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Convergence of Nondiagonal Sequences of Linear Padé-Orthogonal Approximants[1]

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Abstract : A convergence in capacity of nondiagonal sequences of linear Padé-orthogonal approximants corresponding to a measure supported on a general compact subset of the complex plane is proved. This result generalizes the result of Suetin [1] which was the convergence in capacity of nondiagonal sequences of linear Padé-orthogonal approximants corresponding to a measure supported on a finite interval.

Keywords : linear Padé approximants of orthogonal expansions; Padé-orthogonal approximants; Fourier-Padé approximants; orthogonal polynomials; orthogonal Padé approximants.
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1 Introduction

Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. There exists a unique exterior conformal mapping Φ from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \{w : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. We assume that E is such that the inverse function $\Psi = \Phi^{-1}$ can be extended continuously to $\overline{\mathbb{C}} \setminus \{w : |w| < 1\}$. Note that the closure of a bounded Jordan region and a finite interval satisfy the above conditions. In this whole paper, E is as described above.

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Let μ be a finite positive Borel measure with infinite support $\text{supp}(\mu)$ contained in E. We write $\mu \in \mathcal{M}(E)$ and define the associated inner product,

$$\langle g, h\rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu).$$

Let

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \dots,$$

be the orthonormal polynomial of degree n with respect to μ having positive leading coefficient; that is, $\langle p_n, p_m\rangle_\mu = \delta_{n,m}$. Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E.

Definition 1.1. Let $F \in \mathcal{H}(E)$, $\mu \in \mathcal{M}(E)$, and a pair of nonnegative integers (n, m) be given. A rational function $[n/m]_F^\mu := P_{n,m}^\mu/Q_{n,m}^\mu$ is called an (n, m) *(linear) Padé-orthogonal approximant* of F with respect to μ if $P_{n,m}^\mu$ and $Q_{n,m}^\mu$ are polynomials satisfying

$$\deg(P_{n,m}^\mu) \le n, \quad \deg(Q_{n,m}^\mu) \le m, \quad Q_{n,m}^\mu \not\equiv 0,$$

$$\langle Q_{n,m}^\mu F - P_{n,m}^\mu, p_j\rangle_\mu = 0, \quad \text{for } j = 0, 1, 2, \dots, n + m.$$

Since $Q_{n,m}^\mu \not\equiv 0$, we normalize it to have leading coefficient equal to 1.

These rational functions always exist because finding an ordered pair of $(P_{n,m}^\mu, Q_{n,m}^\mu)$ is the same as solving a system of $n + m + 1$ linear equations with $n + m + 2$ unknowns. But in general they may not be unique (see [2, Example 1.2]). Moreover, it is not difficult to see that if $E = \{z \in \mathbb{C} : |z| \le 1\}$ and $d\mu = d\theta/2\pi$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, then the linear Padé-orthogonal approximants are exactly the classical Padé approximants (see the G. Frobenius definition [3] for the definition of the classical Padé approximants). Note that linear Padé-orthogonal approximants have been called by several names such as linear Padé approximants of orthogonal expansions [4], α-Padé approximants [1, 5], Fourier-Padé approximants [6–8], and orthogonal Padé approximants [9,10]. Furthermore, we would like to emphasize that there is another construction called nonlinear Padé approximants of orthogonal expansions (see e.g. [11]) which is intimately connected with linear Padé-orthogonal approximants. However, since we restrict our consideration to the linear case, we will omit the word "linear" when we refer to linear Padé-orthogonal approximants.

Almost all results in the subject of Padé-orthogonal approximation have been mainly concentrated on the case when the measure μ is supported on a finite interval (see e.g. [1,5,9,10,12–15]). S.P. Suetin [1] was the first to prove the convergence of row sequences of Padé-orthogonal approximants for a general class of measures supported on $[-1, 1]$ for which the corresponding sequence of orthonormal polynomials has ratio asymptotic behavior. Moreover, he also proved an inverse result [5] for row sequences of Padé-orthogonal approximants with respect to a measure supported on $[-1, 1]$ under the assumption that the denominators of the approximants converge with geometric rate to a certain polynomial of degree m. For measures satisfying Szegő's condition, V.I. Buslaev [9,10] obtained inverse type results without the requirement that the denominators converge geometrically. Some problems on the convergence of diagonal sequences of Padé-orthogonal approximants with respect to a measure supported on $[-1,1]$ were considered in [12–15]. Some papers which consider measures μ supported on the unit circle are [6–8,16]. N. Bosuwan, G. López Lagomasino, and E.B. Saff [2] and N. Bosuwan and G. López Lagomasino [17] gave direct and inverse results for row sequences of Padé-orthogonal approximants corresponding to measures supported on a general compact E as described above. The results in [2,17] generalized the results in [1,5,9,10]. The object of this paper is to investigate a convergence behavior of Padé-orthogonal approximants $[n/m]_F^\mu$ with respect to a measure supported on a general compact set as $n \to \infty$ and $m \to \infty$, particularly as $m := m_n \to \infty$, with $m_n = o(n)$ as $n \to \infty$, and $n \to \infty$. The sequences like these were called *nondiagonal sequences* by D.S. Lubinsky and A. Sidi in [14, Section 4.]. S.P. Suetin [1, Theorem 3] was the first to prove the convergence in capacity of nondiagonal sequences of Padé-orthogonal approximants with respect to a measure supported on $[-1,1]$. Later, D.S. Lubinsky and A. Sidi [14, Theorem 4.1] also considered a convergence in capacity of nondiagonal sequences of Padé-orthogonal approximants with respect to a measure supported on $[-1,1]$ but in a very different way, for example, the condition on a measure μ, the condition of the approximated function, and a region of convergence. Our main result in this paper generalizes the result of S.P. Suetin [1, Theorem 3].

