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Stability of Generalized Euler Differential Equations of First Order with Variable Coefficients[1]

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Abstract : In this paper, using the method of the exponential dichotomy of fundamental solution matrix, we prove the Hyers-Ulam stability of generalized Euler differential equations of first order with variable coefficients. Our results can be applied to Euler differential equations of first order so that the related results by Jung *et al.* [S.-M. Jung, B. Kim, Th.M. Rassias, On the Hyers-Ulam stability of a system of Euler differential equations of first order, Tamsui Oxford J. Math. Sci. 24 (2008) 381–388] are generalized.

Keywords : Hyers-Ulam stability; Euler differential equations; exponential dichotomy.

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1 Introduction

Let X be a normed space over a scalar field $\mathbb{K}$, let I be an open interval, and let $a_0, a_1, \dots, a_{n-1}$ be fixed elements of $\mathbb{K}$. Consider the differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) + g(t) = 0, \qquad \forall t \in I, \tag{1.1}$$

where the continuous function $g : I \to X$ is given and the n times continuously differentiable function $y : I \to X$ is unknown. As usual, equation (1.1) is said to be Hyers-Ulam stable if for any n times continuously differentiable function $\tilde{y} : I \to X$ satisfying the inequality

$$\|\tilde{y}^{(n)}(t) + a_{n-1}\tilde{y}^{(n-1)}(t) + \cdots + a_1\tilde{y}'(t) + a_0\tilde{y}(t) + g(t)\| \le \varepsilon, \qquad \forall t \in I,$$

for some constant $\varepsilon > 0$, there is a solution $y_0 : I \to X$ of equation (1.1) such that

$$\|y(t) - y_0(t)\| \le K(\varepsilon), \qquad \forall t \in I,$$

where $K(\varepsilon)$ is a function of ε satisfying $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$. For more detailed definition of the Hyers-Ulam stability, we may refer to [1, 2, 3, 4, 5, 6, 7, 8].

Applications of Hyers-Ulam stability to certain types of ordinary differential equations were firstly investigated by Alsina and Ger [9]. They proved that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \le 3\varepsilon$. Using the methods given in [9], Miura [10], Miura *et al.* [11], Miura *et al.* [12] and Takahasi *et al.* [13] proved that the differential equation $y'(t) = \lambda y(t)$ is Hyers-Ulam stable. In 2004, Jung [14] proved a similar result for the differential equation $\varphi(t)y'(t) = y(t)$. Further results for the nonhomogeneous linear differential equation of first order in the form of

$$y' + p(t)y + q(t) = 0.$$

have been investigated by Miura, Takahasi and Jung [15, 16, 17, 18]. In 2006, using matrix method, Jung [19] proved the Hyers-Ulam stability of first order linear differential equations with constant coefficients in the form of

$$\overrightarrow{y}'(t) = A\overrightarrow{y}(t) + \overrightarrow{b}(t), \tag{1.2}$$

where

$$\overrightarrow{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad \overrightarrow{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

By adopting the idea of [19], Jung *et al.* [20] proved the Hyers-Ulam stability of Euler differential equations of first order in the form of

$$t\overrightarrow{y}'(t) = A\overrightarrow{y}(t) + \overrightarrow{b}(t). \tag{1.3}$$

In this paper we generally consider generalized Euler differential equations of first order with variable coefficients in the form of

$$t\overrightarrow{y}'(t) = A(t)\overrightarrow{y}(t) + \overrightarrow{b}(t), \tag{1.4}$$

where

$$\overrightarrow{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \overrightarrow{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$A(t) = (a_{jk}(t))_{n\times n}$ and $a_{jk}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ are continuous and uniformly bounded functions for all $j, k = 1, ..., n$. Following the idea of [20, 21] we prove the Hyers-Ulam stability of equation (1.4). Furthermore, our results can be applied to equation (1.3) so that the related results by Jung *et al.* [20] are generalized.

