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Hybrid Iterative Method for Solving a System of Generalized Equilibrium Problems, Generalized Mixed Equilibrium Problems and Common Fixed Point Problems in Hilbert Spaces

Benjawan Rodjanadid

School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima, 30000, Thailand
e-mail : benjawan@sut.ac.th

Abstract : In this paper, we propose a hybrid iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem (GMEP), the solutions of a general system of equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Then, we prove that the sequence converges strongly to a common element of the above three sets. Furthermore, we apply our result to prove four new strong convergence theorems in fixed point problems, mixed equilibrium problems, generalized equilibrium problems , equilibrium problems and variational inequality.

Keywords : generalized mixed equilibrium problems; equilibrium problems; nonexpansive mappings; fixed point; inverse-strongly monotone mapping; variational inequality; hybrid iterative method.
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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is

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called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$, and $\rho-$ *Lipschitzian mapping* if there exists a constant $\rho \ge 0$ such that $\|Tx - Ty\| \le \rho\|x - y\|$ for all $x, y \in C$, and a mapping $f : H \to H$ is called a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in H$. Denote the set of fixed point of T by $Fix(T)$, i.e., $Fix(T) = \{x \in C : Tx = x\}$. It is well known that if C is a nonempty closed bounded convex subset of H and $T : C \to C$ is nonexpansive, then $Fix(T) \neq \emptyset$. Recall also that a mapping $A : C \to H$ is called

- (i) *monotone* if $\langle Ax - Ay, x - y\rangle \ge 0$ for all $x, y \in C$,
- (ii) $\eta-$*strongly monotone* if there exists a constant $\eta \ge 0$ such that $\langle Ax-Ay, x-y\rangle \ge \eta\|x - y\|^2$ for all $x, y \in C$,
- (iii) $\delta-$*inverse strongly monotone* if there exists a positive real number δ such that $\langle Ax - Ay, x - y\rangle \ge \delta\|Ax - Ay\|^2$ for all $x, y \in C$.

We can see that if A is $\delta-$*inverse strongly monotone*, then A is monotone mapping.

In 2008, Peng and Yao [1] considered the following generalized mixed equilibrium problem of finding $x^* \in C$ such that

$$(GMEP) : \qquad \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \ge 0, \qquad \forall y \in C. \quad (1.1)$$

Where $A : C \to H$ is a nonlinear mapping, and $\varphi : C \to \mathbb{R}$ be a real value function and $\Phi : C \times C \to \mathbb{R}$ be a bifunction, i.e., $\Phi(x, x) = 0$ for each $x \in C$. The set of solutions for problem (1.1) is denoted by Ω.

In the case of $A \equiv 0$, problem (1.1) reduces to the classical mixed equilibrum problem (for short, MEP) of finding $x^* \in C$ such that

$$\Phi(x^*, y) + \varphi(y) - \varphi(x^*) \ge 0, \qquad \forall y \in C, \tag{1.2}$$

which was considered by Ceng and Yao [2]. Ω is denoted by $MEP(\Phi, \varphi)$.

In the case of $\varphi \equiv 0$, problem (1.1) reduces to the generalized equilibrium problem (for short, GEP) of finding $x^* \in C$ such that

$$\Phi(x^*, y) + \langle Ax^*, y - x^*\rangle \ge 0, \qquad \forall y \in C, \tag{1.3}$$

which was studied by Takahashi and Takahashi [3] and many other for instance, ([4],[5],[6]). The set of solution (1.3) is denoted by EP.

In the case of $\varphi \equiv 0$ and $A \equiv 0$, problem (1.1) reduces to the classical equilibrium problem (for short, EP) of finding $x^* \in C$ such that

$$\Phi(x^*, y) \ge 0, \qquad \forall y \in C. \tag{1.4}$$

The set of solution (1.4) is denoted by $EP(\Phi)$. Given a mapping $T : C \to H$, let $\Phi(x, y) = \langle Tx, y-x\rangle$ for all $x, y \in C$. Then $z \in EP(\Phi)$ if and only if $\langle Tz, y-z\rangle \ge 0$ for all $y \in C$.

In the case of $\Phi \equiv 0$ and $\varphi \equiv 0$, problem (1.1) reduces to the classical variational inequality of finding $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \ge 0, \qquad \forall y \in C. \tag{1.5}$$

The set of solution (1.5) is denoted by $VI(A,C)$.

The generalized mixed equilibrium problems includes, optimization problems, variational inequalities, the Nash equilibrium problem in noncooperative games and others; see, for example ([2],[3],[7]). Peng and Yao [1] obtained some strong convergence theorems for iterative schemes based on the hybrid method and the extragradient method for finding a common element of the set of solutions of problem (1.1), the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality.

Let $F : H \to H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator with constants $\kappa, \eta > 0$, and let $T : H \to H$ be nonexpansive such that $Fix(T) \neq \emptyset$. In 2001, Yamoda [8] studied the variational inequality problem and proposed a hybrid steepest - descent algorithm :

$$x_{n+1} = (I - \mu\lambda_n F)Tx_n \tag{1.6}$$

and proved the strong convergence, where $0 < \mu < \frac{2\eta}{\kappa^2}$ and the sequence $\{\lambda_n\}$ in $(0,1)$.

Recently, Ceng et al. [9] introduced and considered the following a hybrid iterative schemes below for finding a common element of the set of solution (1.1) and the set of fixed points of a finite family of nonexpansive mapping in a real Hilbert space:

$$\begin{cases}\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C\\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)W_n u_n, \forall n \geq 1,\end{cases} \tag{1.7}$$

where W_n is the $W-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. The concept of $W-$mapping was introduced in Alsushiba and Takahashi [10].

Very recently, Jeong [11] considered the generalized equilibrium problem $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases}G_1(\overline{x}, x) + \langle F_1\overline{y}, x - \overline{x}\rangle + \frac{1}{\mu_1}\langle \overline{x} - \overline{y}, x - \overline{x}\rangle \geq 0, \quad \forall x \in C,\\ G_2(\overline{y}, y) + \langle F_2\overline{x}, y - \overline{y}\rangle + \frac{1}{\mu_2}\langle \overline{y} - \overline{x}, y - \overline{y}\rangle \geq 0, \quad \forall y \in C,\end{cases} \tag{1.8}$$

where $G_1, G_2 : C \times C \to \mathbb{R}$ are two bifunctions, $F_1, F_2 : C \to H$ are two nonlinear and $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

In this paper, we introduce a hybrid iterative scheme by the general iterative method (3.1) for finding an element of the set of solutions of the generalized mixed equilibrium problem (1.1), the set of solutions of the generalized equilibrium problem (1.8) and the set of common fixed points of finitely many nonexpansive mappings in real Hilbert space, where$A, F_1, F_2 : C \to H$ be $\eta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively, and $F : C \to H$ is a $\kappa-$Lipschitzian and $\eta-$strongly monotone and then obtain a strong convergence theorem.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\|, \ \forall y \in C$. The mapping $P_C : x \to P_C(x)$ is called the *metric projection* of H onto C. We know that P_C is nonexpansive.

The following characterizes the projection P_C.

Lemma 2.1 ([12])**.** *Given $x \in H$ and $y \in C$. Then $P_C(x) = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

The following lemmas will be useful for proving our main results.

Lemma 2.2 ([12])**.** *For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Lemma 2.3 ([12])**.** *In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda) y\|,$$

for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Lemma 2.4 ([13])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} = (1 - \alpha_n) a_n + \alpha_n \beta_n, \ \forall n \geq 0$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions*

(i) $\{\alpha_n\} \subset [0, 1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n \to \infty} \beta_n \leq 0$.