An outline of this paper is as follows. In the section 2, we introduce some notation and auxiliary lemmas. The statement of the main result and its proof are in the section 3.

2 Notation and auxiliary lemmas

First of all, we introduce some needed notation. For any $\rho > 1$, we denote by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\}, \qquad \text{and} \qquad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

a *level curve of index* ρ and a *canonical domain of index* ρ, respectively. We denote by $\rho_0(F)$ the index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a holomorphic function, and by $\rho_m(F)$ the index ρ of the largest canonical domain D_ρ to which F can be extended as a meromorphic function with at most m poles (counting multiplicities). Basically, the notation $\rho_m(F)$ is just the generalization of the radius of m-meromorphy of F. Moreover, we denote by

$$D_{\rho_\infty(F)} = \bigcup_{m=0}^{\infty} D_{\rho_m(F)}$$

the maximal canonical domain in which F can be continued to a meromorphic function.

Let $\mu \in \mathcal{M}(E)$ be such that

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{2.1}$$

uniformly inside $\mathbb{C} \setminus E$. Such measures are called *regular* (see e.g. [18]). Here and in what follows, the phrase "uniformly inside a domain" means "uniformly on each compact subset of the domain". The Fourier coefficient of F with respect to p_n is given by

$$F_n := \langle F, p_n \rangle_\mu = \int F(z)\overline{p_n(z)}d\mu(z). \tag{2.2}$$

As for Taylor series (see, for example, [18, Theorem 6.6.1]), it is easy to show that

$$\rho_0(F) = \Big(\overline{\lim_{n\to\infty}} |F_n|^{1/n} \Big)^{-1}.$$

Additionally, the series $\sum_{n=0}^{\infty} F_n p_n(z)$ converges to $F(z)$ uniformly inside $D_{\rho_0(F)}$ and diverges pointwise for all $z \in \mathbb{C} \setminus \overline{D_{\rho_0(F)}}$. Therefore, if (2.1) holds, then

$$Q^\mu_{n,m}(z)F(z) - P^\mu_{n,m}(z) = \sum_{k=n+m+1}^{\infty} \langle Q^\mu_{n,m} F, p_k\rangle_\mu \, p_k(z)$$

for all $z \in D_{\rho_0(F)}$ and $P^\mu_{n,m} = \sum_{k=0}^{n} \langle Q^\mu_{n,m} F, p_k\rangle_\mu \, p_k$ is uniquely determined by $Q^\mu_{n,m}$.

In this paper, we restrict ourselves to a class of measures $\mathcal{R}(E) \subset \mathcal{M}(E)$. We write $\mu \in \mathcal{R}(E)$ when the corresponding sequence of orthonormal polynomials has *ratio asymptotics*; that is,

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)}, \tag{2.3}$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. It is not difficult to see that if $\mu \in \mathcal{R}(E)$, then μ is regular.

The second type functions s_n defined by

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z-\zeta} d\mu(\zeta), \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu),$$

are very useful our proof. The following lemma (see [2, Lemma 3.1]) is the asymptotic relation between the orthogonal polynomials p_n and the second type functions s_n.

Lemma 2.1. *If $\mu \in \mathcal{R}(E)$, then*

$$\lim_{n\to\infty} p_n(z)s_n(z) = \frac{\Phi'(z)}{\Phi(z)},$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. Consequently, for any compact set $K \subset \mathbb{C} \setminus E$, there exists n_0 (n_0 depends on K) such that $s_n(z) \neq 0$ for all $z \in K$ and $n \geq n_0$.

