aligned}$$

where $K_2 > 0$ is a constant such that $K_2 = \sup_{t>0} \{\|x_t - z\|, \|x_t - z\|^2\}$.

Next, we show that $\{x_t\}$ is relatively norm-compact. Assume that $\{t_n\} \subset (0, 1)$ is a sequence such that $t_n \rightarrow 0^+$ as $n \rightarrow \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$, $\lambda_n := \lambda_{t_n}$ and $u_n := u_{t_n}$. From (3.6), we have

$$\|x_n - z\|^2 \leq \langle u' - z, x_n - z \rangle + \left(\|u_n - u'\| + \frac{t_n}{2} \right) K_2. \quad (3.8)$$

Since $\{x_n\}$ is bounded, without loss of generality, we assume that there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$ as $i \rightarrow \infty$. From (3.6), we have $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$. It follows from Lemma 2.7 that $x^* \in F(S)$. Further, we show that $x^* \in (A + B)^{-1}0$. Let $v \in Bu$. Note that

$$y_n = J_{\lambda_n}^B(t_n u_n + (1 - t_n)x_n - \lambda_n A x_n).$$

Then, we have

$$t_n u_n + (1 - t_n)x_n - \lambda_n A x_n \in (I + \lambda_n B)y_n \iff \frac{1}{\lambda_n}(t_n u_n + (1 - t_n)x_n - \lambda_n A x_n - y_n) \in B y_n.$$

Since B is maximal monotone, we have $(u, v) \in B$,

$$\begin{aligned} & \left\langle \frac{1}{\lambda_n}(t_n u_n) + (1 - t_n)x_n - \lambda_n A x_n - y_n - v, y_n - u \right\rangle \geq 0 \\ & \iff \langle t_n u_n + (1 - t_n)x_n - \lambda_n A x_n - y_n - \lambda_n v, y_n - u \rangle \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \langle A x_n + v, y_n - u \rangle & \leq \frac{1}{\lambda_n} \langle x_n - y_n, y_n - u \rangle + \frac{t_n}{\lambda_n} \langle u_n - x_n, y_n - u \rangle \\ & \leq \frac{1}{\lambda_n} \|x_n - y_n\| \|y_n - u\| + \frac{t_n}{\lambda_n} \|u_n - x_n\| \|y_n - u\| \\ & \leq (\|x_n - y_n\| + t_n) K_3, \end{aligned} \quad (3.9)$$

where $K_3 > 0$ is a constant such that $K_3 = \sup_{n \geq 1} \left\{ \frac{1}{\lambda_n} (\|y_n - u\|, \|u_n - x_n\| \|y_n - u\|) \right\}$.

Since $\|x_n - y_n\| \rightarrow 0$ and $t_n \rightarrow 0^+$ as $n \rightarrow \infty$, then from (3.9), we obtain that $\langle A x^* + v, x^* - u \rangle \leq 0$, that is $\langle -A x^* - v, x^* - u \rangle \geq 0$, this implies that $-A x^* \in B x^*$, that is $x^* \in (A + B)^{-1}0$. Hence $x^* \in \Omega := F(S) \cap (A + B)^{-1}0$.

Now, replacing z in (3.8) with x^* , we have

$$\|x_n - x^*\|^2 \leq \langle u' - x^*, x_n - x^* \rangle + \left(\|u_n - u'\| + \frac{t_n}{2} \right) K_2. \quad (3.10)$$

Since $x_n \rightharpoonup x^*$. Then, we get that $x_n \rightarrow x^*$. This proved the relatively norm compactness of the net $\{x_t\}$ as $t \rightarrow 0^+$.

