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Extendability of the Complementary Prism of Extendable Graphs[1]

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Abstract : A connected graph G of order at least $2k+2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. Let $\overline{G}$ denote the complement of a simple graph G. The complementary prism of G denoted by $G\overline{G}$ can be obtained by taking a copy of G and a copy of $\overline{G}$ and then joining corresponding vertices by an edge. In this paper, we show that for positive integers l_1 and l_2, there exists a non-bipartite graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable. Further, we show that if G is l_1-extendable and $\overline{G}$ is l_2-extendable non-bipartite graphs for $l_1 \geq 2$ and $l_2 \geq 2$, then $G\overline{G}$ is $(l+1)$-extendable where $l = min\{l_1, l_2\}$. Finally, we provide an example of graph $G\overline{G}$ such that $G\overline{G}$ is not 2-extendable when both G and $\overline{G}$ are 1-extendable non-bipartite.

Keywords : matching; extendable; complement; complementary prism.

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1 Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. A neighbor set of a vertex v in G is denoted by $N_G(v) = \{u \in V(G)|uv \in E(G)\}$. For $v \in V(G)$ and $T \subseteq V(G)$, a neighbor set of a vertex v in T is denoted by $N_T(v) = \{u \in T|uv \in E(G)\}$ and if $X \subseteq V(G)$, $N_G(X)$ denotes $\bigcup_{v\in X} N_G(v)$. Observe that $N_T(v) = N_G(v) \cap T$. The degree of a vertex u in G is denoted by $deg_G(u) = |N_G(u)|$. The minimum degree and maximum degree in a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. The distance between two vertices x and y in a graph G is the number of edges in a shortest path connecting them and denoted by $d_G(x, y)$. The number of odd components of G is denoted by $c_o(G)$. A complete graph of order r is denoted by K_r. Let $H \subseteq V(G)$, a subgraph of G induced by H is denoted by $G[H]$. For graphs H and G, G is called H-free if G does not contain H as an induced subgraph. A subgraph H is called a clique if $H \cong K_r$, for some r.

A set $S \subseteq V(G)$ is called an independent vertex set if no two of which are adjacent. The maximum cardinality of an independent set of G is denoted by $\alpha(G)$. For graphs H_1 and H_2, the join of H_1 and H_2, denoted by $H_1 + H_2$ is the graph with vertex set $V(H_1)\cup V(H_2)$ and edge set $E(H_1)\cup E(H_2)\cup\{uv|u \in V(H_1)$ and $v \in V(H_2)\}$. A subset M of $E(G)$ is called a matching if no two edges of M have common end vertex. A vertex u is saturated by M if there is an edge in M incident with u. For simplicity, a set of all vertices saturated by M is denoted by $V(M)$. M is called a maximum matching in G if G contains no matching of size greater than $|M|$. A perfect matching in G is a matching that saturates all vertices of G. If M_1, M_2 are matching in a graph G, then a symmetric different of M_1 and M_2, denoted by $M_1 \triangle M_2$, is an induced subgraph $G[(M_1 - M_2) \cup (M_2 - M_1)]$.

For a positive integer k, a connected graph G of order at least $2k + 2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. A graph G is k-factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For $k = 1$ and $k = 2$, k-factor-critical graph is also called factor-critical and bicritical, respectively. For simplicity, a graph with a perfect matching is called 0-extendable and 0-factor-critical. Observe that if G is k-extendable, then $|V(G)|$ is even and if G is k-factor-critical, then $|V(G)| \equiv k \pmod 2$.

In 1980, Plummer [1] introduced the concept of k-extendable graphs. He gave a sufficient condition for a graph to be k-extendable in terms of minimum degree. He also established a fundamental theorem (see Theorem 3.2) that mainly used in studying matching extension. One main problem is to establish sufficient conditions for special classes of graphs to be k-extendable. (see, e.g. [2, 3]). Since 1980, the concept of extendable graphs have been recieved attention from many researchers, see Plummer [4]-[6].

The concept of k-factor-critical graphs was introduced in 1996 by Favaron [7]. She established the necessary and sufficient condition for a graph to be k-factor-critical. She also provided a relationship between n-extendable graphs and k-factor - critical graphs.

The *complementary prism* of G, denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of $\overline{G}$ and then joining corresponding vertices by an edge. A complementary prism is a special case of complementary product of graphs introduced by Haynes et al. [8] in 2007. They pointed out that the famous Petersen graph is $C_5\overline{C}_5$ and a corona $K_n \cdot K_1$ which is the graph obtained by adding a vertex and an edge to every vertex of a complete graph K_n, is $K_n\overline{K}_n$. Some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number and the domination number have been investigated (see [8]-[14]).

A problem that arises is that of investigating properties of G so that $G\overline{G}$ is k-extendable for some positive integer k. In our two previous papers [15, 16], we establish that if G is a 2-regular H-free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable and if G is either connected 3-regular F-free or connected r_0-regular graph of order $p \geq 2r_0+1$ where $r_0 \geq 4$, then $G\overline{G}$ is 2-extendable where the graph F is shown in Figure 1. In this paper, we scope our attention to G and $\overline{G}$ which are both non-bipartite graphs. A new strategy for approaching the problem when G or $\overline{G}$ is bipartite is required. In fact, we show, in Section 4, that if G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs for $l_1 \geq 2$ and $l_2 \geq 2$, then $G\overline{G}$ is $(l+1)$-extendable where $l = min\{l_1, l_2\}$. One might ask whether there exist such graphs G and $\overline{G}$. We affirm this in Section 2 by providing some constructions of a non-bipartite graph G such that G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs, respectively, where l_1 and l_2 are positive integers. Section 3 contains some results on extendability and factor-criticality graphs that we make use of in establishing our results in Section 4.

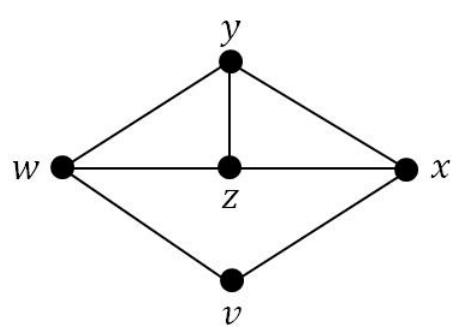

Figure 1: the graph F

2 Some constructions of extendable non-bipartite graphs

In this section, we provide two constructions of extendable non-bipartite graphs in which their complement graphs are also extendable by using cartesian and lexicographic products of two extendable graphs. We first give definitions of cartesian and lexicographic products.

The cartesian product $G \times H$ of two graphs G and H has the vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The lexicographic product $G \circ H$ of two graphs G and H has the vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent either $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

The first two results in this section concern the extendability of graphs obtained from a cartesian product, established by Györi and Plummer [2] and a lexicographic product established by Bai et al. [17].

Theorem 2.1 ([2])**.** *For non-negative integers l_1 and l_2, let G_i be a l_i -extendable graph for $1 \leq i \leq 2$. Then $G_1 \times G_2$ is $(l_1 + l_2 + 1)$-extendable.* □

Theorem 2.2 ([17])**.** *For non-negative integers l_1 and l_2, let G_i be a l_i -extendable graph for $1 \leq i \leq 2$. Then $G_1 \circ G_2$ is $2(l_1+1)(l_2+1)$-factor-critical. In particular, $G_1 \circ G_2$ is $(l_1 + 1)(l_2 + 1)$-extendable.* □

In 1980, Plummer[1] gave a sufficient condition for a graph to be k-extendable in terms of minimum degree.

Theorem 2.3 ([1])**.** *Let G be a graph of order $2p$. If $\delta(G) \geq p + k$, for a non-negative integer k, then G is k-extendable.* □

We are now ready for our constructions.