Finally, we state a lemma due to A.A. Gonchar which is quite useful in the theory of rational approximation. We recall a definition of the logarithmic capacity of a compact set K :

$$\operatorname{cap}(K) := e^{-\gamma(K)},$$

where

$$\gamma(K) := \inf\left\{\int\int \log\frac{1}{|z-t|}d\mu(z)d\mu(t)\ :\ \mu\geq 0,\quad \operatorname{supp}(\mu)\subset K,\quad \mu(K)=1\right\}.$$

Moreover, the logarithmic capacity can be extended to a noncompact set G by

$$\operatorname{cap}(G) := \sup\{\operatorname{cap}(K) : K\subset G,\ K \text{ is compact}\}.$$

Let W and W_n, $n\in\mathbb{N}$, be functions defined on an open region Ω. We say that the sequence $\{W_n\}_{n\in\mathbb{N}}$ *converges to* W *in capacity inside* Ω, if for any $\varepsilon>0$ and for any compact subset K of Ω,

$$\lim_{n\to\infty}\operatorname{cap}(\{z\in K : |W_n(z)-W(z)|\geq\varepsilon\})=0.$$

The following is a part of Lemma 1 in [19] or in Section §2., subsection 2, part b in [20] proved by A.A. Gonchar.

Lemma 2.2. *Suppose that the sequence of functions $\{W_n\}_{n\in\mathbb{N}}$ converges to the function W in capacity inside an open region Ω. If the functions W_n are meromorphic and have no more than $m<\infty$ poles in Ω, then the limit function W is also meromorphic in Ω and has at most m poles in Ω. Hence, in particular, if W has a pole of order ν at the point $a\in\Omega$, then at least ν poles of W_n tend to a as $n\to\infty$.*

3 Main result

The following theorem is our main result. This result extends Suetin's result in [1, Theorem 3] from the interval $[-1,1]$ to a general compact set E as described above.

Theorem 3.1. *Let $F\in\mathcal{H}(E)$ and $\mu\in\mathcal{R}(E)$. Denote by $D_{\rho_\infty(F)}$ the maximal canonical domain in which F can be continued to a meromorphic function. Then every sequence $[n/m_n]_F^\mu$ with $m_n\to\infty, m_n=o(n)$ as $n\to\infty$ converges in capacity to F inside $D_{\rho_\infty(F)}$. Moreover, if F has a pole of order ν at a point $a\in D_{\rho_\infty(F)}$, then at least ν poles of $[n/m_n]_F^\mu$ tend to a as $n\to\infty$ according to their multiplicities.*

Proof of Theorem 3.1. From $\mu\in\mathcal{R}(E)$, it follows that

$$\lim_{n\to\infty}\frac{p_n(z)}{p_{n+l}(z)}=\frac{1}{\Phi(z)^l},\quad l=0,1,2,\ldots, \tag{3.1}$$

uniformly inside $\overline{\mathbb{C}}\setminus E$. By (3.1) and Lemma 2.1, for any $l=0,1,2,\ldots$, we have

$$\lim_{n\to\infty}\frac{s_{n+l}(z)}{s_n(z)}=\lim_{n\to\infty}\frac{p_n(z)}{p_{n+l}(z)}\frac{p_{n+l}(z)s_{n+l}(z)}{p_n(z)s_n(z)}=\frac{1}{\Phi(z)^l}\frac{\Phi'(z)/\Phi(z)}{\Phi'(z)/\Phi(z)}=\frac{1}{\Phi(z)^l}, \tag{3.2}$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. Furthermore, by using the equalities (3.1) and (3.2), we have

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{3.3}$$

and

$$\lim_{n\to\infty} |s_n(z)|^{1/n} = \frac{1}{|\Phi(z)|}, \tag{3.4}$$

uniformly inside $\mathbb{C} \setminus E$, respectively.

Let K be a compact subset of $D_{\rho_\infty(F)}$ and $\sigma := \max\{|\Phi(z)| : z \in K\}$. Denote by d the number of poles of F in $\overline{D_\sigma}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be the poles of F in $\overline{D_\sigma}$ and $\omega_d(z) := \prod_{j=1}^{d}(z-\lambda_j)$. By the way that we take $m_n \to \infty$ and $n \to \infty$, without loss of generality, we assume that $d \leq m_n \leq n$. From the definition of Padé-orthogonal approximants and the condition (3.3), we have

$$Q^{\mu}_{n,m_n}(z)F(z) - P^{\mu}_{n,m_n}(z) = \sum_{k=n+m_n+1}^{\infty} a_{k,n}p_k(z), \quad z \in D_{\rho_0(F)}, \tag{3.5}$$

where

$$a_{k,n} := \langle Q^{\mu}_{n,m_n}F, p_k\rangle_\mu, \quad k = 0, 1, 2, \ldots,$$

and

$$a_{k,n} = 0, \quad k = n+1, n+2, \ldots, n+m_n.$$

Applying Cauchy's integral formula and Fubini's theorem, we obtain, for $k = 0, 1, 2, \ldots,$

$$a_{k,n} := \langle Q^{\mu}_{n,m_n}F,\, p_k\rangle_\mu = \int \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} \frac{Q^{\mu}_{n,m_n}(t)F(t)}{t-z}dt\overline{p_k(z)}d\mu(z)$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m_n}(t)F(t)\int \frac{\overline{p_k(z)}}{t-z}d\mu(z)dt = \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m_n}(t)F(t)s_k(t)dt, \tag{3.6}$$

where $1 < \rho_1 < \rho_0(F)$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_\gamma\}$ be the set of the distinct poles of F in $\overline{D_\sigma}$ and d_k be the multiplicity of α_k so that

$$\omega_d(z) = \prod_{j=1}^{d}(z-\lambda_j) = \prod_{k=1}^{\gamma}(z-\alpha_k)^{d_k}, \quad d = \sum_{k=1}^{\gamma} d_k.$$