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.5 ([14])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n,$$

for all integer $n \geq 0$ and

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.6 ([2]). *Let C be a nonempty closed convex subset of H , $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and let Φ be a bifunction of $C \times C$ in to $\mathbb{R}$ satisfy*

(A1) $\Phi(x, x) = 0$ *for all* $x \in C$*;*

(A2) Φ *is monotone, i.e.,* $\Phi(x, y) + \Phi(y, x) \le 0, \quad \forall x, y \in C$*;*

(A3) for all $x, y, z \in C, \quad \lim_{t\to 0} \Phi(tz + (1-t)x, y) \le \Phi(x, y)$*;*

(A4) for all $x \in C, \quad y \mapsto \Phi(x, y)$ *is convex and lower semicontinuous;*

(B1) for each $x \in H$ *and* $r > 0$*, there exists a bounded subset* $D_x \subset C$ *and* $y_x \in C$ *such that for any* $z \in C \backslash D_x$*,*

$$\Phi(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) C *is bounded set.*

Assume that either (B1) or (B2) holds. For $x \in C$ *and* $r > 0$*, define a mapping* $T_r^{(\Phi,\varphi)} : H \to C$ *as follows.*

$$T_r^{(\Phi,\varphi)}(x) := \{z \in C : \Phi(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z), \ \ \forall y \in C\}$$

for all $x \in H$*. Then , the following conditions hold:*

(i) For each $x \in H, T_r^{(\Phi,\varphi)}(x) \neq \emptyset$;

(ii) $T_r^{(\Phi,\varphi)}$ *is single-valued;*

(iii) $T_r^{(\Phi,\varphi)}$ *is firmly nonexpansive, i.e.,*

$$\|T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y\|^2 \le \langle T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y, x - y\rangle, \ \ \forall x, y \in H;$$

(iv) $Fix(T_r^{(\Phi,\varphi)}) = MEP(\Phi, \varphi)$ *;*

(v) $MEP(\Phi, \varphi)$ *is closed and convex.*

Remark 2.7. *If* $\varphi \equiv 0$ *then* $T_r^{(\Phi,\varphi)}$ *is rewritten as* T_r^{Φ}*.*

Lemma 2.8 ([11]). *Let C be a nonempty closed convex subset of H. let $G_1, G_2 : C \times C \longrightarrow \mathbb{R}$ be two a bifunctions satisfying conditions (A1)-(A4) and let the mapping $F_1, F_2 : C \longrightarrow H$ be ζ_1- inverse strongly monotone and ζ_2- inverse strongly monotone, respectively. Then, for given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution* (1.8) *if and only if $\overline{x}$ is a fixed point of the mapping $\Gamma : C \longrightarrow C$ defined by*

$$\Gamma(x) = T_{\mu_1}^{G_1}(T_{\mu_2}^{G_2}(x - \mu_2 F_2 x) - \mu_1 F_1 T_{\mu_2}^{G_2}(x - \mu_2 F_2 x)), \quad \forall x \in C,$$

where $\overline{y} = T_{\mu_2}^{G_2}(\overline{x} - \mu_2 F_2 \overline{x})$*.*

The set of fixed points of the mapping Γ is denoted by O.

Proposition 2.9 ([3])**.** *Let C, H, Φ, φ and $T_r^{(\Phi,\varphi)}$ be as in Lemma 2.6. Then the following holds:*

$$\|T_s^{(\Phi,\varphi)}x - T_t^{(\Phi,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Phi,\varphi)}x - T_t^{(\Phi,\varphi)}x, T_s^{(\Phi,\varphi)}x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.10 ([15])**.** *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of H. If T has a fixed point, then $I - T$ is demi-closed, that is, when $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I-T)x_n\}$ converges strongly to some y, it follows that $(I-T)x = y$.*

Let H be a real Hilbert space and C a nonempty closed convex subset of H. For a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_N$ and sequence $\{\lambda_{n,i}\}_{i=1}^N$ in $[0,1]$, Kangtunyakarn and Suantai [4] defined the mapping $K_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})U_{n,1},\\
U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1-\lambda_{n,3})U_{n,2},\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})U_{n,N-2},\\
K_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}
\end{aligned} \tag{2.1}$$

Such a mapping K_n is called the $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.

Definition 2.11 ([4])**.** Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, \ldots, N$. They define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \lambda_1T_1 + (1-\lambda_1)I,\\
U_2 &= \lambda_2T_2U_1 + (1-\lambda_2)U_1,\\
U_3 &= \lambda_3T_3U_2 + (1-\lambda_3)U_2,\\
&\vdots\\
U_{N-1} &= \lambda_{N-1}T_{N-1}U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
K = U_N &= \lambda_NT_NU_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}$$

Such a mapping K is called the $K-$*mapping* generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$.

Lemma 2.12 ([4]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N Fix(T_i) \neq \emptyset$ and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the $K-$mapping generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then $Fix(K) = \cap_{i=1}^N Fix(T_i)$.*

Lemma 2.13 ([4]). *Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$ $(i = 1, 2, \ldots, N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the $K-$mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for every $x \in C$,*

$$\lim_{n\to\infty} \|K_n x - Kx\| = 0.$$

Lemma 2.14 ([16]). *Let $\{x_n\}$ be a bounded sequence in a Hilbert space H. Then there exits $L > 0$ such that*

$$\|K_{n+1}x_{n+1} - K_n x_n\| \leq \|x_{n+1} - x_n\| + L\sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|, \quad \forall n \geq 0. \tag{2.2}$$

Lemma 2.15 ([17]). *Let λ be a number in $(0,1]$ and let $\mu > 0$. Let $F : H \to H$ be an operator on H such that, for some constants $\kappa, \eta > 0$, F is $\kappa-$Lipschitzain and $\eta-$strongly monotone. Associating with a nonexpansive mapping $T : H \to H$, define the mapping $T^\lambda : H \to H$ by*

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{\kappa^2}$, that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

3 Main Results

We are now in a position to prove the main result of this paper.

Theorem 3.1. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R}$ a lower semicontinuous and convex function with assumption (B1) or (B2). Let $A, F_1, F_2 : C \to H$ be $\delta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\Delta = \cap_{i=1}^N Fix(T_i) \cap \Omega \cap O \neq \emptyset$. Let $F : C \to H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator with constants $\kappa, \eta > 0$ and $f : C \to C$ a $\rho-$Lipschitzian mapping with constant*

$\rho \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, $\{r_n\}$ is a sequence in $(0, 2\delta]$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a,b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $x_1 \in C$ arbitrarily. Suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$\begin{cases} u_n = T_{r_n}^{(\Phi,\varphi)}(x_n - r_n A x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F) K_n y_n, \forall n \geq 1, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

Then both $\{x_n\}$ and $\{u_n\}$ converges strongly to $x^* = P_\Delta(I - \mu F + \gamma f)x^*$ which solves the following variational inequality

$$\langle(\mu F - \gamma f)x^*, x^* - x\rangle \leq 0, \qquad \forall x \in \Delta. \tag{3.2}$$

and (x^*, y^*) is a solution of problem (1.8) where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. Let $x, y \in C$. Since A is $\delta-$inverse strongly monotone and $r_n \in (0, 2\delta)$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - 2r_n\delta\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &= \|x-y\|^2 + r_n(r_n - 2\delta)\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2, \end{aligned}$$

then the mapping $I - r_n A$ is a nonexpansive mapping, and so are $I - \mu_1 F_1$ and $I - \mu_2 F_2$, provided $\mu_1 \in (0, 2\zeta_1)$ and $\mu_2 \in (0, 2\zeta_2)$, respectively.
Since $F : C \to H$ is a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator and $f : C \to C$ is a $\rho-$Lipschitzian mapping, we have

$$\begin{aligned} \|(I - \mu F)x - (I - \mu F)y\|^2 &= \|x-y\|^2 - 2\mu\langle x - y, Fx - Fy\rangle + \mu^2\|Fx - Fy\|^2 \\ &\leq \|x-y\|^2 - 2\mu\eta\|x-y\|^2 + \mu^2\kappa^2\|x-y\|^2 \\ &= (1 - 2\mu\eta + \mu^2\kappa^2)\|x-y\|^2 \\ &= (1-\tau)^2\|x-y\|^2, \end{aligned}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$, and hence

$$\begin{aligned}
&\|P_\Delta(I - \mu F + \gamma f)(x) - P_\Delta(I - \mu F + \gamma f)(y)\| \\
&\leq \|(I - \mu F + \gamma f)(x) - (I - \mu F + \gamma f)(y)\| \\
&\leq \|(I - \mu F)x - (I - \mu F)y\| + \gamma\|f(x) - f(y)\| \\
&\leq (1 - \tau)\|x - y\| + \gamma\rho\|x - y\| \\
&= (1 - (\tau - \gamma\rho))\|x - y\|,
\end{aligned}$$

for all $x, y \in C$. Since $0 \leq \gamma\rho < \tau < 1$, $1 - (\tau - \gamma\rho) \in [0, 1)$. Therefore $P_\Delta(I - \mu F + \gamma f)$ is a contraction of H into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = P_\Delta(I - \mu F + \gamma f)x^*$.