2 Main Results

Throughout this paper, let $(\mathbb{C}^n, \|\cdot\|)$ be a complex normed space and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. Define the vector norm $\|\cdot\|$ as $\|\overrightarrow{x}\| = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$ for all $\overrightarrow{x} \in \mathbb{C}^n$. Then it is easy to see that $(\mathbb{C}^n, \|\cdot\|)$ is a Banach space and the matrix norm being subject to the vector norm $\|\cdot\|$ can be obtained as

$$\|A\| = \sup_{\|\overrightarrow{x}\|=1} \|A\overrightarrow{x}\| = \max_{1\le j\le n} \sum_{k=1}^{n} |a_{jk}|, \qquad \forall A := (a_{jk})_{n\times n} \in \mathbb{C}^{n\times n}.$$

Definition 2.1 (cf. [22])**.** Let $A(t)$ be a piecewise continuous $n \times n$ matrix valued function defined on an interval $J = (-\infty, \infty)$. The linear differential equation

$$\overrightarrow{y}'(t) = A(t)\overrightarrow{y}(t) \tag{2.1}$$

is said to have an exponential dichotomy on J if there are projections $p(t)$, for all $t \in J$, and positive constants K_1, K_2, α_1, α_2 such that

$$Y(t)Y^{-1}(s)P(s) = P(t)Y(t)Y^{-1}(s), \quad \forall t, s \in J, \tag{2.2}$$

$$\|Y(t)Y^{-1}(s)P(s)\| \le K_1 e^{-\alpha_1(t-s)}, \quad \forall t, s \in J, \ \ t \ge s, \tag{2.3}$$

and

$$\|Y(t)Y^{-1}(s)(I - P(s))\| \le K_2 e^{-\alpha_2(s-t)}, \quad \forall t, s \in J, \ \ t \le s. \tag{2.4}$$

Here $Y(t)$ is any fundamental matrix for equation (2.1). Note that K_1, K_2 are called constants and α_1, α_2 exponents associated with the dichotomy.

Now, we give our main result as follows:

Theorem 2.1. *Suppose that the linear differential equation $\overrightarrow{y}'(\tau) = A(e^\tau)\overrightarrow{y}(\tau)$ has an exponential dichotomy on J with projections $\widetilde{p}(s)$, constants $\widetilde{K}_1, \widetilde{K}_2$ and exponents $\widetilde{\alpha}_1, \widetilde{\alpha}_2$. If $\overrightarrow{y}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ is a continuously differentiable vector function satisfying the differential inequality*

$$\|t\overrightarrow{y}'(t) - A(t)\overrightarrow{y}(t) - \overrightarrow{b}(t)\| \leq \varepsilon, \tag{2.5}$$

for all $t \in \mathbb{R}^+$, for some $\varepsilon > 0$, where $\overrightarrow{b}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ is a continuous vector function, $A(t) = (a_{jk}(t))_{n\times n}$, $a_{jk}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ are continuous and uniformly bounded functions, then there exists a unique solution $\overrightarrow{y_0}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ of (1.4) and a positive constant $\widetilde{L}$ such that

$$\|\overrightarrow{y}(t) - \overrightarrow{y_0}(t)\| \leq \widetilde{L}\varepsilon, \tag{2.6}$$

for all $t \in \mathbb{R}^+$, where $\widetilde{L} = \widetilde{K}_1\widetilde{\alpha}_1^{-1} + \widetilde{K}_2\widetilde{\alpha}_2^{-1}$.

Before providing the proof of this theorem, we first present the following Lemma 2.2 (for a proof see [21]).