Now, we show that the solution of (3.2) is singleton. Assume that $\hat{x}, x^* \in \Omega$ are solutions of (3.2). Then, we have

$$\langle u' - \hat{x}, x^* - \hat{x} \rangle \leq 0$$

and

$$\langle u' - x^*, \hat{x} - x^* \rangle \leq 0.$$

Adding up above two inequalities, we have

$$\|x^* - \hat{x}\|^2 \leq 0.$$

This implies that $\hat{x} = x^*$ and the uniqueness is proved. In summary, we have shown that each cluster point of $\{x_t\}$ equal to x^* as $t \rightarrow 0^+$.

Finally, if we take $u_t = 0$ then (3.2) is reduced to

$$\langle -x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega.$$

It follows that

$$\|x^*\|^2 \leq \langle z, x^* \rangle \leq \|z\| \|x^*\|, \quad \forall z \in \Omega,$$

that is

$$\|x^*\| \leq \|z\|, \quad \forall z \in \Omega.$$

Therefore, x^* is a minimum-norm common element of Ω . This completes the proof. \square

3.2 An explicit iteration scheme

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let B be a maximal monotone operator in H such that the domain of B is included in C and let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent operator of B for all $\lambda > 0$. Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A + B)^{-1}0 \neq \emptyset$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J_{\lambda_n}^B(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, \quad \forall n \geq 1, \end{cases} \quad (3.11)$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{u_n\} \subset H$ is a perturbation for the n -step iteration, which satisfy the following conditions:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C2) $0 < a \leq \beta_n \leq b < 1$;

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(C3) $0 < a' \leq \lambda_n \leq b' < 2\alpha$ with $\lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0$ and $0 < a'' \leq \frac{\lambda_n}{1-\alpha_n} \leq b'' < 2\alpha$;

(C4) $\lim_{n \rightarrow \infty} u_n = u' \in H$.

Then $\{x_n\}$ defined by (3.11) converges strongly to a point $x^* \in \Omega$, where x^* is the unique solution of the variational inequality (3.2). As a special case, if we take $u_n = 0$, then the sequence $\{x_n\}$ converges strongly to the minimum-norm common element of Ω .

Proof. First, we show that $\{x_n\}$ is bounded. It implies from (C4) that $\{u_n\}$ is bounded sequence. Take $p \in \Omega$, then there exists a constant $M_1 > 0$ such that $M_1 = \sup_{n \geq 1} \{\|u_n - p\|\}$. We observe that

$$p = Sp = J_{\lambda_n}^B(p - \lambda_n Ap) = J_{\lambda_n}^B\left(\alpha_n p + (1 - \alpha_n)\left(p - \frac{\lambda_n}{1 - \alpha_n} Ap\right)\right).$$

Since $J_{\lambda_n}^B$, S and $I - \frac{\lambda_n}{1 - \alpha_n} A$ are nonexpansive (see Lemma 2.3), we have

$$\begin{aligned} & \|y_n - p\| \\ = & \left\| J_{\lambda_n}^B\left(\alpha_n u_n + (1 - \alpha_n)\left(I - \frac{\lambda_n}{1 - \alpha_n} A\right)x_n\right) - J_{\lambda_n}^B\left(\alpha_n p + (1 - \alpha_n)\left(I - \frac{\lambda_n}{1 - \alpha_n} A\right)p\right) \right\| \\ \leq & \left\| \alpha_n(u_n - p) + (1 - \alpha_n)\left[\left(I - \frac{\lambda_n}{1 - \alpha_n} A\right)x_n - \left(I - \frac{\lambda_n}{1 - \alpha_n} A\right)p\right] \right\| \\ \leq & \alpha_n \|u_n - p\| + (1 - \alpha_n) \left\| \left(I - \frac{\lambda_n}{1 - \alpha_n} A\right)x_n - \left(I - \frac{\lambda_n}{1 - \alpha_n} A\right)p \right\| \\ \leq & \alpha_n \|u_n - p\| + (1 - \alpha_n) \|x_n - p\|. \end{aligned} \tag{3.12}$$

Then, it follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1 - \beta_n)(Sy_n - p)\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|Sy_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \left[\alpha_n \|u_n - p\| + (1 - \alpha_n) \|x_n - p\| \right] \\ &= (1 - (1 - \beta_n)\alpha_n) \|x_n - p\| + (1 - \beta_n)\alpha_n \|u_n - p\| \\ &\leq \max\{\|x_n - p\|, M_1\}. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, M_1\}, \quad \forall n \geq 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Ax_n\}$ and $\{Sx_n\}$.