Theorem 2.4. *For non-negative integers l_1, l_2, $p_1 \geq 2l_1 + 2$ and $p_2 \geq 2l_2 + 2$ and $1 \leq i \leq 2$, let H_i be l_i-extendable of order p_i. Further, let $G = H_1 \times H_2$. If $\Delta(H_1) = p_1 - 1 - t_1$ and $\Delta(H_2) = p_2 - 1 - t_2$ for some non-negative integers t_1 and t_2, then G is $(l_1 + l_2 + 1)$-extendable and $\overline{G}$ is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$-extendable.*

Proof. By Theorem 2.1, $G = H_1 \times H_2$ is $(l_1 + l_2 + 1)$-extendable as required. We need only show that $\overline{G}$ is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$-extendable. Clearly, G and $\overline{G}$ are of order p_1p_2. Since $N_G((u, v)) = \{(x, v)|xu \in E(H_1)\} \cup \{(u, y)|vy \in E(H_2)\}$, $deg_G((u, v)) = deg_{H_1}(u) + deg_{H_2}(v)$. Thus $\Delta(G) = \Delta(H_1) + \Delta(H_2) = p_1 + p_2 - 2 - t_1 - t_2$. Therefore, $\delta(\overline{G}) = p_1p_2 - p_1 - p_2 + 2 + t_1 + t_2 - 1 = \frac{1}{2}p_1p_2 + \frac{1}{2}p_1p_2 - p_1 - p_2 + t_1 + t_2 + 1 = \frac{1}{2}p_1p_2 + \frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1$. By Theorem 2.3, $\overline{G}$ is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$-extendable as required. This proves our theorem. □

Corollary 2.5. *Let H_1, H_2 and G be graphs defined in Theorem 2.4. If either H_1 or H_2 is non-bipartite, then G and $\overline{G}$ are also non-bipartite.* □

Theorem 2.6. *For non-negative integers $h_1, h_2, \bar{h}_1, \bar{h}_2$, let H_i be a h_i-extendable and let $\overline{H}_i$ be a $\bar{h}_i$-extendable for $1 \leq i \leq 2$. Then $G = H_1 \circ H_2$ is $(h_1+1)(h_2+1)$-extendable graph and $\overline{G}$ is $(\bar{h}_1+1)(\bar{h}_2+1)$-extendable graph.*

Proof. By Theorem 2.2, $G = H_1 \circ H_2$ is $(h_1+1)(h_2+1)$-extendable. We first show that $\overline{G} = \overline{H}_1 \circ \overline{H}_2$. Clearly, $V(\overline{G}) = V(\overline{H_1 \circ H_2}) = V(H_1) \times V(H_2) = V(\overline{H}_1) \times V(\overline{H}_2) = V(\overline{H}_1 \circ \overline{H}_2)$. Let $(u_1, v_1), (u_2, v_2) \in V(H_1) \times V(H_2)$ and let $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1 \circ H_2})$. Thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$.

If $u_1 = u_2$, then $v_1 v_2 \notin E(H_2)$ and thus $(u_1, v_1)(u_2, v_2) \in E(\overline{H}_1 \circ \overline{H}_2)$. Further, if $u_1 \neq u_2$, then $u_1 u_2 \notin E(H_1)$. And again $(u_1, v_1)(u_2, v_2) \in E(\overline{H}_1 \circ \overline{H}_2)$. Hence, $E(\overline{H_1 \circ H_2}) \subseteq E(\overline{H}_1 \circ \overline{H}_2)$.

We now suppose that $(u_1, v_1)(u_2, v_2) \in E(\overline{H}_1 \circ \overline{H}_2)$. If $u_1 = u_2$, then $v_1 v_2 \in E(\overline{H}_2)$. Thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$. Further, if $u_1 \neq u_2$, then $u_1 u_2 \in E(\overline{H}_1)$ and thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$. In either case $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1 \circ H_2})$. Hence, $E(\overline{H}_1 \circ \overline{H}_2) \subseteq E(\overline{H_1 \circ H_2})$. Therefore, $E(\overline{H}_1 \circ \overline{H}_2) = E(\overline{H_1 \circ H_2})$. Thus $\overline{H_1 \circ H_2} = \overline{H}_1 \circ \overline{H}_2$. It follows by Theorem 2.2 that $\overline{G}$ is $(\bar{h}_1+1)(\bar{h}_2+1)$-extendable graph as required. This proves our theorem. □

Corollary 2.7. *For $1 \leq i \leq 2$, let $H_i, \overline{H}_i$ and G be graphs defined in Theorem 2.6. If H_1 is connected, $E(H_2) \neq \phi$ and $E(\overline{H}_2) \neq \phi$ then G and $\overline{G}$ are non-bipartite.* □

According to Theorems 2.4 and 2.6, we have shown that there exists a graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable for some integers l_1 and l_2. Theorem 2.9 establishes that for any positive integers l_1 and l_2, there is a graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable.

Lemma 2.8. *Let P_t be a path of order t. If $t \geq 4$ is an even integer, then P_t is 0-extendable and $\overline{P}_t$ is $(t-4)$-factor-critical. Further, $\overline{P}_t$ is $\frac{1}{2}(t-4)$-extendable.*

Proof. Clearly, P_t contains a perfect matching. We only show that $\overline{P}_t$ is $(t-4)$-factor-critical. Let $T \subseteq V(\overline{P}_t)$ such that $|T| = t-4$. Clearly, $\overline{P}_t - T$ is connected and contains P_4 as a subgraph. Thus $\overline{P}_t - T$ is one of a graph in $\{K_4, K_4 - e, C_4, K_4 - \{e_1, e_2\}, P_4\}$, where e_1 and e_2 have a common end vertex. In either case, $\overline{P}_t - T$ contains a perfect matching. Thus $\overline{P}_t - T$ is $(t-4)$-factor-critical as required. It then follows by definition of k-extendable that $\overline{P}_t$ is $\frac{1}{2}(t-4)$-extendable. This proves our lemma. □

Theorem 2.9. *For positive integers l_1 and l_2, there is a graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable. Further, G and $\overline{G}$ are non-bipartite.*

Proof. Let $H_1 = \overline{P}_{2l_1+2}$ and $H_2 = P_{2l_2+2}$. By Lemma 2.8, H_1 is (l_1-1)-extendable, $\overline{H}_1$ is 0-extendable, H_2 is 0-extendable and $\overline{H}_2$ is (l_2-1)-extendable. Let $G = H_1 \circ H_2$. By Theorem 2.6, G is l_1-extendable and $\overline{G}$ is l_2-extendable as required. Further, it is clear that G and $\overline{G}$ are non-bipartite. This proves our theorem. □

3 Results on extendability and k-factor critical graphs

In this section, we establish some results which are used in establishing our results in Section 4. Our first result is a well known theorem for studying an existence of a perfect matching in graphs established by Tutte.

Theorem 3.1 ([18]). *(Tutte's Theorem) A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G - S) \leq |S|$.* □

In 1980, Plummer [1] established a fundamental theorem on k-extendable graphs as following.

Theorem 3.2 ([1]). *Let G be a graph of order $p \geq 2k + 2$ and $k \geq 1$. If G is k-extendable, then*
(a) G is $(k - 1)$-extendable, and
(b) G is $(k + 1)$-connected. □

A necessary and sufficient condition for a graph to be k-extendable and to be k-factor-critical was provided by Yu [19] and Favaron [7], respectively.

Theorem 3.3 ([19]). *A graph G is k-extendable ($k \geq 1$) if and only if for any $S \subseteq V(G)$,*
(a) $c_o(G - S) \leq |S|$ and
(b) $c_o(G - S) = |S| - 2t, (0 \leq t \leq k - 1)$ implies that $F(S) \leq t$, where $F(S)$ is the size of a maximum matching in $G[S]$. □

Theorem 3.4 ([7]). *A graph G is k-factor-critical if and only if $|V(G)| \equiv k$ (mod 2) and for $S \subseteq V(G)$ with $|S| \geq k, c_o(G - S) \leq |S| - k$.* □

Some following properties of k-factor-critical graphs were proved in [7].

Theorem 3.5 ([7]). *Let G be a k-factor-critical graph. Then G is $(k-2)$-factor-critical.* □

Theorem 3.6 ([7]). *If G is a $2k$-extendable non-bipartite graph for $2k \geq 2$, then G is a $2k$-factor-critical graph.* □

Maschlanka and Volkmann [20] gave a relationship between k-extendable non-bipartite graph and the independence number.

Theorem 3.7 ([20]). *Let G be a k-extendable non-bipartite graph of order p. Then $\alpha(G) \leq \frac{1}{2}p - k$.* □

In Phd. Thesis of Yu [21], he gave the following observation.

Observation 3.1. *A graph G is k-extendable if and only if for any matching M of size i $(1 \leq i \leq k)$, $G - V(M)$ is a $(k - i)$-extendable graph.* □

An observation on k-factor-critical graphs which is similar to Observation 3.1 can be stated as following.