Multiplying the equation (3.5) by ω_d and expanding

$$\sum_{k=n+m_n+1}^{\infty} a_{k,n}\omega_d p_k (= \omega_d Q^{\mu}_{n,m_n}F - \omega_d P^{\mu}_{n,m_n} \in \mathcal{H}(\overline{D_\sigma}))$$

in terms of the Fourier series corresponding to the orthonormal system $\{p_\nu\}_{\nu=0}^{\infty}$, we obtain that for $z \in \overline{D_\sigma}$,

$$\omega_d(z)Q^\mu_{n,m_n}(z)F(z) - \omega_d(z)P^\mu_{n,m_n}(z) = \sum_{k=n+m_n+1}^{\infty} a_{k,n}\omega_d(z)p_k(z) = \sum_{\nu=0}^{\infty} b_{\nu,n}p_\nu(z), \tag{3.7}$$

where

$$b_{\nu,n} := \sum_{k=n+m_n+1}^{\infty} a_{k,n}\langle \omega_d p_k,\, p_\nu\rangle_\mu, \quad \nu = 0, 1, 2, \ldots. \tag{3.8}$$

First of all, we will estimate $|a_{k,n}|$ in terms of $|\tau_{k,n}|$ where

$$\tau_{k,n} := \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^\mu_{n,m_n}(t)F(t)s_k(t)dt, \quad \rho_{d-1}(F) < \rho_2 < \rho_d(F), \quad k = 0, 1, 2, \ldots. \tag{3.9}$$

Notice that the only difference between the integral in (3.9) and the last integral in (3.6) is the domains of the integrals. The greater number ρ of Γ_ρ will allow to have a better bound on $|s_k|$. For each $k \geq 0$, the function $Q^\mu_{n,m_n}Fs_k$ is meromorphic on $\overline{D_{\rho_2}} \setminus D_{\rho_1} = \{z \in \mathbb{C} : \rho_1 \leq |\Phi(z)| \leq \rho_2\}$ and has poles at $\alpha_1, \alpha_2, \ldots, \alpha_\gamma$ with multiplicities at most $d_1, d_2, \ldots, d_\gamma$, respectively. Applying Cauchy's residue theorem to the function $Q^\mu_{n,m_n}Fs_k$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^\mu_{n,m_n}(t)F(t)s_k(t)dt - \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} Q^\mu_{n,m_n}(t)F(t)s_k(t)dt\\ &= \sum_{j=1}^{\gamma}\operatorname{res}(Q^\mu_{n,m_n}Fs_k,\, \alpha_j),\end{aligned} \tag{3.10}$$

for $k \geq 0$. Recall that the limit formula for the residue of $Q^\mu_{n,m_n}Fs_k$ at α_j is

$$\operatorname{res}(Q^\mu_{n,m_n}Fs_k,\, \alpha_j) = \frac{1}{(d_j-1)!}\lim_{z\to\alpha_j}((z-\alpha_j)^{d_j}Q^\mu_{n,m_n}(z)F(z)s_k(z))^{(d_j-1)}. \tag{3.11}$$

By the Leibniz formula and the fact that for n sufficiently large, $s_n(z) \neq 0$ for $z \in \mathbb{C} \setminus E$ (see Lemma 2.1), we can transform the expression under the limit sign as follows

$$\begin{aligned}((z-\alpha_j)^{d_j}Q^\mu_{n,m_n}(z)F(z)s_k(z))^{(d_j-1)} &= \left((z-\alpha_j)^{d_j}Q^\mu_{n,m_n}(z)F(z)s_n(z)\frac{s_k(z)}{s_n(z)}\right)^{(d_j-1)}\\ &= \sum_{p=0}^{d_j-1}\binom{d_j-1}{p}((z-\alpha_j)^{d_j}Q^\mu_{n,m_n}(z)F(z)s_n(z))^{(d_j-1-p)}\left(\frac{s_k(z)}{s_n(z)}\right)^{(p)}.\end{aligned}$$

To avoid long expressions, let us introduce the following notation:

$$\beta_n(j,p) := \frac{1}{(d_j-1)!}\binom{d_j-1}{p}\lim_{z\to\alpha_j}((z-\alpha_j)^{d_j}Q^\mu_{n,m_n}(z)F(z)s_n(z))^{(d_j-1-p)},$$

for $j = 1, 2, \ldots, \gamma$ and $p = 0, 1, 2, \ldots, d_j - 1$ (notice that the $\beta_n(j,p)$ do not depend on k), so we can rewrite the equality (3.10) as

$$a_{k,n} = \tau_{k,n} - \sum_{j=1}^{\gamma} \left(\sum_{p=0}^{d_j-1} \beta_n(j,p) \left(\frac{s_k}{s_n} \right)^{(p)} (\alpha_j) \right), \quad n \geq n_0 \quad \text{and} \quad k = 0, 1, 2, \ldots. \tag{3.12}$$