We shall divide the proof into several steps.

step 1. We shall show that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded.

Let $p \in \Delta = \cap_{i=1}^N Fix(T_i) \cap \Omega \cap O$, arbitrarily. Since $p = T_{r_n}^{(\Phi,\varphi)}(p - r_n Ap)$ and $T_{r_n}^{(\Phi,\varphi)}$ and $(I - r_n A)$ are nonexpansive, we obtain that for any $n \geq 1$,

$$\begin{aligned}
\|u_n - p\| &= \|T_{r_n}^{(\Phi,\varphi)}(x_n - r_n Ax_n) - T_{r_n}^{(\Phi,\varphi)}(p - r_n Ap)\| \\
&\leq \|(x_n - r_n Ax_n) - (p - r_n Ap)\| \\
&\leq \|x_n - p\|. \qquad (3.3)
\end{aligned}$$

Putting $z_n = T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)$ and $z = T_{\mu_2}^{G_2}(p - \mu_2 F_2 p)$, we have

$$\begin{aligned}
\|z_n - z\| &= \|T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - T_{\mu_2}^{G_2}(p - \mu_2 F_2 p)\| \\
&\leq \|(u_n - \mu_2 F_2 u_n) - (p - \mu_2 F_2 p)\| \\
&\leq \|u_n - p\|. \qquad (3.4)
\end{aligned}$$

And since $p = T_{\mu_1}^{G_1}(z - \mu_1 F_1 z)$, we know that for any $n \geq 1$,

$$\begin{aligned}
\|y_n - p\| &= \|T_{\mu_1}^{G_1}(z_n - \mu_1 F_1 z_n) - T_{\mu_1}^{G_1}(z - \mu_1 F_1 z)\| \\
&\leq \|(z_n - \mu_1 F_1 z_n) - (z - \mu_1 F_1 z)\| \\
&\leq \|z_n - z\| \\
&\leq \|u_n - p\|. \qquad (3.5)
\end{aligned}$$

Furthermore, from (3.1), (3.3) and (3.5) we have

$$\begin{aligned}
&\|x_{n+1} - p\| = \|(\alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n\mu F)K_n y_n) - p\| \\
&= \|\alpha_n(\gamma f(x_n) - \mu Fp) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n\mu F)K_n y_n - ((1 - \beta_n)I \\
&\qquad - \alpha_n\mu F)K_n p\| \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Fp\| + \beta_n\|x_n - p\| + \|((1 - \beta_n)I - \alpha_n\mu F)K_n y_n - ((1 - \beta_n)I \\
&\qquad - \alpha_n\mu F)K_n p\| \\
&= \alpha_n\|\gamma f(x_n) - \mu Fp\| + \beta_n\|x_n - p\| + (1 - \beta_n)\|I - \frac{\alpha_n}{1 - \beta_n}\mu F)K_n y_n \\
&\qquad - (I - \frac{\alpha_n}{1 - \beta_n}\mu F)K_n p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)(1-\frac{\alpha_n\tau}{1-\beta_n}\|y_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-\mu Fp\| \\
&\leq (1-\beta_n-\alpha_n\tau)\|u_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-\mu Fp\| \\
&\leq (1-\alpha_n\tau)\|x_n-p\|+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-\mu Fp\| \\
&\leq (1-\alpha_n\tau)\|x_n-p\|+\alpha_n\gamma\rho\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Fp\| \\
&= (1-(\tau-\gamma\rho)\alpha_n)\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Fp\| \\
&= (1-(\tau-\gamma\rho)\alpha_n)\|x_n-p\|+(\tau-\gamma\rho)\alpha_n\frac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma\rho} \qquad (3.6)
\end{aligned}
$$

It follows from (3.6) induction that

$$\|x_n-p\|\leq M, \quad \forall n\geq 1$$

where $M=\max\{\|x_0-p\|,\frac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma\rho}\}$. Hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\},\{y_n\},\{Ax_n\},\{K_ny_n\}$ and $\{f(x_n)\}$ are all bounded.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Set $x_{n+1}=\beta_nx_n+(1-\beta_n)V_n$ for all $n\geq 1$. Then from the definition of V_n, we obtain

$$
\begin{aligned}
&\|V_{n+1}-V_n\| \\
&= \|\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}\mu F)K_{n+1}y_{n+1}}{1-\beta_{n+1}} \\
&\quad -\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n\mu F)K_ny_n}{1-\beta_n}\| \\
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)+K_{n+1}y_{n+1}-K_ny_n \\
&\quad +\frac{\alpha_n}{1-\beta_n}\mu FK_ny_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\mu FK_{n+1}y_{n+1}\| \\
&\leq \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-\mu FK_{n+1}y_{n+1})+\frac{\alpha_n}{1-\beta_n}(\mu FK_ny_n-\gamma f(x_n))\| \\
&\quad +\|K_{n+1}y_{n+1}-K_ny_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-\mu FK_{n+1}y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\mu FK_ny_n-\gamma f(x_n)\| \\
&\quad +\|K_{n+1}y_{n+1}-K_ny_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|f(x_{n+1})\|+\mu\|FK_{n+1}y_{n+1}\|) \\
&\quad +\frac{\alpha_n}{1-\beta_n}(\mu\|FK_ny_n\|+\gamma\|f(x_n)\|)+\|K_{n+1}y_{n+1}-K_ny_n\|. \qquad (3.7)
\end{aligned}
$$

From Lemma 2.14, there exist $L > 0$ such that

$$\begin{aligned}
&\|V_{n+1} - V_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|f(x_{n+1})\| + \mu\|FK_{n+1}y_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\mu\|FK_n y_n\| + \gamma\|f(x_n)\|) \\
&\quad + \|y_{n+1} - y_n\| + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \qquad (3.8)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\|y_{n+1} - y_n\|^2 \\
&= \|T_{\mu_1}^{G_1}(z_{n+1} - \mu_1 F_1 z_{n+1}) - T_{\mu_1}^{G_1}(z_n - \mu_1 F_1 z_n)\|^2 \\
&\le \|(z_{n+1} - z_n) - \mu_1(F_1 z_{n+1} - F_1 z_n)\|^2 \\
&= \|z_{n+1} - z_n\|^2 - 2\mu_1\langle z_{n+1} - z_n, F_1 z_{n+1} - F_1 z_n\rangle + \mu_1^2\|F_1 z_{n+1} - F_1 z_n\|^2 \\
&\le \|z_{n+1} - z_n\|^2 - 2\mu_1\zeta_1\|F_1 z_{n+1} - F_1 z_n\|^2 + \mu_1^2\|F_1 z_{n+1} - F_1 z_n\|^2 \\
&= \|z_{n+1} - z_n\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_{n+1} - F_1 z_n\|^2 \\
&\le \|z_{n+1} - z_n\|^2 \\
&= \|T_{\mu_2}^{G_2}(u_{n+1} - \mu_2 F_2 u_{n+1}) - T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)\|^2 \\
&\le \|(u_{n+1} - u_n) - \mu_2(F_2 u_{n+1} - F_2 u_n)\|^2 \\
&= \|u_{n+1} - u_n\|^2 - 2\mu_2\langle u_{n+1} - u_n, F_2 u_{n+1} - F_2 u_n\rangle + \mu_2^2\|F_2 u_{n+1} - F_2 u_n\|^2 \\
&\le \|u_{n+1} - u_n\|^2 - 2\mu_2\zeta_2\|F_2 u_{n+1} - F_2 u_n\|^2 + \mu_2^2\|F_2 u_{n+1} - F_2 u_n\|^2 \\
&= \|u_{n+1} - u_n\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|F_2 u_{n+1} - F_2 u_n\|^2 \\
&\le \|u_{n+1} - u_n\|^2. \qquad (3.9)
\end{aligned}$$