Observation 3.2. *Let G be a k-factor-critical graph and $S \subseteq V(G)$ where $|S| < k$. Then $G - S$ is $(k - |S|)$-factor-critical.* □

A following lemma follows from Theorem 3.7.

Lemma 3.8. *Let G be a k-extendable non-bipartite graph and $S \subseteq V(G)$ where $|S| \leq 2k - 2$. Then $G - S$ is a non-bipartite graph.*

Proof. Suppose to the contrary that $G - S$ is a bipartite graph. Then $\alpha(G) \geq \alpha(G - V(S)) \geq \frac{1}{2}(|V(G)| - (2k - 2)) = \frac{1}{2}|V(G)| - k + 1$. But this contradicts Theorem 3.7 and completes the proof of our lemma. □

Our next corollary follows immediately by Observation 3.1 and Lemma 3.8

Corollary 3.9. *Let G be a k-extendable non-bipartite graph and let M be a matching in G where $|M| = l \leq k - 1$. Then $G - V(M)$ is $(k - l)$-extendable non-bipartite.* □

Note that the upper bound on $|M|$ in Corollary 3.9 is best possible. Let $G = K_{2k} + K_{t,t}$ for some positive integers $k, t \geq 2$. It is easy to see that G is k-extendable. Clearly, there is a matching M of size k in $G[K_{2k}]$ such that $G - V(M)$ is a bipartite graph.

4 Main results

In this section, we establish the extendability of the complementary prism $G\overline{G}$ of G where G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs, respectively. We begin with some lemmas. To simplify our discussion of complementary prisms, G and $\overline{G}$ are referred to subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. For a vertex v of G, there is exactly one vertex of $\overline{G}$ which is adjacent to v in $G\overline{G}$. This vertex is denoted by $\overline{v}$. That is $\{\overline{v}\} = N_{\overline{G}}(v)$. Conversely, v is the only vertex of G which is adjacent to $\overline{v}$. Similarly, for a set $\phi \neq X = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\} \subseteq V(\overline{G})$ is denoted by $\overline{X}$ and vice versa.

In 2015, Janseana and Ananchuen [16] gave a relationship between the number of odd components and a size of a cutset in complementary prism.

Lemma 4.1 ([16]). *Let $G\overline{G}$ be a complementary prism and let $S = A \cup \overline{B}$ be a cutset of $G\overline{G}$, where $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Then*

a) $c_o(G\overline{G} - S) = |S| - 2t$, for some $t \geq 0$.

b) $c_o(G\overline{G} - S) = |A| + |B| - 2t \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |A| + |B| - 2|A \cap B|$. Consequently, $|A \cap B| \leq t$.

c) If $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |A| + |B| - 2|A \cap B|$, then each component of $G[B - A]$ and $\overline{G}[\overline{A} - \overline{B}]$ is singleton and hence $G[A - B]$ is a clique. □

Lemma 4.2. *Let G be a k-extendable graph for some integer $k \geq 2$ and let $S \subseteq V(G)$ be a cutset of G. If $G[S]$ contains $t \leq k-1$ independent edges, then $|S| \geq k+t+1$.*

Proof. Let $S' = S - V(F)$ where F is a matching of size t in $G[S]$. By Observation 3.1, $G' = G - V(F)$ is $(k-t)$-extendable. Observe that $k - t \geq 1$. By Theorem 3.2(b), G' is $(k-t+1)$-connected. Since S' is a cutset of G', $|S'| \geq k-t+1$ and thus $|S| \geq 2t + k - t + 1 = k + t + 1$ as required. This proves our lemma. □

Let x be a real number, $\lfloor x \rfloor_e$ is denoted a greatest even integer less than x and $\lfloor x \rfloor_o$ is denoted a greatest odd integer less than x. Clearly, $\lfloor x \rfloor_e = 2\lfloor \frac{x}{2} \rfloor$ and $\lfloor x \rfloor_o = 2\lfloor (x-1)/2 \rfloor + 1$. Note that for an even integer k, if x is an integer and $\lfloor x \rfloor_e = k$ then $x = k$ or $x = k+1$.

Lemma 4.3. *Let G be a k-extendable non-bipartite graph for $k \geq 2$. Further, let $M \subseteq E(G)$ be a matching of size m and let $S = \phi$ or $S \subseteq V(G) - V(M)$ be an independent set such that $k - m - |S| = t \geq 0$ for some integer t. Then*

(a) If $|S|$ is even, then $G-(V(M) \cup S)$ is t-extendable. Further $G-(V(M) \cup S)$ is $\lfloor t \rfloor_e$-factor-critical. Consequently, there is a perfect matching in $G-(V(M) \cup S)$.

(b) If $|S|$ is odd and $t \geq 1$, then $G - (V(M) \cup S)$ is $\lfloor t \rfloor_o$-factor-critical. Thus $G - (V(M) \cup S)$ is 1-factor-critical.

(c) If $|S|$ is odd, $t = 0$ and there is a vertex $v \in V(G) - (V(M) \cup S)$ such that $vs \in E(G)$ for some $s \in S$, then $G - (V(M) \cup S \cup \{v\})$ contains a perfect matching.

Proof. We first suppose $m = k$. So $S = \phi$ and thus $G - (V(M) \cup S) = G - V(M)$ contains a perfect matching by Theorem 3.2(a) and it is 0-factor-critical as required. We now suppose that $m \leq k-1$. By Corollary 3.9, $G - V(M)$ is $(k-m)$-extendable non-bipartite. Since $k - m = |S| + t$, $G - V(M)$ is $(|S| + t)$-extendable non-bipartite.

(a) $|S|$ is even. By Theorem 3.2(a), $G - V(M)$ is $(|S| + \lfloor t \rfloor_e)$-extendable and thus it is $(|S| + \lfloor t \rfloor_e)$-factor-critical by Theorem 3.6. Hence, by Observation 3.2, $G - (V(M) \cup S)$ is $\lfloor t \rfloor_e$-factor-critical as required. It then follows by Theorem 3.2(a) that $G - (V(M) \cup S)$ contains a perfect matching. This proves (a).

(b) $|S|$ is odd and $t \geq 1$. By Theorem 3.2(a), $G - V(M)$ is $(|S| + \lfloor t \rfloor_o)$-extendable and thus it is $(|S| + \lfloor t \rfloor_o)$-factor-critical by Theorem 3.6. By Observation 3.2, $G - (V(M) \cup S)$ is $\lfloor t \rfloor_o$-factor-critical. Since $t \geq 1$, $\lfloor t \rfloor_o \geq 1$. Further, by Theorem 3.5, $G - (V(M) \cup S)$ is 1-factor-critical as required. This proves (b).

(c) Let $M' = M \cup \{vs\}$ and $S' = S - \{s\}$. Hence, our result follows from (a). This completes the proof of our lemma. □

Lemma 4.4. *Let G be a k-extendable graph for some integer k and let $S \subseteq V(G)$ be a cutset of G. If $G[S]$ contains t independent edges for $t \leq k$, then $c_o(G-S) \leq |S| - 2t$. Further, if $1 \leq t \leq k-1$ and $c_o(G-S) = |S| - 2t$ then $G - S$ contains no even components.*

Proof. Let F be a matching of size t in $G[S]$. Since G is a k-extendable graph, $G - V(F)$ contains a perfect matching by Theorem 3.2(a). By Theorem 3.1, $c_o(G - S) = c_o((G - V(F)) - (S - V(F))) \leq |S - V(F)| = |S| - 2t$, as required. We now suppose that $1 \leq t \leq k - 1$ and $c_o(G - S) = |S| - 2t$. Let D be an even component of $G - S$. By Lemma 4.2 and the fact that $t \leq k - 1 < k + 1$, $V(F)$ is not a cutset of G. Then there is an edge $e = sd$ joining a vertex s in $S - V(F)$ and a vertex d in D. Since G is k-extendable and $F \cup \{e\}$ is a matching of size $t + 1 \leq k$, it follows that there is a perfect matching in $G' = G - (V(F) \cup \{s, d\})$. Let $S' = S - (V(F) \cup \{s\})$. Clearly, $c_o(G' - S') = c_o(G - S) + 1 = |S| - 2t + 1$. Since G' contains a perfect matching, by Theorem 3.1, $|S| - 2t + 1 \leq c_o(G' - S') \leq |S| - (|V(F)| + 1) = |S| - 2t - 1$, a contradiction. Hence, there is no even component in $G - S$. This proves our lemma. □

Lemma 4.5. *Let G be a l-extendable graph and let M be a matching of size $l + t$ where $t \geq 1$. Then there is a maximum matching in $G - V(M)$ saturates all except at most $2t$ non-adjacent vertices in $G - V(M)$.*

Proof. Let $T \subseteq M$ where $|T| = t$. Thus $M - T$ is a matching of size l in G. So there is a perfect matching F in $G - V(M - T)$. Clearly, $|V(F) \cap V(T)| = 2t$. Let $F_1 = \{xy \in F | \{x, y\} \cap V(M) = \phi\}$ and $F_2 = \{xy \in F | x \in V(M) \text{ and } y \notin V(M)\}$. Further, let F_2' be a maximum matching in $G[V(F_2) - V(M)]$. Then, $F_1 \cup F_2'$ is a matching in $G - V(M)$ saturates all except at most $2t$ non-adjacent vertices as required. □

By similar arguments as in the proof of Lemma 4.5, the next lemma follows.