Since $a_{k,n} = 0$, for $k = n+1, n+2, \ldots, n+m_n$, it follows from (3.12) and $d \leq m_n$ that

$$\sum_{j=1}^{\gamma} \sum_{p=0}^{d_j-1} \beta_n(j,p) \left(\frac{s_k}{s_n} \right)^{(p)} (\alpha_j) = \tau_{k,n}, \quad k = n+1, n+2, \ldots, n+d. \tag{3.13}$$

We will view this as a system of d equations with d unknowns $\beta_n(j,p)$. If we can show that

$$\Lambda_n := \begin{vmatrix} \left(\frac{s_{n+1}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+1}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+1}}{s_n}\right)^{(d_j-1)}(\alpha_j) \\ \left(\frac{s_{n+2}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+2}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+2}}{s_n}\right)^{(d_j-1)}(\alpha_j) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{s_{n+d}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+d}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+d}}{s_n}\right)^{(d_j-1)}(\alpha_j) \end{vmatrix}_{j=1,2,\ldots,\gamma} \neq 0, \tag{3.14}$$

(this expression represents the determinant of order d in which the indicated groups of columns are successively written out for $j = 1, 2, \ldots, \gamma$), then we can express $\beta_n(j,p)$ in terms of $(s_k/s_n)^{(p)}(\alpha_j)$ and $\tau_{k,n}$, for $k = n+1, n+2, \ldots, n+d$. However, since

$$\lim_{n\to\infty} |\Lambda_n| = |\Lambda| := \left| \begin{vmatrix} R(\alpha_j) & R'(\alpha_j) & \cdots & R^{(d_j-1)}(\alpha_j) \\ R^2(\alpha_j) & (R^2)'(\alpha_j) & \cdots & (R^2)^{(d_j-1)}(\alpha_j) \\ \vdots & \vdots & \vdots & \vdots \\ R^d(\alpha_j) & (R^d)'(\alpha_j) & \cdots & (R^d)^{(d_j-1)}(\alpha_j) \end{vmatrix}_{j=1,2,\ldots,\gamma} \right|$$

$$= \left| \prod_{j=1}^{\gamma} (d_j - 1)!! \prod_{j=1}^{\gamma} (\Phi'(\alpha_j))^{d_j(d_j-1)/2} \prod_{j=1}^{\gamma} \Phi(\alpha_j)^{-d_j^2} \prod_{1 \leq i < j \leq \gamma} \left(\frac{1}{\Phi(\alpha_j)} - \frac{1}{\Phi(\alpha_i)} \right)^{d_i d_j} \right|,$$

where $R(z) = 1/\Phi(z)$ and $n!! = 0!1!2! \cdots$ n! (use, for example, [21, Theorem 1] to verify the last equality), for sufficiently large n, $|\Lambda_n| \neq 0$. In fact, for sufficiently large n, $|\Lambda_n| \geq c_1 > 0$ where the number c_1 does not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, c_4, \ldots$ and we will consider only n large enough so that $|\Lambda_n| \geq c_1 > 0$).

Applying Cramer's rule to (3.13), we have

$$\beta_n(j,p) = \frac{\Lambda_n(j,p)}{\Lambda_n} = \frac{1}{\Lambda_n} \sum_{s=1}^{d} \tau_{n+s,n} C_n(s,q), \tag{3.15}$$

where $\Lambda_n(j,p)$ is the determinant obtained from Λ_n replacing the column with index $q = (\sum_{l=0}^{j-1} d_l) + p + 1$ (where we define $d_0 := 0$) with the column

$$[\tau_{n+1,n} \ \ \tau_{n+2,n} \ \ \cdots \ \ \tau_{n+d,n}]^T$$

and $C_n(s,q)$ is the $(s,q)^{\text{th}}$ cofactor of $\Lambda_n(j,p)$. Substituting $\beta_n(j,p)$ in the formula (3.12) with the expression in (3.15), we obtain

$$a_{k,n} = \tau_{k,n} - \frac{1}{\Lambda_n}\sum_{j=1}^{\gamma}\sum_{p=0}^{d_j-1}\sum_{s=1}^{d}\tau_{n+s,n}C_n(s,q)\left(\frac{s_k}{s_n}\right)^{(p)}(\alpha_j), \quad k \ge n+m_n+1. \tag{3.16}$$