And

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&= \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_n}^{(\Phi,\varphi)}(x_n - r_n Ax_n)\| \\
&\le \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_{n+1}}^{(\Phi,\varphi)}(x_n - r_n Ax_n)\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n - r_n Ax_n) - T_{r_n}^{(\Phi,\varphi)}(x_n - r_n Ax_n)\| \\
&\le \|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_n Ax_n)\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n - r_n Ax_n) - T_{r_n}^{(\Phi,\varphi)}(x_n - r_n Ax_n)\| \\
&= \|x_{n+1} - x_n - r_{n+1}Ax_{n+1} + r_{n+1}Ax_n - r_{n+1}Ax_n + r_n Ax_n\| \\
&\quad + \|T_{r_{n+1}}^{(\Phi,\varphi)}(x_n - r_n Ax_n) - T_{r_n}^{(\Phi,\varphi)}(x_n - r_n Ax_n)\| \\
&\le \|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_{n+1}Ax_n)\| + \|r_n Ax_n - r_{n+1}Ax_n\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|T^{(\Phi,\varphi)}_{r_{n+1}}(x_n - r_nAx_n) - T^{(\Phi,\varphi)}_{r_n}(x_n - r_nAx_n)\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| \\
&\quad + \|T^{(\Phi,\varphi)}_{r_{n+1}}(x_n - r_nAx_n) - T^{(\Phi,\varphi)}_{r_n}(x_n - r_nAx_n)\|. \qquad (3.10)
\end{aligned}
$$

It follows from (3.9) and (3.10) that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|u_{n+1} - u_n\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| \\
&\quad + \|T^{(\Phi,\varphi)}_{r_{n+1}}(x_n - r_nAx_n) - T^{(\Phi,\varphi)}_{r_n}(x_n - r_nAx_n)\|. \qquad (3.11)
\end{aligned}
$$

Without loss of generality, let us assume that there exists a real number k such that $r_n > k > 0$ for all n. Utilizing Proposition 2.9, we have

$$
\begin{aligned}
&\|T^{(\Phi,\varphi)}_{r_{n+1}}(x_n - r_nAx_n) - T^{(\Phi,\varphi)}_{r_n}(x_n - r_nAx_n)\| \\
&\quad \leq \frac{|r_{n+1} - r_n|}{r_{n+1}}\|T^{(\Phi,\varphi)}_{r_{n+1}}(I - r_nA)x_n\| \\
&\quad \leq \frac{|r_{n+1} - r_n|}{k}\|T^{(\Phi,\varphi)}_{r_{n+1}}(I - r_nA)x_n\|. \qquad (3.12)
\end{aligned}
$$

By (3.11) and (3.12) , we have

$$
\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \frac{|r_{n+1} - r_n|}{k}\|T^{(\Phi,\varphi)}_{r_{n+1}}(I - r_nA)x_n\|. \qquad (3.13)
$$

Using (3.8) and (3.13), we get

$$
\begin{aligned}
&\|V_{n+1} - V_n\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\gamma\|f(x_{n+1})\| + \mu\|FK_{n+1}y_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n}(\mu\|FK_ny_n\| + \gamma\|f(x_n)\|) \\
&\quad + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| + (\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| \\
&\quad + \frac{|r_{n+1} - r_n|}{k}\|T^{(\Phi,\varphi)}_{r_{n+1}}(I - r_nA)x_n\|).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|V_{n+1} - V_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\gamma\|f(x_{n+1}) + \mu\|FK_{n+1}y_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n}(\mu\|FK_ny_n\| \\
&\quad + \gamma\|f(x_n)\|) + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i} + |r_{n+1} - r_n|\|Ax_n\| \\
&\quad + \frac{|r_{n+1} - r_n|}{k}\|T^{(\Phi,\varphi)}_{r_{n+1}}(I - r_nA)x_n\| \qquad (3.14)
\end{aligned}
$$

Applying the conditions (i), (iii) and (iv) and taking the superior limit as $n \to \infty$ to (3.14), we have

$$\limsup_{n\to\infty}(\|V_{n+1} - V_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.5, we have $\lim_{n\to\infty} \|V_n - x_n\| = 0$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|V_n - x_n\| = 0. \tag{3.15}$$

From (3.10), (3.12), (3.15) and condition(iii), we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0 \tag{3.16}$$

Step 3. We will show that $\lim_{n\to\infty} \|x_n - u_n\| = 0$, $\lim_{n\to\infty} \|u_n - y_n\| = 0$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|K_n y_n - y_n\| = 0$.
Since $x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)K_n y_n$, we obtain

$$\begin{aligned}\|x_n - K_n y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - K_n y_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - \mu F K_n y_n\| + \beta_n\|x_n - K_n y_n\|,\end{aligned}$$

and hence

$$\|x_n - K_n y_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - \mu F K_n y_n\|. \tag{3.17}$$