Next, we show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Set $y_n = J_{\lambda_n}^B z_n$, where $z_n = \alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n$. By the nonexpansivity of the mappings $J_{\lambda_n}^B$ and

$I - \frac{\lambda_n}{1-\alpha_n}A$, we have

$$\begin{aligned}
& \|y_{n+1} - y_n\| \\
&= \|J_{\lambda_{n+1}}^B z_{n+1} - J_{\lambda_n}^B z_n\| \\
&\leq \|J_{\lambda_{n+1}}^B z_{n+1} - J_{\lambda_{n+1}}^B z_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&\leq \|z_{n+1} - z_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&= \|\alpha_{n+1}u_{n+1} + (1 - \alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1} - (\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n)\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&= \left\| \alpha_{n+1}(u_{n+1} - u_n) + (\alpha_{n+1} - \alpha_n)(u_n - x_n) + (1 - \alpha_{n+1}) \left[\left(I - \frac{\lambda_{n+1}}{1 - \alpha_{n+1}} A \right) x_{n+1} - \left(I - \frac{\lambda_n}{1 - \alpha_n} A \right) x_n \right] \right. \\
&\quad \left. + (\lambda_n - \lambda_{n+1})Ax_n \right\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&\leq \alpha_{n+1}(\|u_{n+1}\| + \|u_n\|) + |\alpha_{n+1} - \alpha_n|(\|u_n\| + \|x_n\|) \\
&\quad + (1 - \alpha_{n+1}) \left\| \left(I - \frac{\lambda_{n+1}}{1 - \alpha_{n+1}} A \right) x_{n+1} - \left(I - \frac{\lambda_n}{1 - \alpha_n} A \right) x_n \right\| \\
&\quad + |\lambda_{n+1} - \lambda_n| \|Ax_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|u_n\|) + |\alpha_{n+1} - \alpha_n|(\|u_n\| + \|x_n\|) + |\lambda_{n+1} - \lambda_n| \|Ax_n\| \\
&\quad + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\|.
\end{aligned}$$

By Lemma 2.5, we have

$$\|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J_{\lambda_{n+1}}^B z_n - z_n\|.$$

Then, it follows that

$$\begin{aligned}
& \|y_{n+1} - y_n\| \\
&\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}(\|u_{n+1}\| + \|u_n\|) + |\alpha_{n+1} - \alpha_n|(\|u_n\| + \|x_n\|) \\
&\quad + |\lambda_{n+1} - \lambda_n| \|Ax_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J_{\lambda_{n+1}}^B z_n - z_n\| \\
&\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + \left(\alpha_{n+1} + |\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \right) M_2,
\end{aligned}$$

where $M_2 = \sup_{n \geq 1} \{\|u_{n+1}\| + \|u_n\|, \|u_n\| + \|x_n\|, \|Ax_n\|, \|J_{\lambda_{n+1}}^B z_n - z_n\|\}$. Then, we have

$$\begin{aligned}
& \|S y_{n+1} - S y_n\| \\
&\leq \|y_{n+1} - y_n\| \\
&\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + \left(\alpha_{n+1} + |\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n| + \frac{|\lambda_{n+1} - \lambda_n|}{a'} \right) M_2.
\end{aligned}$$

From (C1) and (C3), we obtain

$$\limsup_{n \rightarrow \infty} (\|S y_{n+1} - S y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.2, we get

$$\lim_{n \rightarrow \infty} \|Sy_n - x_n\| = 0. \quad (3.13)$$

Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|Sy_n - x_n\| = 0. \quad (3.14)$$

