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Coupled Coincidence Point Theorems for a α-ψ-Contractive Mapping in Partially Metric Spaces with M-Invariant Set[1]

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Abstract : In this paper, we introduce the notion M-invariant set for mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$. We showed the existence of a coupled coincidence point theorem for a α-ψ-contractive mapping in partially ordered complete metric spaces without the mixed g-monotone property, using the concept of M-invariant set. We also show the uniqueness of a coupled common fixed point for such mappings and give some examples to show the validity of our result.

Keywords : fixed point; coupled coincidence; invariant set; ddmissible.
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1 Introduction

The first result in the existence of a fixed point for contraction type of mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [1] in 2004. Following this work, Nieto and Lopez [2, 3] extened the results in [1] for non-decreasing mapping. Later, Agarwal *et al.* [4] presented some new results for contractions in partially ordered metric spaces.

A notion of coupled fixed point theorem was defined by Guo and Lakshmikantham [5]. After that, Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone property. Furthermore, they proved the existence and uniqueness

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of a coupled fixed point theorems for mappings which satisfy the mixed monotone property in partially ordered metric space. Since 2006, many authors have studied coupled fixed point theorems in partially ordered metric space and their applications have been established. The results in [6] were extend by Lakshimikantham and Ciric in [7] by defining the mixed g-monotone and to study the existence and uniqueness of coupled coincidence point for such mapping which satisfy the mixed monotone property in partially ordered metric space. As a continuation of this work, several coupled fixed point and coupled coincidence point results have appeared in the recent literature. Work noted in [7–27] are some examples of these works.

In 2012, Samet *et al.* [23] introduced the concept of an α-ψ-contractive and α-admissible mapping and show fixed point theorem for such mapping in complete metric spaces. Later, Mursaleen et al.[20] reconsidered the notion of an α-ψ-contractive and α-admissible and prove some coupled fixed point theorem for generalized contractive mapping in partially ordered metric spaces. Very recently, Kaushik *et al.* [19] defined the notion of an α-ψ-contractive and (α)-admissible for a pair of mapping and prove coupled coincidence point theorem which generalization of the result of Mursaleen *et al.* [20].

In this work, we introduce M-invariant set for mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ and prove the existence of a coupled coincidence point theorem and a coupled common fixed point theorem for a α-ψ-contractive mapping in partially ordered complete metric spaces.

2 Preliminaries

2.1 Instructions to Authors

In this section, we give some definitions, proposition, examples and remarks which are useful for main results in this paper. Throughout this paper, $(X, \leq)$ denotes a partially ordered set with the partial order $\leq$. By $x \leq y$, we mean $y \geq x$. A mapping $f : X \to X$ is said to be non-decreasing (resp., non-increasing) if for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$ (resp. $f(y) \geq f(x)$).

Let $(x, \leq)$ is a partially ordered set, the partial order $\leq_2$ for the product set $X \times X$ defined in the following way

$$(x, y), (u, v) \in X \times X, \quad (x, y) \leq_2 (u, v) \Longleftrightarrow x \leq u \quad \text{and} \quad y \geq v.$$

We say that (x, y) is comparable to (u, v) if either $(x, y) \leq_2 (u, v)$ or $(u, v) \leq_2 (x, y)$.

The concept of a mixed monotone property and a coupled fixed point have been introduced by Bhaskar and Lakshmikantham in [6].

Definition 2.1 ([6]). *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. We say F has the mixed monotone property if for any $x, y \in X$*

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \textit{ implies } F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Definition 2.2 ([6]). *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Lakshmikantham and *Ćirić* in [7] introduced the concept of a mixed g-monotone mapping and a coupled coincidence point.

Definition 2.3 ([7]). *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if for any $x, y \in X$*

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \text{ implies } F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \leq gy_2 \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Definition 2.4 ([7]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 2.5 ([7]). *Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.*

Let Ψ denote the family non-decreasing and right continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\Sigma_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ satisfying the following conditions:

1. $\psi^{-1}(\{0\}) = \{0\}$,
2. $\psi(t) < t$ for all $t > 0$,
3. $\lim_{r \to t^+} \psi(r) < t$ for all $t > 0$.

Lemma 2.6. *If $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and right continuous, then $\psi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Mursaleen *et al.* [20] introduce (α, ψ)-contractive in following way.