Lemma 2.2 (cf. [21])**.** *Suppose that the linear differential equation (2.1) has an exponential dichotomy on J with projections $P(t)$, constants K_1, K_2 and exponents α_1, α_2. If $\overrightarrow{y}(t) : \mathbb{R} \to \mathbb{C}^n$ is a continuously differentiable vector function satisfying the differential inequality*

$$\|\overrightarrow{y}'(t) - A(t)\overrightarrow{y}(t) - \overrightarrow{b}(t)\| \leq \varepsilon, \tag{2.7}$$

for all $t \in \mathbb{R}$ and for some $\varepsilon > 0$, where $\overrightarrow{b}(t) : \mathbb{R} \to \mathbb{C}^n$ is a continuous vector function, $A(t) = (a_{jk}(t))_{n\times n}$, $a_{jk}(t) : \mathbb{R} \to \mathbb{C}^n$ are continuous and uniformly bounded functions, then there exists a unique solution $\overrightarrow{y_0}(t) : \mathbb{R} \to \mathbb{C}^n$ of the linear differential equation $\overrightarrow{y}'(t) = A(t)\overrightarrow{y}(t) + \overrightarrow{b}(t)$ and a positive constant L such that

$$\|\overrightarrow{y}(t) - \overrightarrow{y_0}(t)\| \leq L\varepsilon, \tag{2.8}$$

where $L = K_1\alpha_1^{-1} + K_2\alpha_2^{-1}$.

Proof. **(Proof of the Theorem 2.1)**. Let $t = e^\tau$ and $\overrightarrow{z} : \mathbb{R} \to \mathbb{C}^n$ be given by $\overrightarrow{z}(\tau) = \overrightarrow{y}(e^\tau)$. Then

$$\overrightarrow{z}'(\tau) = \frac{d\overrightarrow{z}(\tau)}{d\tau} = e^\tau \frac{d\overrightarrow{y}}{dt}(e^\tau) = t\overrightarrow{y}'(t)$$

and

$$\overrightarrow{z}'(\tau) - A(e^\tau)\overrightarrow{z}(\tau) - \overrightarrow{b}(e^\tau) = t\overrightarrow{y}'(t) - A(t)\overrightarrow{y}(t) - \overrightarrow{b}(t).$$

From the assumption (2.5), we obtain

$$\|\overrightarrow{z}'(\tau) - A(e^\tau)\overrightarrow{z}(\tau) - \overrightarrow{b}(e^\tau)\| \le \varepsilon,$$

for all $\tau \in \mathbb{R}$ and for some $\varepsilon > 0$.

By the assumption of the Lemma 2.2 and this theorem, there exists a differential vector function $\overrightarrow{z_0}(t) : \mathbb{R} \to \mathbb{C}^n$ such that

$$\overrightarrow{z}'(\tau) = A(e^\tau)\overrightarrow{z}(\tau) + \overrightarrow{b}(e^\tau),$$

and

$$\|\overrightarrow{z}(\tau) - \overrightarrow{z_0}(\tau)\| \le \widetilde{L}\varepsilon, \quad \forall \tau \in \mathbb{R}.$$

Then the function $\overrightarrow{y_0}(t) = \overrightarrow{z_0}(\ln t)$ satisfies

$$\begin{aligned} \overrightarrow{y_0}'(t) &= \frac{1}{t}\frac{d\overrightarrow{z_0}}{d\tau}(\ln t) = \frac{1}{t}[A(e^{\ln t})\overrightarrow{z_0}(\ln t) + \overrightarrow{b}(e^{\ln t})] \\ &= \frac{1}{t}[A(t)\overrightarrow{y_0}(\ln t) + \overrightarrow{b}(t)], \end{aligned}$$

i.e.,

$$t\overrightarrow{y_0}'(t) = A(t)\overrightarrow{y_0}(t) + \overrightarrow{b}(t)$$

with

$$\|\overrightarrow{y}(t) - \overrightarrow{y_0}(t)\| = \|\overrightarrow{z}(\ln t) - \overrightarrow{z_0}(\ln t)\| \le \widetilde{L}\varepsilon$$

for all $t \in \mathbb{R}^+$. This completes the proof of the Theorem. □

To give a corollary of Theorem 2.1, we need the following Lemmas 2.3 and 2.4, which were proved in [21]:

Lemma 2.3 (cf. [21])**.** *Suppose that $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix whose eigenvalues have nonzero real parts. Then the homogeneous differential equation $\overrightarrow{y}'(t) = A\overrightarrow{y}(t)$ has an exponential dichotomy on J.*