Next, we show that $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$. By the convexity of $\|\cdot\|^2$ and (3.12), we have

$$\begin{aligned} & \|y_n - p\|^2 \\ \leq & \left\| (1 - \alpha_n) \left[\left(x_n - \frac{\lambda_n}{1 - \alpha_n} Ax_n \right) - \left(p - \frac{\lambda_n}{1 - \alpha_n} Ap \right) \right] + \alpha_n (u_n - p) \right\|^2 \\ \leq & (1 - \alpha_n) \left\| \left(x_n - \frac{\lambda_n}{1 - \alpha_n} Ax_n \right) - \left(p - \frac{\lambda_n}{1 - \alpha_n} Ap \right) \right\|^2 + \alpha_n \|u_n - p\|^2 \\ = & (1 - \alpha_n) \left\| (x_n - p) - \frac{\lambda_n}{1 - \alpha_n} (Ax_n - Ap) \right\|^2 + \alpha_n \|u_n - p\|^2 \\ = & (1 - \alpha_n) \left[\|x_n - p\|^2 - \frac{2\lambda_n}{1 - \alpha_n} \langle Ax_n - Ap, x_n - p \rangle + \frac{\lambda_n^2}{(1 - \alpha_n)^2} \|Ax_n - Ap\|^2 \right] + \alpha_n \|u_n - p\|^2 \\ \leq & (1 - \alpha_n) \left[\|x_n - p\|^2 - \frac{2\lambda_n \alpha}{1 - \alpha_n} \|Ax_n - Ap\|^2 + \frac{\lambda_n^2}{(1 - \alpha_n)^2} \|Ax_n - Ap\|^2 \right] + \alpha_n \|u_n - p\|^2 \\ = & (1 - \alpha_n) \left[\|x_n - p\|^2 + \frac{\lambda_n}{(1 - \alpha_n)^2} \left(\lambda_n - 2(1 - \alpha_n)\alpha \right) \|Ax_n - Ap\|^2 \right] + \alpha_n \|u_n - p\|^2 \\ \leq & \|x_n - p\|^2 + \frac{\lambda_n}{1 - \alpha_n} \left(\lambda_n - 2(1 - \alpha_n)\alpha \right) + \alpha_n \|u_n - p\|^2. \end{aligned} \quad (3.15)$$

Then, it follows from (3.15) that

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ \leq & \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|Sy_n - p\|^2 \\ \leq & \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 \\ \leq & \beta_n \|x_n - p\|^2 + (1 - \beta_n) \left[\|x_n - p\|^2 + \frac{\lambda_n}{1 - \alpha_n} \left(\lambda_n - 2(1 - \alpha_n)\alpha \right) \|Ax_n - Ap\|^2 + \alpha_n \|u_n - p\|^2 \right] \\ = & \|x_n - p\|^2 + (1 - \beta_n)\alpha_n \|u_n - p\|^2 + \lambda_n(1 - \beta_n) \left(\frac{\lambda_n}{1 - \alpha_n} - 2\alpha \right) \|Ax_n - Ap\|^2, \end{aligned}$$

which implies by (C2) and (C3) that

$$\begin{aligned} & b'(1 - b)(2\alpha - b'') \|Ax_n - Ap\|^2 \\ \leq & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - (1 - \beta_n)\alpha_n \|x_n - p\|^2 + (1 - \beta_n)\alpha_n \|u_n - p\|^2 \\ \leq & (\|x_n - p\| - \|x_{n+1} - p\|) \|x_{n+1} - x_n\| + (1 - \beta_n)\alpha_n [\|u_n - p\|^2 - \|x_n - p\|^2]. \end{aligned}$$