Let $\delta > 0$ be sufficiently small so that $\rho_0(F) - 2\delta > 1$ and $\varepsilon > 0$ be sufficiently small so that for all $j = 1, 2, 3, \ldots, \gamma$,

$$\{z \in \mathbb{C} : |z - \alpha_j| = \varepsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| \ge \rho_0(F) - \delta\}$$

and

$$\left(\frac{s_k}{s_n}\right)^{(p)}(\alpha_j) = \frac{p!}{2\pi i}\int_{|z-\alpha_j|=\varepsilon}\frac{s_k(z)}{s_n(z)(z-\alpha_j)^{p+1}}dz, \tag{3.17}$$

where $k = 0, 1, 2, \ldots,$ and $p = 0, 1, 2, \ldots, d_j - 1$. Using (3.2) and (3.17), we can easily check that for $p = 0, 1, 2, \ldots, d_j - 1$, $j = 1, 2, \ldots, \gamma$, and $k = n+1, n+2, \ldots, n+d$,

$$\left|\left(\frac{s_k}{s_n}\right)^{(p)}(\alpha_j)\right| \le c_2, \tag{3.18}$$

for all $n \ge n_1$, and for $p = 0, 1, 2, \ldots, d_j - 1$, $j = 1, 2, \ldots, \gamma$, and $k \ge n + m_n + 1$,

$$\left|\left(\frac{s_k}{s_n}\right)^{(p)}(\alpha_j)\right| \le \frac{c_3}{(\rho_0(F) - 2\delta)^{k-n}}, \tag{3.19}$$

for all $n \ge n_2$. The equation (3.18) implies that

$$|C_n(s,q)| \le (d-1)!c_2^{d-1} = c_4, \quad s, q = 1, 2, \ldots, d, \tag{3.20}$$

for $n \ge n_3$. Combining the estimates (3.18), (3.19), (3.20), and $|\Lambda_n| \ge c_1 > 0$, we see from (3.16) that

$$\begin{aligned} |a_{k,n}| &\le |\tau_{k,n}| + \frac{dc_3c_4}{c_1}\frac{1}{(\rho_0(F)-2\delta)^{k-n}}\sum_{s=1}^{d}|\tau_{n+s,n}| \\ &\le |\tau_{k,n}| + \frac{c_5}{(\rho_0(F)-2\delta)^{k-n}}\sum_{s=1}^{d}|\tau_{n+s,n}|, \qquad k \ge n+m_n+1, \end{aligned} \tag{3.21}$$

for $n \ge n_4$.

Secondly, we will give an estimate of $|b_{\nu,n}|$ (see the equality (3.8) for the definition of $b_{\nu,n}$) in terms of $|\tau_{k,n}|$. By the Cauchy-Schwarz inequality and the orthonormality of p_ν, we have

$$|\langle \omega_d p_k,\, p_\nu\rangle_\mu|^2 \le \langle \omega_d p_k,\, \omega_d p_k\rangle_\mu \langle p_\nu,\, p_\nu\rangle_\mu \le \max_{z\in E} |\omega_d(z)|^2 = c_6, \quad k,\nu = 0,1,2,\ldots. \tag{3.22}$$

By (3.21), (3.22), and the fact that

$$\sum_{k=n+m_n+1}^{\infty} \frac{1}{(\rho_0(F)-2\delta)^{k-n}} \le \sum_{k=1}^{\infty} \frac{1}{(\rho_0(F)-2\delta)^{k}} < \infty,$$

we obtain, for n sufficiently large and for all $\nu \ge 0$,

$$\begin{aligned} |b_{\nu,n}| &\le \sum_{k=n+m_n+1}^{\infty} |a_{k,n}||\langle \omega_d p_k,\, p_\nu\rangle_\mu| \le \sqrt{c_6} \sum_{k=n+m_n+1}^{\infty} |a_{k,n}| \\ &\le \sqrt{c_6}\left(\sum_{k=n+m_n+1}^{\infty} |\tau_{k,n}| + c_5 \sum_{k=n+m_n+1}^{\infty} \frac{1}{(\rho_0(F)-2\delta)^{k-n}} \sum_{s=1}^{d} |\tau_{n+s,n}|\right) \\ &\le c_7 \sum_{k=n+1}^{\infty} |\tau_{k,n}|. \end{aligned} \tag{3.23}$$

Finally, we will show that

$$\lim_{n\to\infty} \operatorname{cap}\{z \in K : |F(z) - [n/m_n]_F^\mu| \ge \varepsilon\} = 0.$$

Choose $\delta > 0$ so small that

$$\rho_2 := \rho_d(F) - \delta > \rho_{d-1}(F), \quad \rho_0(F) - 2\delta > 1, \quad \text{and} \quad \frac{\sigma+\delta}{\rho_2-\delta} < 1. \tag{3.24}$$

By the triangle inequality, we can rewrite (3.7) in the following form

$$|\omega_d(z) Q^\mu_{n,m_n}(z) F(z) - \omega_d(z) P^\mu_{n,m_n}(z)| \le \sum_{\nu=0}^{n+m_n} |b_{\nu,n}||p_\nu(z)| + \sum_{\nu=n+m_n+1}^{\infty} |b_{\nu,n}||p_\nu(z)|. \tag{3.25}$$