Proof. This Lemma was proved in [21], however we prove it again for completeness and convenience. Assume that A has d distinct eigenvalues λ_μ with algebraic multiplicity n_μ and geometric multiplicity m_μ, where $\mu \in \{1, 2, \cdots, d\}$ and denote by $\mathrm{Re}(\lambda_\mu)$ their real part. Choose a nonsingular matrix $N \in \mathbb{C}^{n\times n}$ such that $J = N^{-1}AN$, where J is the Jordan form matrix of the form

$$J = \begin{pmatrix} J_{11} & & & & \\ & \ddots & & O & \\ & & J_{\mu\nu} & & \\ & O & & \ddots & \\ & & & & J_{dm_\mu} \end{pmatrix}, \quad J_{\mu\nu} = \begin{pmatrix} \lambda_\mu & 1 & 0 & \cdots & 0 \\ 0 & \lambda_\mu & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_\mu \end{pmatrix},$$

and the Jordan block $J_{\mu\nu}$ is an $\nu\times\nu$ matrix for each $\mu\in\{1,2,\cdots,d\}$ and $\nu\in\{1,2,\cdots,m_\mu\}$, and $\sum_{\nu=1}^{m_\mu}\nu=n_\mu$ for any $\mu\in\{1,2,\cdots,d\}$.

Note that $e^A=I+A+\frac{A^2}{2!}+\cdots+\frac{A^n}{n!}+\cdots$, where I denotes the unite matrix in $\mathbb{C}^{n\times n}$. Then the fundamental matrix solution $X(t)$ for the differential equation $\overrightarrow{y}'(t)=A\overrightarrow{y}(t)$ can be expressed in the form

$$X(t)=e^{At}=Ne^{Jt}N^{-1}. \tag{2.9}$$

If we set

$$e^{Jt}=\begin{pmatrix} e^{J_{11}t} & & & & \\ & \ddots & & O & \\ & & e^{J_{\mu\nu}t} & & \\ & O & & \ddots & \\ & & & & e^{J_{dm_\mu}t}\end{pmatrix},\quad e^{J_{\mu\nu}t}=e^{\lambda_\mu t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{\nu-1}}{(\nu-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{\nu-2}}{(\nu-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}.$$

Choose a diagonal matrix $P=\mathrm{diag}(P_{11},\cdots,P_{\mu\nu},\cdots,P_{dm_\mu})$, where the Jordan block $P_{\mu\nu}$ is an $\nu\times\nu$ zero matrix or unite matrix corresponding with $\mathrm{Re}(\lambda_\mu)<0$ or $\mathrm{Re}(\lambda_\mu)>0$ for for each $\mu\in\{1,2,\cdots,d\}$ and $\nu\in\{1,2,\cdots,m_\mu\}$.

Let $Q:=NPN^{-1}$, then

$$\begin{aligned} QX(t)X^{-1}(s) &= QNe^{J(t-s)}N^{-1}=NPN^{-1}Ne^{J(t-s)}N^{-1} \\ &= Ne^{J(t-s)}PN^{-1}=Ne^{J(t-s)}N^{-1}NPN^{-1} \\ &= X(t)X^{-1}(s)Q,\quad \forall t,s\in J, \end{aligned}\tag{2.10}$$

and for all $t\ge s$, it follows that

$$\begin{aligned} \|X(t)X^{-1}(s)Q\| &= \|Ne^{J(t-s)}N^{-1}NPN^{-1}\| \\ &= \|Ne^{J(t-s)}PN^{-1}\| \\ &\le \|N\|\|N^{-1}\|\|e^{J(t-s)}P\| \\ &\le \|N\|\|N^{-1}\|\max_{\mathrm{Re}(\lambda_\mu)<0}e^{\lambda_\mu(t-s)}\max_{\mathrm{Re}(\lambda_\mu)<0}\sum_{i=0}^{m_\mu}\frac{(t-s)^i}{i!}. \end{aligned}\tag{2.11}$$