From (C1) – (C3) and (3.14), we obtain that

$$\lim_{n \rightarrow \infty} \|Ax_n - Ap\| = 0. \quad (3.16)$$

On the other hand, by firmly nonexpansivity of $J_{\lambda_n}^B$, we have

$$\begin{aligned} & \|y_n - p\|^2 \\ &= \|J_{\lambda_n}^B(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n) - J_{\lambda_n}^B(p - \lambda_n Ap)\|^2 \\ &\leq \langle \alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap), y_n - p \rangle \\ &= \frac{1}{2} \left[\|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^2 + \|y_n - p\|^2 \right. \\ &\quad \left. - \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|^2 \right], \end{aligned}$$

which implies that

$$\begin{aligned} & \|y_n - p\|^2 \\ &\leq \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^2 - \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|^2 \\ &\leq \alpha_n \|u_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|^2. \end{aligned}$$

Then, it follows that

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1 + \beta_n) \|y_n - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + (1 + \beta_n) \left[\alpha_n \|u_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|^2 \right] \\ &= (1 - (1 - \beta_n)\alpha_n) \|x_n - p\|^2 + (1 - \beta_n)\alpha_n \|u_n - p\|^2 - (1 - \beta_n) \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|^2, \end{aligned}$$

which implies by (C2) that

$$\begin{aligned} & (1 - b) \|\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - (1 - \beta_n)\alpha_n \|u_n - p\|^2 + (1 - \beta_n)\alpha_n \|u_n - p\|^2 \\ &\leq (\|x_n - p\| - \|x_{n+1} - p\|) \|x_{n+1} - x_n\| - (1 - \beta_n)\alpha_n \|u_n - p\|^2 + (1 - \beta_n)\alpha_n \|u_n - p\|^2. \end{aligned}$$

Then, by (C1), (C2) and (3.13), we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.17)$$

Consequently,

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\leq \|x_n - Sy_n\| + \|y_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.18)$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle u' - x^*, y_n - x^* \rangle \leq 0,$$

where x^* is the same as in Theorem 3.1. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u' - x^*, y_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle u' - x^*, y_{n_i} - x^* \rangle.$$

By the boundedness of $\{y_n\}$, without loss of generality, we assume that $y_{n_i} \rightarrow z \in C$ as $i \rightarrow \infty$. From (3.17) and (3.18), we also have $y_n - Sy_n \rightarrow 0$. Then from Lemma 2.7, we get that $z \in F(S)$. Further, by the similar method in the proof of Theorem 3.1, we can show that $z \in \Omega$. Then, we obtain

$$\limsup_{n \rightarrow \infty} \langle u' - x^*, y_n - x^* \rangle = \langle u' - x^*, z - x^* \rangle \leq 0. \quad (3.19)$$

Finally, we show that $x_n \rightarrow x^*$. From (3.12) and Lemma 2.9, we have

$$\begin{aligned} \|Sy_n - x^*\|^2 &\leq \|y_n - x^*\|^2 \\ &= \left\| \left(1 - \alpha_n\right) \left[\left(I - \frac{\lambda_n}{1 - \alpha_n} A\right) x_n - \left(I - \frac{\lambda_n}{1 - \alpha_n} A\right) x^* \right] + \alpha_n (u_n - x^*) \right\|^2 \\ &\leq (1 - \alpha_n)^2 \left\| \left(I - \frac{\lambda_n}{1 - \alpha_n} A\right) x_n - \left(I - \frac{\lambda_n}{1 - \alpha_n} A\right) x^* \right\|^2 + 2\alpha_n \langle u_n - x^*, y_n - x^* \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle u_n - x^*, y_n - x^* \rangle. \end{aligned}$$