Lemma 4.6. *Let G be a k-factor-critical graph and let $T \subseteq V(G)$ where $|T| = k+t$. Then there is a maximum matching in $G - V(T)$ saturates all except at most t non-adjacent vertices.* □

Lemma 4.7. *Let G be an 1-extendable graph of order $p \geq 6$ and let v be a vertex of degree 2 in G. Then there are perfect matchings M_1, M_2 in G such that v is a vertex of C_{2n} in $M_1 \triangle M_2$ where $n \geq 3$. Further, there is a vertex $x \in V(C_{2n})$ where C_{2n} is a subgraph of $M_1 \triangle M_2$ such that $vx \notin E(G)$ and $G - \{v, x\}$ contains a perfect matching.*

Proof. Let $\{u_1, u_2\} = N_G(v)$. We first suppose $N_G(u_1) \cap N_G(u_2) = \{v\}$. Let M_1 be a perfect matching in G containing vu_1 and M_2 a perfect matching in G containing vu_2. Clearly, $\{vu_1, u_2u_3\} \subseteq M_1$ and $\{vu_2, u_1u_4\} \subseteq M_2$ for some $u_3, u_4 \in V(G)$. Since $\{v\} = N_G(u_1) \cap N_G(u_2), u_3 \neq u_4$. Hence, $u_3u_2vu_1u_4$ is a path of length 4 containing v. It must be contained in an even cycle of order at least 6 in $M_1 \triangle M_2$ as required.

So we now suppose that $N_G(u_1) \cap N_G(u_2) \neq \{v\}$. Then there is a vertex $v \neq u_3 \in N_G(u_1) \cap N_G(u_2)$. Since G is 2-connected by Theorem 3.2(b) and G is of order at least 6, it follows that u_3 is not a cut vertex. Then there is a vertex $u_4 \in N_G(u_1) \cup N_G(u_2)$ where $u_4 \neq u_3$. Without loss of generality, suppose $u_4 \in N_G(u_1)$. Let M_1 be a perfect matching in G containing u_1u_4 and M_2 be a perfect matching in G containing u_2u_3. It is easy to see that $\{u_1u_4, vu_2\} \subseteq M_1$ and $\{u_2u_3, vu_1\} \subseteq M_2$. Hence, $u_4u_1vu_2u_3$ is a path of length 4 containing v. It must be contained in an even cycle of order at least 6 in $M_1 \triangle M_2$ as required. Further, let $x \in V(C_{2n})$ such that the distance between v and x along the cycle C_{2n} is 3. Clearly, $xv \notin E(G)$ and it is easy to see that $G - \{v, x\}$ contains a perfect matching. This completes the proof of our lemma. □

For an induced subgraph H of G, Com_H denotes the set of all components in H. If $X \subseteq V(G)$, then we use Com_X for $Com_{G[X]}$. For a cutset S of $G\overline{G}$, put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup B)$. Thus $S = A \cup \overline{B}$. Further, let $T_{B-A} = \{F | F$ is an odd component of $G[B - A]$ and $N_G(u) - V(F) \subseteq A$ for all $u \in V(F)\}$. $T_{\overline{A}-\overline{B}} = \{F | F$ is an odd component of $\overline{G}[\overline{A} - \overline{B}]$ and $N_{\overline{G}}(\bar{u}) - V(F) \subseteq \overline{B}$ for all $\bar{u} \in V(F)\}$. Finally, let $L = L_G \cup L_{\overline{G}}$, where $L_G = \{F | F$ is an odd component in $G[B - A]$ and $N_{G\overline{G}}(V(F)) \cap C \neq \phi\}$ and $L_{\overline{G}} = \{F | F$ is an odd component in $\overline{G}[\overline{A} - \overline{B}]$ and $N_{G\overline{G}}(V(F)) \cap \overline{C} \neq \phi\}$. Note that if $C = \phi$, then $L = \phi$. Clearly, $T_{B-A} \cap L_G = \phi$ and $T_{\overline{A}-\overline{B}} \cap L_{\overline{G}} = \phi$. It is easy to see that, if G is connected and $G[B - A]$ contains only odd components, then $Com_{B-A} = T_{B-A} \cup L_G$. Similarly, if $\overline{G}$ is connected and $\overline{G}[\overline{A} - \overline{B}]$ contains only odd components, then $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}}$. In what follows, the symbols Com_H, S, A, $\overline{B}$, C, T_{B-A}, $T_{\overline{A}-\overline{B}}$, L, L_G and $L_{\overline{G}}$ are referred to these set up.

Lemma 4.8. *For a graph G, let $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Suppose $c_o(G - A) = |A| - t_1$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - t_2$, for some non-negative integers t_1, t_2. Then $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2)$. Further, if $A \cup B \neq V(G)$ and $G - A$ and $\overline{G} - \overline{B}$ contain no even components, then $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2) - 2$.*

Proof. It is easy to see that $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2)$. We now suppose that $A \cup B \neq V(G)$ and $G - A$, $\overline{G} - \overline{B}$ contain no even components. Let $x \in V(G) - (A \cup B)$. Then x is in an odd component of $G - A$, say C. Clearly, $\bar{x} \notin \overline{A} \cup \overline{B}$ and thus $\bar{x}$ is in an odd component of $\overline{G} - \overline{B}$, say D. Hence, $G\overline{G}[V(C) \cup V(D)]$ forms an even component in $G\overline{G} - (A \cup \overline{B})$. Therefore $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2) - 2$ as required. This proves our lemma. □

Lemma 4.9. *Let G and $\overline{G}$ be l_1-extendable and l_2-extendable graphs, respectively where l_1 and l_2 are positive integers. Further, let M be a matching of size $l + 1$ in $G\overline{G}$ where $l = min\{l_1, l_2\}$. If either $M = \{x_i\bar{x}_i | x_i \in V(G)$ for $1 \leq i \leq l + 1\}$ or $M \subseteq (E(G) \cup E(\overline{G}))$, then $G\overline{G}$ has a a perfect matching containing M.*

Proof. Clearly, if $M = \{x_i\bar{x}_i | x_i \in V(G)$ for $1 \leq i \leq l + 1\}$, then $\{v\bar{v} | v \in V(G)\}$ is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that $M \subseteq (E(G) \cup E(\overline{G}))$. Put $M_G = M \cap E(G)$ and $M_{\overline{G}} = M \cap E(\overline{G})$. If $1 \leq |M_G| \leq l$ and $1 \leq |M_{\overline{G}}| \leq l$, then it is easy to see that $M = M_G \cup M_{\overline{G}}$ can be extended to a perfect matching in $G\overline{G}$, by Theorem 3.2(a), since G is l_1-extendable and $\overline{G}$ is l_2-extendable. Hence, we suppose without loss of generality that $|M_G| = l + 1$. Suppose there is no perfect matching in G containing M_G. By Lemma 4.5, there is a maximum matching F_1 in $G - V(M)$ saturates all except two non-adjacent vertices, say x and y. So $\bar{x}\bar{y} \in E(\overline{G})$. Since $\overline{G}$ is l_2-extendable where $l_2 \geq 1$ and by Theorem 3.2(a), it follows that there is a perfect matching F_2 in $\overline{G}$ containing $\bar{x}\bar{y}$. Hence, $M \cup F_1 \cup (F_2 - \{\bar{x}\bar{y}\}) \cup \{x\bar{x}, y\bar{y}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our lemma. □

We are now ready to prove our main result. We begin with the extendability of $G\overline{G}$ where G is l_1-extendable and $\overline{G}$ is l_2-extendable for $l_1 \geq 4$ and $l_2 \geq 4$.