For a sufficiently small nonnegative number δ, we know that

$$M_1:=e^{-\delta(t-s)}\max_{\mathrm{Re}(\lambda_\mu)<0}\sum_{i=0}^{m_\mu}\frac{(t-s)^i}{i!}$$

is bounded for all $t\ge s$. Let $\lambda^-:=\max_{\mathrm{Re}(\lambda_\mu)<0}\{\lambda_\mu\}+\delta$, then by (2.11), we have

$$\|X(t)X^{-1}(s)Q\|\le\|N\|\|N^{-1}\|M_1e^{(t-s)\lambda^-},\quad t\ge s. \tag{2.12}$$

Similarly for all $t \leq s$ and $\delta > 0$, we have

$$\begin{aligned} \|X(t)X^{-1}(s)(I-Q)\| &= \|Ne^{J(t-s)}N^{-1}(I-NPN^{-1})\| \\ &= \|Ne^{J(t-s)}(I-P)N^{-1}\| \\ &\leq \|N\|\|N^{-1}\|M_2 e^{(t-s)\lambda^+}, \end{aligned} \tag{2.13}$$

where

$$M_2 := e^{-\delta(s-t)} \max_{\mathrm{Re}(\lambda_\mu)>0} \sum_{i=0}^{m_\mu} \frac{(s-t)^i}{i!} \quad and \quad \lambda^+ := \max_{\mathrm{Re}(\lambda_\mu)>0} \{\lambda_\mu\} - \delta.$$

Thus, by (2.10), (2.11), (2.12) and (2.13), we know that $\overrightarrow{y}'(t) = A\overrightarrow{y}(t)$ has an exponential dichotomy on J with projections $P(t) \equiv Q$, constants $\widehat{K}_1 = \|N\|\|N^{-1}\|M_1, \widehat{K}_2 = \|N\|\|N^{-1}\|M_2$ and exponents $\widehat{\alpha}_1 = -\lambda^-$, $\widehat{\alpha}_2 = \lambda^+$. This completes the proof of the Lemma. □

Lemma 2.4 (cf. [21])**.** *Suppose that $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix whose eigenvalues have nonzero real parts. If $\overrightarrow{y}(t) : \mathbb{R} \to \mathbb{C}^n$ is a continuously differentiable vector function satisfying the differential inequality*

$$\|\overrightarrow{y}'(t) - A\overrightarrow{y}(t) - \overrightarrow{b}(t)\| \leq \varepsilon, \tag{2.14}$$

for all $t \in \mathbb{R}$, for some $\varepsilon > 0$, where $\overrightarrow{b}(t) : \mathbb{R} \to \mathbb{C}^n$ is a continuous vector function, then there exists a unique solution $\overrightarrow{y_0}(t) : \mathbb{R} \to \mathbb{C}^n$ of (1.2) and a positive constant M such that

$$\|\overrightarrow{y}(t) - \overrightarrow{y_0}(t)\| \leq M\varepsilon, \tag{2.15}$$

for all $t \in \mathbb{R}$.

Corollary 2.5. *Suppose that $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix whose eigenvalues have nonzero real parts. If $\overrightarrow{y}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ is a continuously differentiable vector function satisfying the differential inequality*

$$\|t\overrightarrow{y}'(t) - A\overrightarrow{y}(t) - \overrightarrow{b}(t)\| \leq \varepsilon, \tag{2.16}$$

for all $t \in \mathbb{R}^+$, for some $\varepsilon > 0$, where $\overrightarrow{b}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ is a continuous vector function, then there exists a unique solution $\overrightarrow{y_0}(t) : \mathbb{R}^+ \to \mathbb{C}^n$ of (1.3) and a positive constant $\widetilde{M}$ such that

$$\|\overrightarrow{y}(t) - \overrightarrow{y_0}(t)\| \leq \widetilde{M}\varepsilon, \tag{2.17}$$

for all $t \in \mathbb{R}^+$.