Then, it follows that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|Sy_n - x^*\|^2 \\ &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left[(1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle u_n - x^*, y_n - x^* \rangle \right] \\ &= (\beta_n + (1 - \beta_n)(1 - \alpha_n)^2) \|x_n - x^*\|^2 + 2\alpha_n (1 - \beta_n) \langle u_n - x^*, y_n - x^* \rangle \\ &\leq (1 - \alpha_n(1 - \beta_n)) \|x_n - x^*\|^2 + 2\alpha_n (1 - \beta_n) \langle u_n - u', y_n - x^* \rangle + 2\alpha_n (1 - \beta_n) \langle u' - x^*, y_n - x^* \rangle \\ &\leq (1 - \alpha_n(1 - \beta_n)) \|x_n - x^*\|^2 + 2\alpha_n (1 - \beta_n) \|u_n - u'\| \|y_n - x^*\| + 2\alpha_n (1 - \beta_n) \langle u' - x^*, y_n - x^* \rangle \\ &= (1 - \gamma_n) \|x_n - x^*\|^2 + \gamma_n \delta_n, \end{aligned}$$

where $\gamma_n = \alpha_n(1 - \beta_n)$ and $\delta_n = 2\|u_n - u'\| \|y_n - x^*\| + 2\langle u' - x^*, y_n - x^* \rangle$. From (C1), (C4) and (3.18), it is easily seen that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Therefore, by Lemma 2.8, we conclude that $x_n \rightarrow x^*$. This completes the proof. \square

4 Some applications

4.1 Application to variational inequalities

Let H be a real Hilbert space and $g : H \rightarrow (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the *subdifferential* ∂g of g is defined as follows:

$$\partial g(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y \rangle, \forall z \in H\}, \quad \forall x \in H.$$

From [34], we know that ∂g is maximal monotone. Let C be a closed and convex subset of H and let δ_C be the indicator function of C , i.e.,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \quad (4.1)$$

Since δ_C is a proper lower semicontinuous convex function on H , the subdifferential $\partial\delta_C$ of δ_C is a maximal monotone operator. So, we can define the resolvent of $\partial\delta_C$ by

$$J_\lambda^{\partial\delta_C} x = (I + \lambda\partial\delta_C)^{-1}x, \quad \forall x \in H.$$

Lemma 4.1. ([33]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H , P_C be the metric projection from H onto C and $\partial\delta_C$ be the subdifferential of δ_C , where δ_C is as defined in (4.5) and $J_\lambda^{\partial\delta_C} = (I + r\partial\delta_C)^{-1}$. Then*

$$y = J_\lambda^{\partial\delta_C} x \iff y = P_C x, \quad \forall x \in H, y \in C.$$

Using Theorems 3.1, 3.2 and Lemma 4.1, we obtain the following results.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap VI(C, A) \neq \emptyset$. Let λ be a positive constant such that $\lambda \in [a, b] \subset (0, 2\alpha)$ and $\{u_t\} \subset H$ be a perturbation satisfy $\lim_{t \rightarrow 0^+} u_t = u' \in H$. For each $t \in (0, 1 - \frac{\lambda}{2\alpha})$, the net sequence $\{x_t\}$ define by*

$$x_t = SP_C(tu_t + (1-t)x_t - \lambda Ax_t), \quad (4.2)$$

converges strongly to a point $x^ \in \Omega$. As a special case, if we take $u_t = 0$, then the sequence $\{x_t\}$ converges strongly to the minimum-norm common element of Ω .*

Theorem 4.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap VI(C, A) \neq \emptyset$. For initial guess $x_1 \in C$, then the sequence $\{x_n\}$ define by*

$$\begin{cases} y_n = P_C(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S y_n, \quad \forall n \geq 1, \end{cases} \quad (4.3)$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{u_n\} \subset H$ is a perturbation for the n -step iteration, which satisfy the conditions (C1) – (C4). Then $\{x_n\}$ defined by (4.3) converges strongly to a point $x^ \in \Omega$. As a special case, if we take $u_n = 0$, then the sequence $\{x_n\}$ converges strongly to the minimum-norm common element of Ω .*

4.2 Application to equilibrium problems

Let C be a nonempty, closed convex subset of a real Hilbert space H and $\Theta : C \times C \rightarrow \mathbb{R}$ be a bifunction, where \mathbb{R} is the set of all real numbers. The equilibrium problem is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \quad (4.4)$$

The set of solutions of the equilibrium problem (4.4) is denoted by $EP(\Theta)$.