Define

$$A_n^1(z) := \frac{\sum_{\nu=0}^{n+m_n} |b_{\nu,n}||p_\nu(z)|}{|\omega_d(z) Q^\mu_{n,m_n}(z)|} \quad \text{and} \quad A_n^2(z) := \frac{\sum_{\nu=n+m_n+1}^{\infty} |b_{\nu,n}||p_\nu(z)|}{|\omega_d(z) Q^\mu_{n,m_n}(z)|},$$

and let $Q^\mu_{n,m_n}(z) := \prod_{j=1}^{u_n}(z-\lambda_{n,j})$. Therefore, the relation (3.25) implies

$$\left|F(z) - \frac{P^\mu_{n,m_n}(z)}{Q^\mu_{n,m_n}(z)}\right| \le A_n^1(z) + A_n^2(z),$$

for all $z \in \hat{D}_\sigma := \overline{D}_\sigma \setminus (\cup_{n=0}^{\infty}\{\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,u_n}\} \cup \{\lambda_1, \lambda_2, \ldots, \lambda_d\})$.
Let us bound $A_n^1(z)$ from above. We will first estimate $|\tau_{k,n}/Q^\mu_{n,m_n}(z)|$ for $z \in \hat{D}_\sigma$ and for $k \geq n+1$. By definition of $\tau_{k,n}$,

$$\frac{\tau_{k,n}}{Q^\mu_{n,m_n}(z)} = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} s_k(t)F(t)\frac{Q^\mu_{n,m_n}(t)}{Q^\mu_{n,m_n}(z)}dt, \quad k \geq n+1. \tag{3.26}$$

Then, we shall approximate the factors multiplying $F(t)$ in the integral sign separately. For n sufficiently large,

$$|s_k(t)| \leq \frac{c_8}{(\rho_2-\delta)^k}, \quad k \geq n+1.$$

Define

$$Q^\mu_{n,m_n,\rho_2}(t) := \prod_{\lambda_{n,j}\in D_{\rho_2}} (t-\lambda_{n,j}).$$

It is easy to see that

$$\left|\frac{t-\zeta}{z-\zeta}\right| \leq c_9,$$

for all $t \in \Gamma_{\rho_2}, z \in \hat{D}_\sigma$, and $\zeta \in \mathbb{C} \setminus D_{\rho_2}$ (note that the last inequality of (3.24) implies that $\rho_2 > \sigma$). Then,

$$\left|\frac{Q^\mu_{n,m_n}(t)}{Q^\mu_{n,m_n}(z)}\right| \leq {c_9}^{m_n}\left|\frac{Q^\mu_{n,m_n,\rho_2}(t)}{Q^\mu_{n,m_n,\rho_2}(z)}\right| \leq \frac{c_{10}^{m_n}}{|Q^\mu_{n,m_n,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma, \quad t \in \Gamma_{\rho_2}. \tag{3.27}$$

By (3.26), we obtain

$$\left|\frac{\tau_{k,n}}{Q^\mu_{n,m_n}(z)}\right| \leq \frac{c_{11}^{m_n}}{|Q^\mu_{n,m_n,\rho_2}(z)|(\rho_2-\delta)^k}, \quad z \in \hat{D}_\sigma, \quad k \geq n+1, \quad n \geq n_5,$$

which implies

$$\left|\frac{b_{\nu,n}}{Q^\mu_{n,m_n}(z)}\right| \leq \frac{c_{12}^{m_n}}{|Q^\mu_{n,m_n,\rho_2}(z)|(\rho_2-\delta)^n}, \quad z \in \hat{D}_\sigma, \quad n \geq n_6. \tag{3.28}$$

Applying (3.3) and the maximum modulus principle, we have

$$|p_\nu(z)| \leq c_{13}(\sigma+\delta)^\nu, \quad z \in \overline{D_\sigma}, \quad \nu \geq 0. \tag{3.29}$$

Using (3.28) and (3.29), we obtain the estimate:

$$A_n^1(z) = \frac{1}{|\omega_d(z)|}\sum_{\nu=0}^{n+m_n}\frac{|b_{\nu,n}||p_\nu(z)|}{|Q^\mu_{n,m_n}(z)|} \leq \frac{c_{13}c_{12}^{m_n}(n+m_n+1)(\sigma+\delta)^{n+m_n}}{|\omega_d(z)Q^\mu_{n,m_n,\rho_2}(z)|(\rho_2-\delta)^n}, \quad z \in \hat{D}_\sigma.$$

We choose $\theta > 0$ such that $(\sigma+\delta)/(\rho_2-\delta) < \theta < 1$. Since $m_n = o(n)$ as $n \to \infty$, for n sufficiently large, thus, we have

$$A_n^1(z) \leq \frac{c_{14}\theta^n}{|\omega_d(z)Q^\mu_{n,m_n,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma. \tag{3.30}$$