Form (3.15) and condition (i), we obtain

$$\lim_{n\to\infty} \|x_n - K_n y_n\| = 0. \tag{3.18}$$

On the other hand, from (3.3) and (3.5) we get

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)K_n y_n - p\|^2 \\ &= \|(\alpha_n \gamma f(x_n) - \alpha_n \mu F p) + (\beta_n x_n - \beta_n p) + ((1 - \beta_n)I - \alpha_n \mu F)K_n y_n \\ &\quad - ((1 - \beta_n)I - \alpha_n \mu F)K_n p\|^2 \\ &\leq [\|\alpha_n \gamma f(x_n) - \alpha_n \mu F p\| + \beta_n\|x_n - p\| + (1 - \beta_n)\|(I - \frac{\alpha_n}{1-\beta_n}\mu F)K_n y_n \\ &\quad - (I - \frac{\alpha_n}{1-\beta_n}\mu F)K_n p\|]^2 \\ &\leq (\alpha_n\|\gamma f(x_n) - \mu F p\| + \beta_n\|x_n - p\| + (1 - \beta_n)(1 - \frac{\alpha_n \tau}{1-\beta_n})\|y_n - p\|)^2 \\ &\leq \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n \tau)\|y_n - p\|^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)\|T^{G_1}_{\mu_1}(z_n - \mu_1 F_1 z_n) \\
&\quad - T^{G_1}_{\mu_1}(z - \mu_1 F_1 z)\|^2 \\
&\le \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)\|(z_n - z) - \mu_1(F_1 z_n - F_1 z)\|^2 \\
&\le \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|z_n - z\|^2 \\
&\quad - 2\mu_1\langle z_n - z, F_1 z_n - F_1 z\rangle + \mu_1^2\|F_1 z_n - F_1 z\|^2] \\
&\le \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|z_n - z\|^2 \\
&\quad - 2\mu_1\zeta_1\|F_1 z_n - F_1 z\|^2 + \mu_1^2\|F_1 z_n - F_1 z\|^2] \\
&= \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|z_n - z\|^2 \\
&\quad + \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_n - F_1 z\|^2] \\
&= \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|T^{G_2}_{\mu_2}(u_n - \mu_2 F_2 u_n) \\
&\quad - T^{G_2}_{\mu_2}(p - \mu_2 F_2 p)\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_n - F_1 z\|^2] \\
&\le \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|u_n - p\|^2 \\
&\quad + \mu_2(\mu_2 - 2\zeta_2)\|F_2 u_n - F_2 p\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_n - F_1 z\|^2] \\
&= \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|T^{(\Phi,\varphi)}_{r_n}(x_n - r_n A x_n) \\
&\quad - T^{(\Phi,\varphi)}_{r_n}(p - r_n A p)\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|F_2 u_n - F_2 p\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_n - F_1 z\|^2] \\
&\le \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)[\|x_n - p\|^2 + r_n(r_n - 2\delta) \\
&\quad \times \|A x_n - A p\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|F_2 u_n - F_2 p\|^2 + \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_n - F_1 z\|^2] \\
&= \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\tau)r_n \times (r_n - 2\delta)\|A x_n - A p\|^2 + (1-\beta_n-\alpha_n\tau)\mu_2(\mu_2 - 2\zeta_2)\|F_2 u_n - F_2 p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\tau) \times \mu_1(\mu_1 - 2\zeta_1)\|F_1 z_n - F_1 z\|^2 \\
&= \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + (1-\alpha_n\tau)\|x_n - p\|^2 + (1-\beta_n-\alpha_n\tau)r_n(r_n - 2\delta)\|A x_n - A p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\tau)\mu_2(\mu_2 - 2\zeta_2)\|F_2 u_n - F_2 p\|^2 + (1-\beta_n-\alpha_n\tau)\mu_1(\mu_1 - 2\zeta_1) \\
&\quad \times \|F_1 z_n - F_1 z\|^2 \\
&\le \|x_n - p\|^2 + \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 + (1-\beta_n-\alpha_n\tau)r_n(r_n - 2\delta)\|A x_n - A p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\tau)\mu_2(\mu_2 - 2\zeta_2)\|F_2 u_n - F_2 p\|^2 + (1-\beta_n-\alpha_n\tau)\mu_1(\mu_1 - 2\zeta_1) \\
&\quad \times \|F_1 z_n - F_1 z\|^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
& r_n(2\delta - r_n)(1-\beta_n-\alpha_n\tau)\|A x_n - A p\|^2 + (1-\beta_n-\alpha_n\tau)\mu_2(2\zeta_2 - \mu_2)\|F_2 u_n - F_2 p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\tau)\mu_1(2\zeta_1 - \mu_1)\|F_1 z_n - F_1 z\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2 \\
&\quad = (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \frac{\alpha_n}{\tau}\|\gamma f(x_n) - \mu F p\|^2.
\end{aligned}
$$

From conditions (i), (ii), (iii) and (3.15), we have

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0,\ \lim_{n\to\infty}\|F_1z_n-F_1z\|=0\quad\text{and}\ \lim_{n\to\infty}\|F_2u_n-F_2p\|=0 \tag{3.19}$$

Indeed, from (3.3), (3.4) and Lemma 2.6, we have

$$\begin{aligned}
\|z_n-z\|^2 &= \|T^{G_2}_{\mu_2}(u_n-\mu_2F_2u_n)-T^{G_2}_{\mu_2}(p-\mu_2F_2p)\|^2\\
&\le \langle T^{G_2}_{\mu_2}(u_n-\mu_2F_2u_n)-T^{G_2}_{\mu_2}(p-\mu_2F_2p),(u_n-\mu_2F_2u_n)-(p-\mu_2F_2p)\rangle\\
&= \langle (u_n-\mu_2F_2u_n)-(p-\mu_2F_2p),z_n-z\rangle\\
&= \frac{1}{2}(\|(u_n-\mu_2F_2u_n)-(p-\mu_2F_2p)\|^2+\|z_n-z\|^2\\
&\quad -\|(u_n-\mu_2F_2u_n)-(p-\mu_2F_2p)-(z_n-z)\|^2)\\
&\le \frac{1}{2}(\|u_n-p\|^2+\|z_n-z\|^2-\|(u_n-z_n)-\mu_2(F_2u_n-F_2p)-(p-z)\|^2)\\
&\le \frac{1}{2}(\|x_n-p\|^2+\|z_n-z\|^2-\|(u_n-z_n)-(p-z)\|^2\\
&\quad +2\mu_2\langle (u_n-z_n)-(p-z),F_2u_n-F_2p\rangle-\mu_2^2\|F_2u_n-F_2p\|^2),
\end{aligned}$$

and

$$\begin{aligned}
\|y_n-p\|^2 &= \|T^{G_1}_{\mu_1}(z_n-\mu_1F_1z_n)-T^{G_1}_{\mu_1}(z-\mu_1F_1z)\|^2\\
&\le \langle T^{G_1}_{\mu_1}(z_n-\mu_1F_1z_n)-T^{G_1}_{\mu_1}(z-\mu_1F_1z),(z_n-\mu_1F_1z_n)-(z-\mu_1F_1z)\rangle\\
&= \langle (z_n-\mu_1F_1z_n)-(z-\mu_1F_1z),y_n-p\rangle\\
&= \frac{1}{2}(\|(z_n-\mu_1F_1z_n)-(z-\mu_1F_1z)\|^2+\|y_n-p\|^2\\
&\quad -\|(z_n-\mu_1F_1z_n)-(z-\mu_1F_1z)-(y_n-p)\|^2)\\
&\le \frac{1}{2}(\|z_n-z\|^2+\|y_n-p\|^2-\|(z_n-y_n)+(p-z)\|^2\\
&\quad +2\mu_1\langle (z_n-y_n)+(p-z),F_1z_n-F_1z\rangle-\mu_1^2\|F_1z_n-F_1z\|^2)\\
&\le \frac{1}{2}(\|x_n-p\|^2+\|y_n-p\|^2-\|(z_n-y_n)+(p-z)\|^2\\
&\quad +2\mu_1\langle (z_n-y_n)+(p-z),F_1z_n-F_1z\rangle,
\end{aligned}$$

which implies that

$$\|z_n-z\|^2\le\|x_n-p\|^2-\|(u_n-z_n)-(p-z)\|^2+2\mu_2\|(u_n-z_n)-(p-z)\|\|F_2u_n-F_2p\| \tag{3.20}$$

and

$$\|y_n-p\|^2\le\|x_n-p\|^2-\|(z_n-y_n)+(p-z)\|^2+2\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|. \tag{3.21}$$

It follows from (3.21) that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\alpha_n(\gamma f(x_n)-\mu Fp)+\beta_n(x_n-K_ny_n)+(I-\alpha_n\mu F)K_ny_n-(I-\alpha_n\mu F)K_np\|^2\\
&\le \|(I-\alpha_n\mu F)K_ny_n-(I-\alpha_n\mu F)K_np+\beta_n(x_n-K_ny_n)\|^2+2\alpha_n\langle\gamma f(x_n)-\mu Fp,x_{n+1}-p\rangle\\
&\le \|(I-\alpha_n\mu F)K_ny_n-(I-\alpha_n\mu F)K_np+\beta_n(x_n-K_ny_n)\|^2+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le [(1-\alpha_n\tau)\|y_n-p\|+\beta_n\|x_n-K_ny_n\|]^2+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le (\|y_n-p\|+\|x_n-K_ny_n\|)^2+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&=\|y_n-p\|^2+\|x_n-K_ny_n\|^2+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le \|x_n-p\|^2-\|(z_n-y_n)+(p-z)\|^2+2\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|\\
&\quad+\|x_n-K_ny_n\|^2+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\|(z_n-y_n)+(p-z)\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|\\
&\quad+\|x_n-K_ny_n\|^2+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\mu_1\|F_1z_n-F_1z\|\|(z_n-y_n)+(p-z)\|\\
&\quad+\|x_n-K_ny_n\|^2+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|
\end{aligned}
\tag{3.22}
$$