Proof. The proof of Corollary 2.5 is similar to the proof of Theorem 2.1, and by Lemmas 2.3 and 2.4, the differential equation (1.3) has the Hyers-Ulam stability with $\widetilde{M} = \|N\|\|N^{-1}\|(\frac{M_1}{\lambda^-} + \frac{M_2}{\lambda^+})$. This completes the proof of the corollary. □

3 Some Examples

Example 3.1. *Consider a system of generalized Euler differential equations of first order with variable coefficients in the following form*

$$t\overrightarrow{y}'(t) = A(t)\overrightarrow{y}(t) + \overrightarrow{b}(t), \tag{3.1}$$

where

$$\overrightarrow{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 2+\cos\ln t & 4 \\ 0 & -2+\cos\ln t \end{pmatrix}, \quad \overrightarrow{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

By Theorem 2.1, putting $t = e^\tau$, we obtain

$$A(e^\tau) = \begin{pmatrix} 2+\cos\tau & 4 \\ 0 & -2+\cos\tau \end{pmatrix}.$$

Following some computation, we have

$$Y(\tau) = \begin{pmatrix} e^{2\tau+\sin\tau} & e^{-2\tau+\sin\tau} - e^{2\tau+\sin\tau} \\ 0 & e^{-2\tau+\sin\tau} \end{pmatrix},$$

and

$$Y(\tau)Y^{-1}(s) = \begin{pmatrix} e^{2(\tau-s)+\sin\tau-\sin s} & e^{-2(\tau-s)+\sin\tau-\sin s} - e^{2(\tau-s)+\sin\tau-\sin s} \\ 0 & e^{-2(\tau-s)+\sin\tau-\sin s} \end{pmatrix}.$$

Choosing $P(\tau) \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, we can verify that

$$P(\tau)Y(\tau)Y^{-1}(s) = Y(\tau)Y^{-1}(s)P(s) = \begin{pmatrix} 0 & e^{-2(\tau-s)+\sin\tau-\sin s} \\ 0 & e^{-2(\tau-s)+\sin\tau-\sin s} \end{pmatrix},$$

and

$$\|Y(\tau)Y^{-1}(s)P(s)\| \le e^2 e^{-2(\tau-s)}, \quad \tau \ge s,$$

$$\|Y(\tau)Y^{-1}(s)(I-P(s))\| \le 2e^2 e^{2(\tau-s)}, \quad \tau \le s.$$

Thus the differential equation $\overrightarrow{y}'(\tau) = A(e^\tau)\overrightarrow{y}(\tau)$ has an exponential dichotomy on J with $\widetilde{K}_1 = e^2, \widetilde{K}_2 = 2e^2, \widetilde{\alpha}_1^{-1} = \widetilde{\alpha}_2^{-1} = 2$. By Theorem 2.1, equation (3.1) satisfies the Hyers-Ulam stability with $\widetilde{L} = \widetilde{K}_1\widetilde{\alpha}_1^{-1} + \widetilde{K}_2\widetilde{\alpha}_2^{-1} = \frac{3}{2}e^2$.

Example 3.2. *Consider a system of Euler differential equations of first order in the following form*

$$t\overrightarrow{y}'(t) = A\overrightarrow{y}(t) + \overrightarrow{b}(t), \tag{3.2}$$

where

$$\overrightarrow{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}, \quad A = \begin{pmatrix} -3 & -4 & 2 \\ -3 & -5 & 3 \\ -7 & -10 & 6 \end{pmatrix}, \quad \overrightarrow{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix}.$$

Since the matrix A has three eigenvalues $-1, 1$, and -2, there exist nonsingular matrices

$$N = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad N^{-1} = \begin{pmatrix} -1 & -4 & 2 \\ -1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

such that $J = N^{-1}AN$. According to Corollary 2.5, equation (3.2) satisfies the Hyers-Ulam stability with $\|N\|\|N^{-1}\|(\frac{M_1}{\lambda^-} + \frac{M_2}{\lambda^+}) = 84$.

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