Theorem 4.10. *For positive integers $l_1 \geq 4$, $l_2 \geq 4$, let G and $\overline{G}$ be l_1-extendable and l_2-extendable non-bipartite graphs of order $p \geq 2l + 2$, respectively, where $l = min\{l_1, l_2\}$. Then $G\overline{G}$ is $(l + 1)$-extendable.*

Proof. Let $M \subseteq E(G\overline{G})$ be a matching of size $l+1$ in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Note that $M_{G\overline{G}} = \{x\bar{x}|$ for some $x \in V(G)\}$. If $M_{G\overline{G}} = M$ or $M_{G\overline{G}} = \phi$, then, by Lemma 4.9, there is a perfect matching in $G\overline{G}$ containing M as required. We now suppose that $M_{G\overline{G}} \neq M$ and $M_{G\overline{G}} \neq \phi$. Without loss of generality, we may suppose that $|M_G| \geq |M_{\overline{G}}|$. Hence, $M_G \neq \phi$.

Put $S = V(G) \cap V(M_{G\overline{G}})$. Let N_S be a maximum matching in $G[S]$. Put $I_S = S - V(N_S)$. Clearly, I_S is an independent set. Similarly, let $N_{\overline{S}}$ be a maximum matching in $\overline{G}[\overline{S}]$ and put $I_{\overline{S}} = \overline{S} - V(N_{\overline{S}})$. For simplicity, we denote the cardinalities of each set by its small letter, i.e., $m_G = |M_G|$, $m_{\overline{G}} = |M_{\overline{G}}|$, $m_{G\overline{G}} = |M_{G\overline{G}}|$, $s = |S|$, $i_S = |I_S|$, etc.

Clearly, $1 \leq m_G \leq l$, $s = \bar{s}$, $n_S + i_S \geq 1$ and $n_{\overline{S}} + i_{\overline{S}} \geq 1$ since $s = \bar{s} = m_{G\overline{G}} \geq 1$. Therefore,

$$l + 1 = m_G + m_{\overline{G}} + m_{G\overline{G}} \tag{4.1}$$

$$l + 1 = m_G + m_{\overline{G}} + s \tag{4.2}$$

$$l + 1 = m_G + m_{\overline{G}} + 2n_S + i_S \tag{4.3}$$

$$l + 1 = m_G + m_{\overline{G}} + 2n_{\overline{S}} + i_{\overline{S}}. \tag{4.4}$$

Consequently, $m_G + n_S = l + 1 - (m_{\overline{G}} + n_S + i_S) \leq l$ since $n_S + i_S \geq 1$ and $m_{\overline{G}} + n_{\overline{S}} = l + 1 - (m_G + n_{\overline{S}} + i_{\overline{S}}) \leq l$ since $n_{\overline{S}} + i_{\overline{S}} \geq 1$. Further, $s \equiv i_S$ (mod 2) and $\bar{s} \equiv i_{\overline{S}}$ (mod 2) because $s = 2n_S + i_S$ and $s = \bar{s} = 2n_{\overline{S}} + i_{\overline{S}}$.

We first suppose that $i_S = 0$. Since $m_G + n_S \leq l$, by Theorem 3.2(a), there is a perfect matching in $G - (V(M_G) \cup N_S)$, say F_G. Now consider $\overline{G}$. By Equation 4.4, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq 0$ since $m_G \geq 1$. By Lemma 4.3(a), there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}}$ is a perfect matching in $G\overline{G}$ containing M as required.

So we now suppose that $i_S \geq 1$. We distinguish 2 cases according to parity of s.

Case 1 : s is even. So $i_S \geq 2$ and $i_{\overline{S}} \geq 0$ are also even. We distinguish 2 subcases according to $m_{\overline{G}} + n_S$.

Subcase 1.1 : $m_{\overline{G}}+n_S \geq 1$. So, by Equation 4.3, $l-(m_G+n_S+i_S) = m_{\overline{G}}+n_S-1 \geq 0$. By Lemma 4.3(a), there is a perfect matching in $G-(V(M_G \cup N_S) \cup I_S)$, say F_G. Further, by Equation 4.4 and the fact that $m_G \geq 1$, $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = m_G+n_{\overline{S}}-1 \geq 0$. So, by Lemma 4.3(a), there is a perfect matching in $\overline{G}-(V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}}$ is a perfect matching in $G\overline{G}$ containing M as required.

Subcase 1.2 : $m_{\overline{G}} = n_S = 0$. We first show that $n_{\overline{S}} \leq \frac{l}{2}$. Since $n_S = 0$, $G[S]$ is independent and thus $\overline{G}[\overline{S}]$ is a complete graph. Because s is even, $n_{\overline{S}} = \frac{1}{2}\bar{s} = \frac{1}{2}s$. So, by Equation 4.2 and the fact that $m_G \geq 1$, $n_{\overline{S}} = \frac{1}{2}s = \frac{1}{2}(l+1-m_G-m_{\overline{G}}) = \frac{1}{2}(l+1-m_G) \leq \frac{l}{2}$ as required. By Equation 4.3, $l-m_G = m_{\overline{G}}+2n_S+i_S-1 = i_S-1$ and $\lfloor i_S-1 \rfloor_e = i_S-2 \geq 0$ since $i_S = s$ is even. It follows by Lemma 4.3(a) that $G' = G-V(M_G)$ is (i_S-2)-factor-critical. By Lemma 4.6, there is a maximum matching F_G in $G'-I_S$ saturates all except at most 2 vertices in $G'-I_S$.

We next consider $\overline{G}$. By Equation 4.4 and the fact that $m_G \geq 1$, $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = m_G+n_{\overline{S}}-1 \geq n_{\overline{S}} \geq 0$. By Lemma 4.3(a), there is a perfect matching in $\overline{G}-(V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$, say $F_{\overline{G}}$. Clearly, if F_G is a perfect matching in $G'-I_S$, then $M \cup F_G \cup F_{\overline{G}}$ is a perfect matching in $G\overline{G}$ as required.

We now suppose that F_G is not a perfect matching. Let $x, y \in V(G')-I_S$ where x and y are unsaturated by F_G. Clearly, $xy \notin E(G)$. So $\bar{x}\bar{y} \in E(\overline{G})$. Because $n_{\overline{S}} \leq \frac{l}{2}$ and $l \geq 4$, it follows that $m_{\overline{G}}+n_{\overline{S}}+1 = n_{\overline{S}}+1 \leq \frac{l}{2}+1 \leq \frac{l}{2}+(\frac{l}{2}-1) \leq l-1$. By Theorem 3.2(a), there is a perfect matching in $\overline{G}-V(M_{\overline{G}} \cup N_{\overline{S}} \cup \{\bar{x}\bar{y}\})$, say $F'_{\overline{G}}$. Hence, $M \cup F_G \cup (F'_{\overline{G}}-\{\bar{x}\bar{y}\}) \cup \{x\bar{x}, y\bar{y}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This proves Case 1.

Case 2 : s is odd. So i_S and $i_{\overline{S}}$ are also odd. We distinguish 3 subcases according to $m_{\overline{G}}+n_S$.

Subcase 2.1 : $m_{\overline{G}} = n_S = 0$. By Equation 4.3, $l-(m_G+(i_S-1)) = m_{\overline{G}}+2n_S = 0$. Let $i \in I_S$, by Lemma 4.3(a), there is a perfect matching in $G-(V(M_G) \cup (I_S-\{i\}))$, say F_G. Let $iv \in F_G$. We now consider $\overline{G}$. Since $n_S = 0$, $G[S]$ is independent and thus $\overline{G}[\overline{S}]$ is a complete graph of odd order s. Therefore, $n_{\overline{S}} = \frac{1}{2}(s-1)$ and $i_{\overline{S}} = 1$. By Equation 4.2, $l = m_G+s-1$. So $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = l-(n_{\overline{S}}+i_{\overline{S}}) = m_G+s-1-(\frac{1}{2}(s-1)+1) = m_G+\frac{1}{2}(s-3)$.