Next, let us approximate $A_n^2(z)$. Since $\deg(\omega_d P^{\mu}_{n,m_n}) \le n + d \le n + m_n$, by a computation similar to (3.6), we obtain

$$\begin{aligned} b_{\nu,n} &= \langle \omega_d Q^{\mu}_{n,m_n} F - \omega_d P^{\mu}_{n,m_n},\, p_\nu \rangle_\mu = \langle \omega_d Q^{\mu}_{n,m_n} F,\, p_\nu \rangle_\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \omega_d(t) Q^{\mu}_{n,m_n}(t) F(t) s_\nu(t) dt, \qquad \nu \ge n + m_n + 1. \end{aligned} \tag{3.31}$$

As before, from (3.4) and (3.31), we have

$$\frac{|b_{\nu,n}|}{|\omega_d(z) Q^{\mu}_{n,m_n}(z)|} \le \frac{c_{15}^{m_n}}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|(\rho_2 - \delta)^\nu}, \quad z \in \hat{D}_\sigma, \quad \nu \ge n + m_n + 1, \tag{3.32}$$

for $n \ge n_7$. Then, using (3.29) and (3.32), for n sufficiently large, we obtain, for $z \in \hat{D}_\sigma$, footnotesize

$$\begin{aligned} A_n^2(z) &= \sum_{\nu=n+m_n+1}^{\infty} \frac{|b_{\nu,n}||p_\nu(z)|}{|\omega_d(z) Q^{\mu}_{n,m_n}(z)|} = \sum_{\nu=n+m_n+1}^{\infty} \frac{c_{13} c_{15}^{m_n} (\sigma + \delta)^\nu}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|(\rho_2 - \delta)^\nu} \\ &\le \frac{c_{16}^{m_n}}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|} \sum_{\nu=n+m_n+1}^{\infty} \frac{(\sigma + \delta)^\nu}{(\rho_2 - \delta)^\nu} \\ &\le \frac{c_{16}^{m_n}}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|} \sum_{\nu=n+1}^{\infty} \frac{(\sigma + \delta)^\nu}{(\rho_2 - \delta)^\nu} \le \frac{c_{17}^{m_n}}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|} \left(\frac{\sigma + \delta}{\rho_2 - \delta} \right)^n. \end{aligned}$$

Therefore, for n sufficiently large,

$$A_n^2(z) \le \frac{c_{18} \theta^n}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|}. \tag{3.33}$$

Combining (3.30) and (3.33), we have, for n sufficiently large,

$$\left| F(z) - \frac{P^{\mu}_{n,m_n}(z)}{Q^{\mu}_{n,m_n}(z)} \right| \le \frac{c_{19} \theta^n}{|\omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)|}, \quad z \in \hat{D}_\sigma. \tag{3.34}$$

Let $T_n(z) := \omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z)$. Then, $T_n(z)$ is a monic polynomial of degree at most $2m_n$. Let $\varepsilon > 0$. Clearly,

$$\begin{aligned} e_n &:= \left\{ z \in \hat{D}_\sigma : \left| F(z) - \frac{P^{\mu}_{n,m_n}(z)}{Q^{\mu}_{n,m_n}(z)} \right| \ge \varepsilon \right\} \\ &\subset \left\{ z \in \hat{D}_\sigma : \left| \omega_d(z) Q^{\mu}_{n,m_n,\rho_2}(z) \right| \le \frac{c_{19} \theta^n}{\varepsilon} \right\} =: E_n. \end{aligned}$$

The capacity function is monotonic and has the well-known property,

$$\operatorname{cap} \{ z \in \mathbb{C} : |z^n + a_{n-1} z^{n-1} + \ldots + a_0| \le \rho^n \} = \rho, \quad \rho > 0.$$

Hence, we find that for n sufficiently large

$$\operatorname{cap} e_n \leq \operatorname{cap} E_n \leq \left(\frac{1}{\varepsilon} c_{19} \theta^n\right)^{1/\deg T_n} \leq \left(\frac{1}{\varepsilon} c_{19} \theta^n\right)^{1/2m_n} \leq \frac{c_{19}^{1/2m_n} \theta^{n/2m_n}}{\varepsilon^{1/2m_n}}.$$

This means that

$$\begin{aligned} \operatorname{cap}\{z \in K : |F(z) - [n/m_n]_F^\mu| \geq \varepsilon\} &\leq \operatorname{cap}\{z \in \overline{D_\sigma} : |F(z) - [n/m_n]_F^\mu| \geq \varepsilon\} \\ &= \operatorname{cap} e_n \to 0, \end{aligned}$$

as $n \to \infty$. This proves that $[n/m_n]_F^\mu$ converges in capacity to F inside $D_{\rho_\infty(F)}$, as $n \to \infty$. In addition, by Lemma 2.2, we get that each pole of F in $D_{\rho_\infty(F)}$ attracts at least as many poles of $[n/m_n]_F^\mu$ as the order of that pole. □

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