Since $\alpha_n\to 0$, $\|x_n-x_{n+1}\|\to 0$, $\|F_1z_n-F_1z\|\to 0$ and $\|x_n-K_ny_n\|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|(z_n-y_n)+(p-z)\|=0. \tag{3.23}$$

Also, from (3.4) and (3.20) , we obtain that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \|y_n-p\|^2+\|x_n-K_ny_n\|^2+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le [\|z_n-z\|^2-\|(z_n-y_n)+(p-z)\|^2+2\mu_1\|(z_n-y_n)+(p-z)\|\|F_1z_n-F_1z\|]\\
&\quad+\|x_n-K_ny_n\|^2+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le (\|x_n-p\|^2-\|(u_n-z_n)-(p-z)\|^2+2\mu_2\|(u_n-z_n)-(p-z)\|\|F_2u_n-F_2p\|)\\
&\quad-\|(z_n-y_n)+(p-z)\|^2+2\mu_1\|(z_n-y_n)+(p-z)\|\|F_1z_n-F_1z\|+\|x_n-K_ny_n\|^2\\
&\quad+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|(u_n-z_n)-(p-z)\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\|(u_n-z_n)-(p-z)\|\|F_2u_n-F_2p\|\\
&\quad-\|(z_n-y_n)+(p-z)\|^2+2\mu_1\|(z_n-y_n)+(p-z)\|\|F_1z_n-F_1z\|\\
&\quad+\|x_n-K_ny_n\|^2\\
&\quad+2\|y_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|
\end{aligned}
$$

$$\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\mu_2\|(u_n - z_n) - (p - z)\|\|F_2 u_n - F_2 p\|$$
$$- \|(z_n - y_n) + (p - z)\|^2 + 2\mu_1\|(z_n - y_n) + (p - z)\|\|F_1 z_n - F_1 z\|$$
$$+ \|x_n - K_n y_n\|^2 + 2\|y_n - p\|\|x_n - K_n y_n\| + 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\|.$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, $\|F_2 u_n - F_2 p\| \to 0$ and $\|(z_n - y_n) + (p - z)\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|(u_n - z_n) - (p - z)\| = 0. \tag{3.24}$$

Note that $T_{r_n}^{(\Phi,\varphi)}$ is firmly nonexpansivity. Hence we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{(\Phi,\varphi)}(x_n - r_n A x_n) - T_{r_n}^{(\Phi,\varphi)}(p - r_n A p)\|^2 \\
&\le \langle u_n - p, (x_n - r_n A x_n) - (p - r_n A p)\rangle \\
&= \frac{1}{2}(\|(x_n - r_n A x_n) - (p - r_n A p)\|^2 + \|u_n - p\|^2 \\
&\quad - \|(x_n - r_n A x_n) - (p - r_n A p) - (u_n - p)\|^2) \\
&\le \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(A x_n - A p)\|^2) \\
&= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle A x_n - A p, x_n - u_n\rangle \\
&\quad - r_n^2\|A x_n - A p\|^2,
\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|A x_n - A p\|\|x_n - u_n\|. \tag{3.25}$$

From (3.1), (3.5) and (3.25), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - \mu F p) + \beta_n(x_n - K_n y_n) + (I - \alpha_n\mu F)K_n y_n - (I - \alpha_n\mu F)K_n p\|^2 \\
&\le \|(I - \alpha_n\mu F)K_n y_n - (I - \alpha_n\mu F)K_n p + \beta_n(x_n - K_n y_n)\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(x_n) - \mu F p, x_{n+1} - p\rangle \\
&\le \|(I - \alpha_n\mu F)K_n y_n - (I - \alpha_n\mu F)K_n p + \beta_n(x_n - K_n y_n)\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\| \\
&\le [(1 - \alpha_n\tau)\|y_n - p\| + \beta_n\|x_n - K_n y_n\|]^2 + 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\| \\
&\le (\|y_n - p\| + \|x_n - K_n y_n\|)^2 + 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\| \\
&\le (\|u_n - p\| + \|x_n - K_n y_n\|)^2 + 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\| \\
&= \|u_n - p\|^2 + \|x_n - K_n y_n\|^2 + 2\|u_n - p\|\|x_n - K_n y_n\| \\
&+ 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\| \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|A x_n - A p\|\|x_n - u_n\| + \|x_n - K_n y_n\|^2 \\
&\quad + 2\|u_n - p\|\|x_n - K_n y_n\| + 2\alpha_n\|\gamma f(x_n) - \mu F p\|\|x_{n+1} - p\|.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|x_n-u_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2r_n\|Ax_n-Ap\|\|x_n-u_n\|+\|x_n-K_ny_n\|^2\\
&\quad +2\|u_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2r_n\|Ax_n-Ap\|\|x_n-u_n\|+\|x_n-K_ny_n\|^2\\
&\quad +2\|u_n-p\|\|x_n-K_ny_n\|+2\alpha_n\|\gamma f(x_n)-\mu Fp\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\alpha_n \to 0, \|Ax_n-Ap\| \to 0$, $\|x_n-x_{n+1}\| \to 0$ and $\|x_n-K_ny_n\| \to 0$ as $n\to\infty$, we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.26}$$

From (3.23), (3.24) and (3.26), we obtain that

$$\begin{aligned}
\lim_{n\to\infty}\|u_n-y_n\| &= \lim_{n\to\infty}\|(u_n-z_n)-(p-z)+(z_n-y_n)+(p-z)\|\\
&\le \lim_{n\to\infty}\|(u_n-z_n)-(p-z)\|+\lim_{n\to\infty}\|(z_n-y_n)+(p-z)\|\\
&=0
\end{aligned} \tag{3.27}$$

and

$$\lim_{n\to\infty}\|x_n-y_n\|\le\lim_{n\to\infty}\|x_n-u_n\|+\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.28}$$

Since $\|K_ny_n-y_n\|\le\|K_ny_n-x_n\|+\|x_n-y_n\|$, we also have

$$\lim_{n\to\infty}\|K_ny_n-y_n\|=0. \tag{3.29}$$

Step 4. We shall show that

$$\limsup_{n\to\infty}\langle(\mu F-\gamma f)x^*,x^*-x_n\rangle\le 0,$$

where $x^*=P_\Delta(I-\mu F+\gamma f)x^*$. To show this inequality, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle(\mu F-\gamma f)x^*,x^*-y_{n_i}\rangle=\limsup_{n\to\infty}\langle(\mu F-\gamma f)x^*,x^*-y_n\rangle. \tag{3.30}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to ω. Without loss of generality, we may assume that $y_{n_i}\rightharpoonup\omega$. Let us show $\omega\in\Delta$.