We next show that $m_G+\frac{1}{2}(s-3) \geq 1$. Suppose to the contrary that $m_G+\frac{1}{2}(s-3) = 0$. Since $m_G \geq 1$ and s is a positive odd integer, it follows that $m_G = 1$ and $s = 1$. By Equation 4.2, $l+1 = m_G+m_{\overline{G}}+s = 1+0+1 = 2$. Thus $l = 1$, contradicting the fact that $l \geq 4$. Hence, $m_G+\frac{1}{2}(s-3) \geq 1$ as required.

Therefore, $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = m_G+\frac{1}{2}(s-3) \geq 1$. By Lemma 4.3(b), $\overline{G}-(V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ is 1-factor-critical. Recall that $iv \in F_G$. Clearly, $\bar{v} \notin V(M_{\overline{G}})$ since $m_{\overline{G}} = 0$. So there is a perfect matching in $\overline{G}-((V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}}) \cup \{\bar{v}\})$, say $F_{\overline{G}}$. Hence, $M \cup (F_G-\{iv\}) \cup F_{\overline{G}} \cup \{v\bar{v}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This proves Subcase 2.1.

Subcase 2.2 : $m_{\overline{G}}+n_S \geq 2$. By Equation 4.3, $l-(m_G+n_S+i_S) = m_{\overline{G}}+n_S-1 \geq 1$. By Lemma 4.3(b), $G-(V(M_G \cup N_S) \cup I_S)$ is 1-factor-critical.

We now consider $\overline{G}$. By Equation 4.4, $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = m_G+n_{\overline{S}}-1$.

We first suppose that $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = m_G+n_{\overline{S}}-1 \geq 1$. By Lemma 4.3(b),

$\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ is 1-factor-critical. Let $x \in V(G)$ such that $x, \bar{x} \notin V(M)$. Clearly, x exists because $|V(M_G \cup M_{\overline{G}}) \cup S| \leq 2l + 1$ and G and $\overline{G}$ are of order at least $2l+2$. Since $G - (V(M_G \cup N_S) \cup I_S)$ and $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ are 1-factor-critical, it follows that there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S \cup \{x\})$, say F_G, and there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}} \cup \{\bar{x}\})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{x\bar{x}\}$ is a perfect matching in $G\overline{G}$ containing M as required.

So we next suppose that $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 = 0$. It follows that $n_{\overline{S}} = 0$ and $m_G = 1$ since $m_G \geq 1$. Thus $i_{\overline{S}} = \bar{s}$ and $m_{\overline{G}} \leq m_G = 1$. Put $M_G = \{xy\}$. Since $\overline{G}$ is l_2-extendable, for $l_2 \geq 4$, by Theorem 3.2(b), $\overline{G}$ is 5-connected. So $\{\bar{x}, \bar{y}\} \cup V(M_{\overline{G}})$ is not a cutset of $\overline{G}$ since $|\{\bar{x}, \bar{y}\} \cup V(M_{\overline{G}})| \leq 4$. Hence, there is an edge joining a vertex in $V(\overline{G}) - (\{\bar{x}, \bar{y}\} \cup V(M_{\overline{G}}))$, say $\bar{u}$, and a vertex in $\overline{S}$, say $\bar{w}$. Because $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = 0$ and $i_{\overline{S}} = \bar{s}$, it follows that $l - (m_{\overline{G}} + n_{\overline{S}} + 1 + (i_{\overline{S}} - 1)) = l - (m_{\overline{G}} + n_{\overline{S}} + 1 + (\bar{s} - 1)) = 0$. By Lemma 4.3(a), there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}} \cup \{\bar{u}\bar{w}\}) \cup (\overline{S} - \{\bar{w}\}))$, say $F_{\overline{G}}$. Since $G - (V(M_G \cup N_S) \cup I_S)$ is 1-factor-critical and $u \notin V(M_G)$, it follows that there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S \cup \{u\})$, say F_G. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This proves Subcase 2.2.

Subcase 2.3 : $m_{\overline{G}} + n_S = 1$. By Equation 4.3 and the fact that i_S is odd, $m_G + n_S = l + 1 - (m_{\overline{G}} + n_S + i_S) \leq l - 1$. We distinguish 2 subcases according to $m_{\overline{G}}$ and n_S.

Subcase 2.3.1 : $m_{\overline{G}} = 0$ and $n_S = 1$. Observe that $G[V(M_G \cup N_S)]$ contains $m_G + n_S \leq l - 1$ independent edges and $|V(M_G \cup N_S)| = 2(m_G + n_S) = (m_G + n_S) + m_G + n_S \leq l - 1 + (m_G + n_S)$. It follows by Lemma 4.2 that $V(M_G \cup N_S)$ is not a cutset of G. Then there are a vertex $u \in V(G) - (V(M_G) \cup S)$ and a vertex $z \in I_S$ such that $uz \in E(G)$. Since $l - ((m_G + n_S + 1) + (i_S - 1)) = l - (m_G + n_S + i_S) = m_{\overline{G}} + n_S - 1 = 0$, by Lemma 4.3(a), there is a perfect matching in $G - (V(M_G \cup N_S \cup \{uz\}) \cup (I_S - \{z\}))$, say F_G. We now consider $\overline{G}$. We next show that $m_G + n_{\overline{S}} \geq 2$. Suppose to the contrary that $m_G + n_{\overline{S}} = 1$. Since $m_G \geq 1$, $n_{\overline{S}} = 0$ and $m_G = 1$. By Equation 4.3, $l + 1 = m_G + m_{\overline{G}} + 2n_S + i_S = 3 + i_S$. So $i_S = l - 2 \geq 4 - 2 = 2$. It follows that $\overline{G}[\overline{S}]$ contains K_2 as an induced subgraph. Thus $n_{\overline{S}} \geq 1$, contradicting the fact that $n_{\overline{S}} = 0$. Hence, $m_G + n_{\overline{S}} \geq 2$.

By Equation 4.4, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq 1$. By Lemma 4.3(b), $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ is 1-factor-critical. Recall that $m_{\overline{G}} = 0$. So $\bar{u} \notin V(M_{\overline{G}})$ Therefore, there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}} \cup \{\bar{u}\})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of Subcase 2.3.1.

Subcase 2.3.2 : $m_{\overline{G}} = 1$ and $n_S = 0$. Put $m_{\overline{G}} = \{\bar{x}_1\bar{x}_2\}$. Note that $m_G + n_{\overline{S}} \geq 2$ since $m = l + 1 \geq 5$ and $n_S = 0$. If there is a vertex $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$ such that $uz \in E(G)$ for some $z \in S$, then by applying similar argument as in the proof of Subcase 2.3.1, there is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that there is no vertex $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$ such that $uz \in E(G)$ for some $z \in S$. Thus $V(M_G) \cup \{x_1, x_2\}$ is a cutset of G and $\{x_1, x_2\}$ is a cutset of $G - V(M_G)$. We next show that $s = 1$. Suppose to the contrary that $s \geq 3$. By Equation

4.2, $m_G = l + 1 - m_{\overline{G}} - s = l - s \leq l - 3$. By Observation 3.1, $G - V(M_G)$ is $(l - m_G)$-extendable. Because $l - m_G \geq 3$, by Theorem 3.2(b), $G - V(M_G)$ is 4-connected, contradicting the fact that $\{x_1, x_2\}$ is a cutset of $G - V(M_G)$. Hence, $s = 1$. Put $S = \{z\}$. Therefore, $zu \notin E(G)$ for $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$. So $N_G(z) \subseteq V(M_G) \cup \{x_1, x_2\}$

By Equation 4.2, $m_G = l + 1 - m_{\overline{G}} - s = l - 1$. By Observation 3.1, $G' = G - V(M_G)$ is 1-extendable. By Theorem 3.2(b), G' is 2-connected. Therefore, $N_{G'}(z) = \{x_1, x_2\}$ and $deg_{G'}(z) = 2$. By Lemma 4.7, there is a vertex $u \in V(G')$ such that $uz \notin E(G')$ and $G' - \{u, z\}$ contains a perfect matching, say F_G. We now consider $\overline{G}$. Since $l \geq 4$, $m_{\overline{G}} = 1$ and $\bar{s} = s = 1$, it follows that $l - (m_{\overline{G}} + \bar{s}) = l - 2 \geq 2$. By Lemma 4.3(b), $\overline{G}' = \overline{G} - V(M_{\overline{G}} \cup \overline{S})$ is 1-factor-critical. Then there is a perfect matching in $\overline{G}' - \{\bar{u}\}$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of Subcase 2.3.2. and thus completes the proof of our theorem. □