Definition 2.7 ([20]). *Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ be a mapping. Then a map F is said to be (α, ψ)-contractive if there exist two functions $\alpha : X^2 \times X^2 \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi(\frac{d(x, u) + d(y, v)}{2})$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Definition 2.8 ([20]). *Let $F : X \times X \to X$ and $\alpha : X^2 \times X^2 \to [0, \infty)$. We say that F is (α)-admissible if for all $x, y, u, v \in X$, we have*

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1.$$

Theorem 2.9 ([20]). *Let $(X, \leq)$ be a partially ordered set and let there exist d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be mapping having the mixed monotone property on X. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0, \infty)$, such that the following holds*

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi(\frac{d(x, u) + d(y, v)}{2}) \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Suppose also that

(i) F is (α)-admissible.

(ii) F is continuous.

(iii) there exist $x_0, y_0 \in X$ such that

$$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \text{ and } \alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is F has a coupled fixed point.

Kaushik *et al.* [19] introduced the notion of (α, ψ)-contractive and (α)-admissible for a pair of mapping and prove coupled coincidence point theorem which generalization of the result of Mursaleen *et al.* [20] as follow.

Definition 2.10 ([19]). *Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ and $g : X \to X$ be two mapping. Then F and g are said to be (α, ψ)-contractive if there exist two functions $\alpha : X^2 \times X^2 \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha((gx, gy), (gu, gv))d(F(x, y), F(u, v)) \leq \psi(\frac{d(gx, gu) + d(gy, gv)}{2})$$

for all $x, y, u, v \in X$ with $gx \geq gu$ and $gy \leq gv$.

Definition 2.11 ([19]). *Let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, \infty)$. We say that F and g are (α)-admissible if for all $x, y, u, v \in X$, we have*

$$\alpha((gx, gy), (gu, gv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1.$$

Theorem 2.12 ([19]). *Let $(X, \leq)$ be a partially ordered set and let there exist d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mapping having the mixed g-monotone property on X. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0, \infty)$, such that the following holds*

$$\alpha((gx, gy), (gu, gv))d(F(x, y), F(u, v)) \leq \psi(\frac{d(gx, gu) + d(gy, gv)}{2}) \tag{2.2}$$

for all $x, y, u, v \in X$ with $gx \geq gu$ and $gy \leq gv$.

Suppose also that

(i) F *and* g *are* (α)*-admissible.*

(ii) F *is continuous.*

(iii) $F(X \times X)) \subseteq g(X)$ *and* g *is continuous and commutes with* F.

(iv) *there exist* $x_0, y_0 \in X$ *such that*

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$$
$$\text{and} \quad \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$$

If there exist $x_0, y_0 \in X$ *such that*

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$*, that is* F *has a coupled coincidence point.*

Theorem 2.13 ([19]). *Let* $(X, \leq)$ *be a partially ordered set and let there exist* d *be a metric on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be mapping having the mixed g-monotone property on* X*. Suppose that there exists* $\psi \in \Psi$ *and* $\alpha : X^2 \times X^2 \to [0, \infty)$*, such that the following holds*

$$\alpha((gx, gy), (gu, gv))d(F(x, y), F(u, v)) \leq \psi(\frac{d(gx, gu) + d(gy, gv)}{2})$$

for all $x, y, u, v \in X$ *with* $gx \geq gu$ *and* $gy \leq gv$.

Suppose also that

(i) F *and* g *are* (α)*-admissible.*

(ii) *For any two sequences* (x_n) *and* (y_n) *in* X *with* $\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1$ *and* $\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1$ *for all* n*, if there is* $x, y \in X$ *such that* $lim_{n\to\infty} x_n = x$ *and* $lim_{n\to\infty} y_n = y$*, then* $\alpha((gx_n, gy_n), (gx, gy)) \geq 1$ *and* $\alpha((gy_n, gx_n), (gy, gx)) \geq 1$ *for all* n.

(iii) $F(X \times X)) \subseteq g(X)$ *and* g *is continuous and commutes with* F.

(iv) *there exist* $x_0, y_0 \in X$ *such that*

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$$
$$\text{and} \quad \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1$$

If there exist $x_0, y_0 \in X$ *such that*

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$*, that is* F *has a coupled coincidence point.*

Now, we give the notion of M-invariant which is useful for our main results.