First, we prove that $\omega \in O$. Utilizing Lemma 2.8 , we have

$$
\begin{aligned}
&\|\Gamma(x)-\Gamma(y)\|^2\\
&=\|T^{G_1}_{\mu_1}[T^{G_2}_{\mu_2}(x-\mu_2F_2x)-\mu_1F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)]\\
&\quad -T^{G_1}_{\mu_1}[T^{G_2}_{\mu_2}(y-\mu_2F_2y)-\mu_1F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y)]\|^2\\
&\le \|T^{G_2}_{\mu_2}(x-\mu_2F_2x)-T^{G_2}_{\mu_2}(y-\mu_2F_2y)-\mu_1(F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)\\
&\quad -F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y))\|^2\\
&=\|T^{G_2}_{\mu_2}(x-\mu_2F_2x)-T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2-2\mu_1\langle T^{G_2}_{\mu_2}(x-\mu_2F_2x)\\
&\quad -T^{G_2}_{\mu_2}(y-\mu_2F_2y),F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)-F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y)\rangle\\
&\quad +\mu_1^2\|F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)-F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2\\
&\le \|T^{G_2}_{\mu_2}(x-\mu_2F_2x)-T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2-2\mu_1\zeta_1\|F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)\\
&\quad -F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2+\mu_1^2\|F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)-F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2\\
&=\|T^{G_2}_{\mu_2}(x-\mu_2F_2x)-T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2+\mu_1(\mu_1-2\zeta_1)\|F_1T^{G_2}_{\mu_2}(x-\mu_2F_2x)\\
&\quad -F_1T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2\\
&\le \|T^{G_2}_{\mu_2}(x-\mu_2F_2x)-T^{G_2}_{\mu_2}(y-\mu_2F_2y)\|^2\\
&\le \|(x-\mu_2F_2x)-(y-\mu_2F_2y)\|^2\\
&=\|(x-y)-\mu_2(F_2x-F_2y)\|^2\\
&\le \|x-y\|^2+\mu_2(\mu_2-2\zeta_2)\|F_2x-F_2y\|^2\\
&\le \|x-y\|^2.
\end{aligned}
$$

for all $x,y\in C$. This shows that $\Gamma: C\to C$ is nonexpansive. Note that

$$\|y_n-\Gamma(y_n)\|=\|\Gamma(u_n)-\Gamma(y_n)\|\le\|u_n-y_n\|$$

from (3.27), we have $\lim_{n\to\infty}\|y_n-\Gamma(y_n)\|\le\lim_{n\to\infty}\|u_n-y_n\|=0$. According to Lemma 2.8 and Lemma 2.10, we obtain $\omega\in O$.
Next, we show that $\omega\in\Omega$. From $u_n=T^{(\Phi,\varphi)}_{r_n}(x_n-r_nAx_n)$, we know that

$$\Phi(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C.$$

From (A2) it follows that

$$\varphi(y)-\varphi(u_n)+\langle Ax_n,y-u_n\rangle+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge\Phi(y,u_n),\quad \forall y\in C.$$

Replacing n by n_i, we have

$$\varphi(y)-\varphi(u_{n_i})+\langle Ax_{n_i},y-u_{n_i}\rangle+\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle\ge\Phi(y,u_{n_i}),\quad \forall y\in C. \quad (3.31)$$

Put $u_t = ty + (1-t)\omega$ for all $t \in (0,1]$ and $y \in C$. Then we have $u_t \in C$. From (3.31) we have

$$\begin{aligned}
&\varphi(u_t) - \varphi(u_{n_i}) + \langle u_t - u_{n_i}, Au_t\rangle \\
&\geq \langle u_t - u_{n_i}, Au_t\rangle - \langle u_t - u_{n_i}, Ax_{n_i}\rangle - \langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + \Phi(u_t, u_{n_i}) \\
&= \langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle + \langle u_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\
&\quad + \Phi(u_t, u_{n_i}).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle \geq 0$.
From (A4), the weakly semicontinuity of φ , $u_{n_i} - x_{n_i} \to 0$ and $u_{n_i} \rightharpoonup \omega$, we have

$$\varphi(u_t) - \varphi(\omega) + \langle u_t - \omega, Au_t\rangle \geq \Phi(u_t, \omega) \ \texttt{ as } \ i \to \infty. \tag{3.32}$$

From (A1), (A4), (3.32) and the convexity of φ , we obtain

$$\begin{aligned}
0 &= \Phi(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\
&= \Phi(u_t, (ty + (1-t)\omega)) + \varphi(ty + (1-t)\omega) - \varphi(u_t) \\
&\leq t\Phi(u_t, y) + (1-t)\Phi(u_t, \omega) + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(u_t) \\
&\leq t\Phi(u_t, y) + (1-t)(\varphi(u_t) - \varphi(\omega) + \langle u_t - \omega, Au_t\rangle) + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(u_t) \\
&= t\Phi(u_t, y) - t\varphi(u_t) + (1-t)\langle u_t - \omega, Au_t\rangle + t\varphi(y) \\
&= t[\Phi(u_t, y) - \varphi(u_t) + \varphi(y)] + (1-t)t\langle y - \omega, Au_t\rangle,
\end{aligned} \tag{3.33}$$

and hence

$$\Phi(u_t, y) - \varphi(u_t) + \varphi(y) + (1-t)\langle y - \omega, Au_t\rangle \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, it follows from (A3) and the weakly semicontinuity of φ that

$$\Phi(\omega, y) - \varphi(\omega) + \varphi(y) + \langle y - \omega, A\omega\rangle \geq 0, \quad \forall y \in C. \tag{3.34}$$

This implies that $\omega \in \Omega$. Next, we show that $\omega \in \cap_{i=1}^N Fix(T_i)$. Assume that there exists $j \in \{1, 2, \ldots, N\}$ such that $\omega \neq T_j\omega$. By Lemma 2.12, we have $\omega \neq K\omega$. Since $y_{n_i} \rightharpoonup \omega$ and $\omega \neq K\omega$, by Opial's condition[18] and (3.29) and Lemma 2.13, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - \omega\| &< \liminf_{i\to\infty} \|y_{n_i} - K\omega\| \\
&\leq \liminf_{i\to\infty} (\|y_{n_i} - K_{n_i}y_{n_i}\| + \|K_{n_i}y_{n_i} - K_{n_i}\omega\| + \|K_{n_i}\omega - K\omega\|) \\
&\leq \liminf_{i\to\infty} \|y_{n_i} - \omega\|,
\end{aligned}$$

which derives a contradiction. This implies that $\omega = K\omega$. It follows from $\omega \in Fix(K) = \cap_{i=1}^N Fix(T_i)$, that $\omega \in \cap_{i=1}^N Fix(T_i)$. Hence $\omega \in \Delta$.

Since $x^* = P_\Delta(I - \mu F + \gamma f)x^*$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\mu F - \gamma f)x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle(\mu F - \gamma f)x^*, y_{n_i} - x^*\rangle \\ &= \langle(\mu F - \gamma f)x^*, \omega - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.35}$$

Step 5. Finally, we prove that $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to x^*. From (3.1), utilizing Lemma 2.2 and Lemma 2.15, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - \mu F x^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\mu F)K_n y_n \\ &\quad - ((1-\beta_n)I - \alpha_n\mu F)K_n x^*\|^2 \\ &\leq \|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\mu F)K_n y_n - ((1-\beta_n)I - \alpha_n\mu F)K_n x^*\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(x_n) - \mu F x^*, x_{n+1} - x^*\rangle \\ &\leq [\beta_n\|x_n - x^*\| + \|((1-\beta_n)I - \alpha_n\mu F)K_n y_n - ((1-\beta_n)I - \alpha_n\mu F)K_n x^*\|]^2 \\ &\quad + 2\alpha_n\langle\gamma f(x_n) - \mu F x^*, x_{n+1} - x^*\rangle \\ &\leq [\beta_n\|x_n - x^*\| + \|(1-\beta_n)((I - \frac{\alpha_n}{1-\beta_n}\mu F)K_n y_n - (I - \frac{\alpha_n}{1-\beta_n}\mu F)K_n x^*)\|]^2 \\ &\quad + 2\alpha_n\gamma\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + 2\alpha_n\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\ &\leq [\beta_n\|x_n - x^*\| + (1-\beta_n)(1 - \frac{\alpha_n}{1-\beta_n}\tau)\|x_n - x^*\|]^2 \\ &\quad + 2\alpha_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + \alpha_n\gamma\rho(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\quad + 2\alpha_n\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + \alpha_n\gamma\rho\|x_n - x^*\|^2 + \alpha_n\gamma\rho\|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\leq \frac{(1-\alpha_n\tau)^2 + \alpha_n\gamma\rho}{1 - \alpha_n\gamma\rho}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\ &= [1 - \frac{2(\tau - \gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}]\|x_n - x^*\|^2 + \frac{(\alpha_n\tau)^2}{1-\alpha_n\gamma\rho}\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\ &\leq [1 - \frac{2(\tau - \gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}]\|x_n - x^*\|^2 + \frac{2(\tau - \gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho} \\ &\quad \times\{\frac{(\alpha_n\tau)^2 M_1}{2(\tau - \gamma\rho)} + \frac{1}{\tau - \gamma\rho}\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle\} \\ &= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\sigma_n, \end{aligned} \tag{3.36}$$