We now turn our attention to the extendability of $G\overline{G}$ when G or $\overline{G}$ is l-extendable for $1 \leq l \leq 3$. We first provide an example of a graph G where both G and $\overline{G}$ are 1-extendable but $G\overline{G}$ is not 2-extendable. Let H be a 1-extendable graph such that $\overline{H}$ is k-extendable for some integer $k \geq 1$. By Theorems 2.4, 2.6 and 2.9, H exists. We now construct a 1-extendable graph G from H. Let $P = u_1, u_2, u_3, u_4$ be a path of order 4 and put $V(G) = V(H) \cup V(P)$ and $E(G) = E(H) \cup V(P) \cup \{u_1h, u_4h | h \in V(H)\}$. It is routine to verify that G and $\overline{G}$ are 1-extendable. However, $G\overline{G}$ is not 2-extendable since $\{u_1\bar{u}_1, u_3u_4\}$ cannot be extended to a perfect matching in $G\overline{G}$.

We now scope our attention to extendability of $G\overline{G}$ where G is l_1-extendable and $\overline{G}$ is l_2-extendable for $l_1 \geq 2$ and $l_2 \geq 2$. We first consider the case $l_1 = 2$ and $l_2 \geq 2$. We begin with the following lemma. Recall that if $\phi \neq \{x_1, \ldots, x_t\} \subseteq V(G)$, then $\{\bar{x}_1, \ldots, \bar{x}_t\} \subseteq V(\overline{G})$ is denoted by $\overline{X}$ and vice versa.

Lemma 4.11. *Let G and $\overline{G}$ be 2-extendable non-bipartite graphs of order $p \geq 10$ and let $M = \{x_1x_2, \bar{y}_1\bar{y}_2, z\bar{z}\}$ be a matching of size 3 in $G\overline{G}$, where $\{x_1, x_2, z\} \subseteq V(G)$ and $\{\bar{y}_1, \bar{y}_2, \bar{z}\} \subseteq V(\overline{G})$. Then there is a perfect matching in $G\overline{G}$ containing M.*

Proof. Suppose to the contrary that there is no perfect matching in $G\overline{G}$ containing M. By Theorem 3.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. By parity, $c_o(G\overline{G} - (V(M) \cup T)) \geq |T| + 2$. Put $S = T \cup V(M)$. So $c_o(G\overline{G} - S) \geq |S| - 4$. Put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup B)$. Observe that $|A| \geq 3$ and $|\overline{B}| \geq 3$.

By Theorem 3.6, G and $\overline{G}$ are bicritical. Thus, by Theorem 3.4, $c_o(G - A) \leq |A| - 2$ and $c_o(\overline{G} - \overline{B}) \leq |\overline{B}| - 2$. We first show that $c_o(G - A) = |A| - 2$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Suppose to the contrary that $c_o(G - A) < |A| - 2$. By parity, $c_o(G - A) \leq |A| - 4$. It then follows by Lemma 4.8 that $c_o(G\overline{G} - S) \leq c_o(G - A) + c_o(\overline{G} - \overline{B}) \leq |A| + |\overline{B}| - 6$, contradicting the fact that $c_o(G\overline{G} - S) \geq |S| - 4$. Hence, $c_o(G - A) = |A| - 2$. Similarly, $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$.

Since G and $\overline{G}$ are 2-extendable, by Theorem 3.3(b), $G[A]$ and $\overline{G}[\overline{B}]$ contain at most one independent edge. Because $\{x_1, x_2, z\} \subseteq A$ and $\{\bar{y}_1, \bar{y}_2, \bar{z}\} \subseteq \overline{B}$, $G[A]$ and $\overline{G}[\overline{B}]$ contain exactly 1 independent edge. By Lemma 4.4, $G - A$ and $\overline{G} - \overline{B}$ contain no even components. If $A \cup B \neq V(G)$, then, by Lemma 4.8, $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - 6 = |S| - 6$, again a contradiction. Hence, $A \cup B = V(G)$. Observe that if $c_o(G - A) \geq 4$, $G[B] = G - A$ contains at least 4 independence vertices and thus $\overline{G}[\overline{B}]$ contains a matching of size at least two, a contradiction. Hence, $c_o(G - A) \leq 3$. Similarly, $c_o(\overline{G} - \overline{B}) \leq 3$ and each component of $\overline{G} - \overline{B}$ is singleton otherwise $G[A] = G - B$ contains at least 2 independent edges, a contradiction. Therefore, $c_o(G[B - A]) = c_o(G - A) \leq 3$ and $\overline{G}[\overline{A} - \overline{B}] = c_o(\overline{G} - \overline{B}) \leq 3$. Since $c_o(G - A) = |A| - 2$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$, it follows that $|A| = 2 + c_o(G - A) \leq 5$ and $|B| = |\overline{B}| = 2 + c_o(\overline{G} - \overline{B}) \leq 5$. Because $z \in A \cap B$, $|A \cup B| = |A| + |B| - |A \cap B| \leq 5 + 5 - 1 \leq 9$, contradicting the fact that $|V(G)| = p \geq 10$. This completes the proof of our lemma. □

The next theorem shows that if G is a 2-extendable non-bipartite graph and $\overline{G}$ is a l-extendable non-bipartite graph of order $p \geq 10$ and $l \geq 2$, then $G\overline{G}$ is 3-extendable.

Theorem 4.12. *Let G be a 2-extendable non-bipartite graph of order $p \geq 10$. If $\overline{G}$ is l-extendable non-bipartite for some positive integer $l \geq 2$, then $G\overline{G}$ is 3-extendable.*

Proof. By Theorem 3.2(b), $\overline{G}$ is 2-extendable non-bipartite graph. Let M be a matching of size 3 in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Further, put $m_G = |M_G|, m_{\overline{G}} = |M_{\overline{G}}|$ and $m_{G\overline{G}} = |M_{G\overline{G}}|$. If $m_{G\overline{G}} = 0$ or $m_{G\overline{G}} = 3$, then, by Lemma 4.9, there is a perfect matching in $G\overline{G}$ containing M as required. So we now consider $1 \leq m_{G\overline{G}} \leq 2$. We distinguish 2 cases according to $m_{G\overline{G}}$.

Case 1: $m_{G\overline{G}} = 1$. If $m_G = m_{\overline{G}} = 1$, then, by Lemma 4.11, there is a perfect matching in $G\overline{G}$ containing M as required. So we suppose without loss of generality that $m_G = 2$, $m_{\overline{G}} = 0$. By applying similar arguments as in the proof of Subcase 2.1 in Theorem 4.10, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 2: $m_{G\overline{G}} = 2$. By applying similar arguments as in the proof of Case 1 in Theorem 4.10, there is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our theorem. □

We point out here that the bound on the order of graphs in Theorem 4.12 is best possible and the hypothesis that G and $\overline{G}$ are non-bipartite is essential. Let G be a 3-regular bipartite graph of order 8 with bipartition (X, Y) where $X = \{x_i | 1 \leq i \leq 4\}$ and $Y = \{y_i | 1 \leq i \leq 4\}$ and $E(G) = \{x_i y_j | 1 \leq i \neq j \leq 4\}$. It is not difficult to show that $\overline{G} \cong K_4 \times K_2$ and both G and $\overline{G}$ are 2-extendable. However, $G\overline{G}$ is not 3-extendable since $\{x_1\bar{x}_1, x_2y_1, \bar{y}_2\bar{y}_3\}$ cannot be extended to a perfect matching in $G\overline{G}$.

We finally turn our attention to 3-extendable graphs.