In the real world, many problems have reformulations which reduces to find a solution of the equilibrium problem (4.4) for instance, optimization and economics (see, e.g., [35] and [36]).

For solving the equilibrium problem, let us assume that a bifunction $\Theta : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions:

- (A1) $\Theta(x, x) = 0$ for all $x \in C$;
- (A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for each $x, y \in C$;
- (A3) Θ is upper-semicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x, y);$$

- (A4) $\Theta(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$.

Lemma 4.4. ([35]) *Let C be a nonempty, closed and convex subset of H and let $\Theta : C \times C \rightarrow \mathbb{R}$ satisfying the conditions (A1) – (A4). Let $\lambda > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{\lambda} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 4.5. ([36]) *Assume that $\Theta : C \times C \rightarrow \mathbb{R}$ satisfies the conditions (A1) – (A4). For $\lambda > 0$ and $x \in H$, define a mapping $T_\lambda : H \rightarrow C$ as follows:*

$$T_\lambda(x) = \left\{ z \in C : \Theta(z, y) + \frac{1}{\lambda} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}, \quad \forall x \in H.$$

Then the following hold:

- (1) T_λ is single-valued.
- (2) T_λ is firmly nonexpansive, i.e., for each $x, y \in H$,

$$\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y \rangle.$$

- (3) $Fix(T_\lambda) = EP(\Theta)$.
- (4) $EP(\Theta)$ is closed and convex.

We call such T_λ the resolvent of Θ for $\lambda > 0$. The following Lemma can be found in [33].

Lemma 4.6. ([33]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $\Theta : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let A_Θ be a multivalued mapping of H into itself defined by*

$$A_\Theta x = \begin{cases} \{z \in H : \Theta(x, y) \geq \langle y - x, z \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \quad (4.5)$$

Then, $EP(\Theta) = A_\Theta^{-1}0$ and A_Θ is a maximal monotone operator with $\text{dom}(A_\Theta) \subset C$. Further, for any $x \in H$ and $\lambda > 0$, the resolvent T_λ of Θ coincides with the resolvent of A_Θ ; i.e., $T_\lambda x = (I + \lambda A_\Theta)^{-1}x$.

Using Theorems 3.1, 3.2, Lemmas 4.6 and 4.5 we obtain the following results.

Theorem 4.7. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let $\Theta : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let T_λ be the resolvent of Θ for $\lambda > 0$ with $\lambda \in [a, b] \subset (0, 2\alpha)$. Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap EP(\Theta) \neq \emptyset$. Let $\{u_t\} \subset H$ be a perturbation satisfy $\lim_{t \rightarrow 0^+} u_t = u' \in H$. For each $t \in (0, 1 - \frac{\lambda}{2\alpha})$, the net sequence $\{x_t\}$ define by*

$$x_t = ST_\lambda(tu_t + (1-t)x_t - \lambda Ax_t). \quad (4.6)$$

converges strongly to a point $x^ \in \Omega$. As a special case, if we take $u_t = 0$, then the sequence $\{x_t\}$ converges strongly to the minimum-norm common element of Ω .*

Theorem 4.8. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping. Let $\Theta : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let T_λ be the resolvent of Θ for $\lambda > 0$. Let $S : C \rightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap EP(\Theta) \neq \emptyset$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = T_{\lambda_n}(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S y_n \end{cases} \quad (4.7)$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{u_n\} \subset H$ is a perturbation for the n -step iteration, which satisfy the conditions (C1)-(C4) Then $\{x_n\}$ defined by (4.7) converges strongly to a point $x^ \in \Omega$. As a special case, if we take $u_n = 0$, then the sequence $\{x_n\}$ converges strongly to the minimum-norm common element of Ω .*

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