where $M_1 = \sup\{\|x_n - p\|^2 : n \geq 1\}$, $\delta_n = \frac{2(\tau-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}$ and $\sigma_n = \frac{(\alpha_n\tau)^2 M_1}{2(\tau-\gamma\rho)} + \frac{1}{\tau-\gamma\rho}\langle\gamma f(x^*) - \mu F x^*, x_{n+1} - x^*\rangle$. It is easy to see that $\delta_n \to 0$, $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \leq 0$. Applying Lemma 2.4 to (3.36), we conclude that $x_n \to x^*$ as $n \to \infty$. Consequently, we can obtain $\|x_n - u_n\| \to 0$ and $\|x_n - y_n\| \to 0$ that $\{u_n\}$ and $\{y_n\}$ are also converges strongly to x^*. This completes the proof. □

Corollary 3.2. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R}$ a lower semicontinuous and convex function with assumption (B1) or (B2). Let $F_1, F_2 : C \to H$ be ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\cap_{i=1}^N Fix(T_i) \cap MEP(\Phi,\varphi) \cap O \neq \emptyset$. Let $F : C \to H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator with constants $\kappa, \eta > 0$ and $f : C \to C$ a $\rho-$Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, $\{r_n\} \subset (0,\infty)$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a,b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $x_1 \in C$ arbitrarily. Suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by*

$$\begin{cases} u_n = \Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C, \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)K_n y_n, \forall n \geq 1, \end{cases} \tag{3.37}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty}|\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

Then both $\{x_n\}$ and $\{u_n\}$ converges strongly to $x^* \in \cap_{i=1}^N Fix(T_i) \cap MEP(\Phi,\varphi) \cap O$, where $x^* = P_{\cap_{i=1}^N Fix(T_i)\cap MEP(\Phi,\varphi)\cap O}(I - \mu F + \gamma f)x^*$ and (x^*, y^*) is a solution of problem (1.8) where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. In Theorem 3.1, for all $n \geq 0$, $u_n = T_{r_n}^{(\Phi,\varphi)}(x_n - r_n A x_n)$ is equivalent to

$$\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.38}$$

Putting $A \equiv 0$, we obtain

$$\Phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

□

Corollary 3.3. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4). Let $A, F_1, F_2 : C \to H$ be $\delta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\cap_{i=1}^N Fix(T_i) \cap EP \cap O \neq \emptyset$. Let $F : C \to H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator with constants $\kappa, \eta > 0$ and $f : C \to C$ a $\rho-$Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{r_n\}$ is a sequence in $(0, 2\delta]$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let K_n be a $K-$mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Let $x_1 \in C$ arbitrarily. Suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by*

$$\begin{cases} u_n = \Phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C, \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)K_n y_n, \forall n \geq 1, \end{cases} \tag{3.39}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$*;*

(iv) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \dots, N\}$.

Then both $\{x_n\}$ and $\{u_n\}$ converges strongly to $x^* \in \cap_{i=1}^N Fix(T_i) \cap EP \cap O$,where $x^* = P_{\cap_{i=1}^N Fix(T_i) \cap EP \cap O}(I - \mu F + \gamma f)x^*$ and (x^*, y^*) is a solution of problem (1.8) *where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.*

Proof. Put $\varphi \equiv 0$ in Theorem 3.1. Then we have from (3.38) that

$$\Phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

□

Corollary 3.4. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $\Phi, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions which satisfying (A1)-(A4). Let $F_1, F_2 : C \to H$ be ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\cap_{i=1}^N Fix(T_i) \cap EP(\Phi) \cap O \neq \emptyset$. Let $F : C \to H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator with constants $\kappa, \eta > 0$ and $f : C \to C$ a $\rho-$Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{r_n\} \subset (0, \infty)$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $x_1 \in C$ arbitrarily. Suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by*

$$\begin{cases} u_n = \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \quad \forall y \in C, \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)K_n y_n, \forall n \geq 1, \end{cases} \tag{3.40}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

Then both $\{x_n\}$ and $\{u_n\}$ converges strongly to $x^* \in \cap_{i=1}^N Fix(T_i) \cap EP(\Phi) \cap O$, where $x^* = P_{\cap_{i=1}^N Fix(T_i)\cap EP(\Phi)\cap O}(I - \mu F + \gamma f)x^*$ and (x^*, y^*) is a solution of problem (1.8) where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. Put $\varphi \equiv 0$ and $A \equiv 0$ in Theorem 3.1. Then we have from (3.38) that

$$\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

□

Corollary 3.5. *Let H be a real Hilbert space, C a closed convex nonempty subset of H. Let $G_1, G_2 : C \times C \to \mathbb{R}$ be two bifunctions which satisfying (A1)-(A4). Let $A, F_1, F_2 : C \to H$ be $\delta-$inverse strongly monotone, ζ_1-inverse strongly monotone and ζ_2-inverse strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H such that $\cap_{i=1}^N Fix(T_i) \cap VI(A, C) \cap O \neq \emptyset$. Let $F : C \to H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator with constants $\kappa, \eta > 0$ and $f : C \to C$ a $\rho-$Lipschitzian mapping with constant $\rho \geq 0$. Let $0 <$*

$\mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma\rho < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, $\{r_n\}$ is a sequence in $(0, 2\delta]$, $\mu_1 \in (0, 2\zeta_1), \mu_2 \in (0, 2\zeta_2)$ and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a,b]$ with $0 < a \le b < 1$. For every $n \ge 1$, let K_n be a $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $x_1 \in C$ arbitrarily. Suppose the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$\begin{cases} u_n = P_C(x_n - r_n A x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n) - \mu_1 F_1 T_{\mu_2}^{G_2}(u_n - \mu_2 F_2 u_n)], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu F)K_n y_n, \forall n \ge 1, \end{cases} \tag{3.41}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}$, $\{r_n\}$ and $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the following conditions:

- *(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- *(ii)* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
- *(iii)* $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;
- *(iv)* $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

Then both $\{x_n\}$ and $\{u_n\}$ converges strongly to $x^* \in \cap_{i=1}^N Fix(T_i) \cap VI(A, C) \cap O$, where $x^* = P_{\cap_{i=1}^N Fix(T_i) \cap VI(A,C) \cap O}(I - \mu F + \gamma f)x^*$ and (x^*, y^*) is a solution of problem (1.8) where $y^* = T_{\mu_2}^{G_2}(x^* - \mu_2 F_2 x^*)$.

Proof. Put $\Phi \equiv 0$ and $\varphi \equiv 0$ in Theorem 3.1. Then we have from (3.38) that

$$\langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C.$$

That is,

$$\langle y - u_n, x_n - r_n A x_n - u_n \rangle \le 0, \quad \forall y \in C.$$

It follows that $u_n = P_C(x_n - r_n A x_n)$ for all $n \ge 1$. Hence the corollary is obtained by Theorem 3.1. □

Example 3.6. The example of our iteration parameters are :

$$\alpha_n := \frac{1}{n},\ \ \beta_n := \frac{n}{n+1},\ \ r_n := \delta(\frac{n}{n+2}),\ \ \lambda_{n,i} := \frac{i}{(N+1)n},\ \ n \ge 1,\ \ i \in \{1, 2, \ldots, N\},$$

$$\mu_1 = \frac{\zeta_1}{4},\ \ \mu_2 = \frac{\zeta_2}{4},\ \ \gamma = \frac{\tau}{4\rho},\ \ \mu = \frac{\eta}{2\kappa^2}.$$

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