Lemma 4.13. *Suppose G and $\overline{G}$ are 3-extendable non-bipartite graphs of order $p \geq 8$. Let $\{x, y, z_1, z_2, z_3\} \subseteq V(G)$ and $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} \subseteq V(\overline{G})$ such that $G[\{z_1, z_2, z_3\}] \cong K_3$. Further, let $M = \{xy, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3\}$ be a matching of size 4 in $G\overline{G}$. Then there is a perfect matching in $G\overline{G}$ containing M.*

Proof. Suppose there is no perfect matching in $G\overline{G} - V(M)$. Then by Theorem 3.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (T \cup V(M))) > |T|$. By parity, $c_o(G\overline{G} - (T \cup V(M))) \geq |T| + 2$. Put $S = T \cup V(M)$. So $c_o(G\overline{G} - S) \geq |S| - 6$. Since $G\overline{G}$ contains a perfect matching, by Theorem 3.1, $c_o(G\overline{G} - S) \leq |S|$. Thus $|S| - 6 \leq c_o(G\overline{G} - S) \leq |S|$. Put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup B)$.

Clearly, $\{z_1, z_2, z_3\} \subseteq A \cap B$. By Lemma 4.1(b), $c_o(G\overline{G} - S) \leq |S| - 6$. So $c_o(G\overline{G} - S) = |S| - 6$.

Since $xy, z_1z_2 \in E(G)$, by Lemma 4.4, $c_o(G - A) \leq |A| - 4$. On the other hand, since $\overline{G}$ is 3-extendable non-bipartite graph, by Theorems 3.2(a) and 3.6, $\overline{G}$ is bicritical. Therefore, by Theorem 3.4, $c_o(\overline{G} - \overline{B}) \leq |\overline{B}| - 2$. We first show that $c_o(G - A) = |A| - 4$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Suppose to the contrary that $c_o(G - A) \neq |A| - 4$. By parity, $c_o(G - A) \leq |A| - 6$. By Lemma 4.8(a), $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| - 6 + |\overline{B}| - 2 = |S| - 8$, a contradiction. Hence, $c_o(G - A) = |A| - 4$. By similar argument, $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. By Lemma 4.4, $G - A$ contains no even components. We next show that $\overline{G} - \overline{B}$ contains no even components. Suppose this is not the case. Then $\overline{G} - \overline{B}$ contains an even component, say $\overline{D}$. Let $\bar{b}\bar{d} \in E(\overline{G})$ such that $\bar{b} \in \overline{B}$ and $\bar{d} \in V(\overline{D})$. By Corollary 3.9, $\overline{G}' = \overline{G} - \{\bar{b}, \bar{d}\}$ is 2-extendable non-bipartite. By Theorem 3.6, $\overline{G}'$ is bicritical. Since $c_o(\overline{G} - (\overline{B} \cup \{\bar{d}\})) = |\overline{B}| - 1$, $c_o(\overline{G}' - (\overline{B} - \{\bar{b}\})) = |\overline{B} - \{\bar{b}\}|$, contradicting Theorem 3.4. Hence, $\overline{G} - \overline{B}$ contains no even components.

If $A \cup B \neq V(G)$, then by Lemma 4.8, $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq c_o(G - A) + c_o(\overline{G} - \overline{B}) - 2 = |A| + |\overline{B}| - 8 = |S| - 8$, a contradiction. So $A \cup B = V(G)$.

Note that $G[A - B]$ contains the edge xy. We first show that $G[A - B]$ contains exactly one independent edge. Suppose $G[A - B]$ contains 2 independent edges. Since $z_1z_2 \in E(G[A \cap B])$, there are at least 3 independent edges in $G[A]$. Therefore, by Lemma 4.4, $c_o(G - A) \leq |A| - 6$, contradicting the fact that $c_o(G - A) = |A| - 4$. Hence, $G[A - B]$ contains exactly one independent edge. We next show that $\overline{G}[\overline{B}]$ contains no edges. Suppose to the contrary that $\overline{B}$ contains an edge $\bar{u}_1\bar{u}_2$. By Corollary 3.9, $\overline{G} - \{\bar{u}_1, \bar{u}_2\}$ is 2-extendable non-bipartite graph. By Theorem 3.6, $\overline{G} - \{\bar{u}_1, \bar{u}_2\}$ is bicritical. Then, by Theorem 3.4, $c_o(\overline{G} - \overline{B}) = c_o((\overline{G} - \{\bar{u}_1, \bar{u}_2\}) - (\overline{B} - \{\bar{u}_1, \bar{u}_2\})) \leq |\overline{B} - \{\bar{u}_1, \bar{u}_2\}| - 2 = |\overline{B}| - 4$, contradicting the fact that $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Hence, $\overline{G}[\overline{B}]$ contains no edges and $\overline{G}[\overline{B}]$ is independent. So $G[B]$ and $G[B - A]$ are clique and thus $c_o(G[B - A]) \leq 1$.

Therefore, $|A| - 4 = c_o(G - A) = c_o(G[B - A]) \leq 1$. So $|A| \leq 5$. If $\overline{G}[\overline{A} - \overline{B}]$ contains at least 4 components, then $G[A - B]$ contains at least two independent edges. But this contradicts the fact that $G[A - B]$ contains exactly one independent edges. Hence, $\overline{G}[\overline{A} - \overline{B}]$ contains at most 3 components. Therefore, $c_o(\overline{G}[\overline{A} - \overline{B}]) =$

$c_o(\overline{G}-\overline{B}) = |\overline{B}|-2 \leq 3$. Hence, $|B| = |\overline{B}| \leq 5$. It follows that $|V(G)| = |A \cup B| = |A| + |B| - |A \cap B| \leq 5 + 5 - 3 = 7$, a contradiction. This proves our lemma. □

Lemma 4.14. *Suppose G and $\overline{G}$ are 3-extendable non-bipartite graphs of order $p \geq 8$. Let $\{x, y, z_1, z_2, z_3\} \subseteq V(G)$ and $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} \subseteq V(\overline{G})$ such that $G[\{z_1, z_2, z_3\}] \ncong K_3$. Further, let $M = \{xy, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3\}$ be a matching of size 4 in $G\overline{G}$. Then there is a perfect matching in $G\overline{G}$ containing M.*

Proof. Suppose $M = \{xy, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3\}$ where $x, y \in V(G)$. Since $G[\{z_1, z_2, z_3\}] \ncong K_3$, we may suppose that $z_1z_2 \notin E(G)$. Since $xy \in E(G)$, by Lemma 4.3(a), there is a perfect matching in $G - \{x, y, z_1, z_2\}$, say F_G. Let $z_3w \in F_G$. Again, because $\bar{z}_1\bar{z}_2 \in E(\overline{G})$, by Lemma 4.3(a), there is a perfect matching in $\overline{G} - \{\bar{z}_1, \bar{z}_2, \bar{w}, \bar{z}_3\}$, say $F_{\overline{G}}$. Thus $M \cup (F_G - \{z_3w\}) \cup F_{\overline{G}} \cup \{w\bar{w}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our lemma. □

Theorem 4.15. *Let G be a 3-extendable non-bipartite graph of order $p \geq 8$. If $\overline{G}$ is l-extendable non-bipartite for some positive integer $l \geq 3$, then $G\overline{G}$ is 4-extendable.*

Proof. By Theorem 3.2(b), $\overline{G}$ is 3-extendable non-bipartite graph. Let M be a matching of size 4 in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Without loss of generallity, suppose $|M_G| \geq |M_{\overline{G}}|$. If $M_{G\overline{G}} = \phi$ or $M_{G\overline{G}} = M$, then, by Lemma 4.9, there is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that $M_{G\overline{G}} \neq \phi$ and $M_{G\overline{G}} \neq M$. Therefore, $1 \leq |M_{G\overline{G}}| \leq 3$. We distinguish 3 cases according to $|M_{G\overline{G}}|$.

Case 1: $|M_{G\overline{G}}| = 1$. By applying similar arguments as in the proof of Subcase 2.1 (if $|M_{\overline{G}}| = 0$) or Subcase 2.3 (if $|M_{\overline{G}}| = 1$) in Theorem 4.10, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 2: $|M_{G\overline{G}}| = 2$. By applying similar arguments as in the proof of Case 1 in Theorem 4.10, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 3: $|M_{G\overline{G}}| = 3$. Then, $|M_G| = 1$ and $|M_{\overline{G}}| = 0$. So, by Lemmas 4.13 and 4.14, there is a perfect matching in $G\overline{G}$ containing M as required.

This completes the proof of our theorem. □

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