Definition 2.14. *Let (X,d) be a metric space and M be a nonempty subset of X^4 and $\alpha : X^2 \times X^2 \to [0,\infty)$. We say that α is M-invariant if for all $x,y,u,v \in X$,*

$$\alpha((x,y),(u,v)) \geq 1 \Rightarrow (x,y,u,v) \in M$$

Example 2.15. *Let $F : X \times X \to X$, $g : X \to X$. Consider a mapping $\alpha : X^2 \times X^2 \to [0,\infty)$ be such that*

$$\alpha((x,y),(u,v)) = 1, \quad \text{for all } x,y,u,v \in X$$

It is easy to see that, α is M-invariant where $M = X^4$ and F is (α)-admissible, also F and g are (α)-admissible.

Next example, we will show that F is (α)-admissible but F and g are not (α)-admissible.

Example 2.16. *Let $X = R$ and $F : X \times X \to X$, $g : X \to X$. Define by $F(x,y) = 1 - x^2$ and $g(x) = x - 1$. Consider a mapping $\alpha : X^2 \times X^2 \to [0,\infty)$ be such that*

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & \text{if } x = u = 0 \text{ and } y = v, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that F and g are (α)-admissible.
Let $y = v$,we have $y - 1 = gy = gv = v - 1$ and $\alpha((0,gy),(0,gv))$=$\alpha((g1,y-1),(g1,v-1)) \geq 1$ implies that
$\alpha((F(1,y-1),F(y-1,1)),(F(1,v-1),(F(v-1,1)))$=$\alpha((0,1-(y-1)^2),(0,1-(v-1)^2)) \geq 1$.
Next, we show that F is not (α)-admissible. Consider $\alpha((0,y),(0,v)) \geq 1$ but
$\alpha((F(0,y),F(y,0)),(F(0,v),F(v,0)))$=$\alpha((1,1-y^2),(1,1-v^2)) = 0$.

3 Main Results

Theorem 3.1. *Let $(X,\leq)$ be a partially ordered set and M be a nonempty subset of X^4 and let there exist d be a metric on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mapping. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0,\infty)$,such that the following holds*

$$\begin{aligned} &\alpha((gx,gy),(gu,gv))[\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}] \\ &\leq \psi(\frac{d(gx,gu) + d(gy,gv)}{2}) \end{aligned} \tag{3.1}$$

for all $x,y,u,v \in X$ with $(gx,gy,gu,gv) \in M$.

Suppose also that

(i) F and g are (α)-admissible.

(ii) F is continuous.

(iii) $F(X \times X)) \subseteq g(X)$ *and* g *is continuous and commutes with* F.

(iv) *there exist* $x_0, y_0 \in X$ *such that*

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1.$$

(v) α *is* M*-invariant.*

Then there exist $(x, y) \in X \times X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$*, that is* F *has a coupled coincidence point.*

Proof *Let* $(x_0, y_0) \in X \times X$. *Since* $F(X \times X) \subseteq g(X)$*, we can choose* $x_1, y_1 \in X$ *such that*

$$gx_1 = F(x_0, y_0) \text{ and } gy_1 = F(y_0, x_0).$$

Again from $F(X \times X) \subseteq g(X)$ *we can choose* $x_2, y_2 \in X$ *such that*

$$gx_2 = F(x_1, y_1) \text{ and } gy_2 = F(y_1, x_1).$$

Continuing this process we can construct sequences $\{(gx_n)\}$ *and* $\{(gy_n)\}$ *in* X *such that*

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 1. \tag{3.2}$$

Since

$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) = \alpha((gx_0, gy_0), (gx_1, gy_1)) \geq 1$$

From F *and* g *are* (α)*-admissible, we have*

$$\alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) = \alpha((gx_1, gy_1), (gx_2, gy_2)) \geq 1$$

Again, using the fact that F *and* g *are* (α)*-admissible, we have*

$$\alpha((F(x_1, y_1), F(y_1, x_1)), (F(x_2, y_2), F(y_2, x_2))) = \alpha((gx_2, gy_2), (gx_3, gy_3)) \geq 1$$

By repeating this argument, we get

$$\begin{aligned} &\alpha((F(x_n, y_n), F(y_n, x_n)), (F(x_{n+1}, y_{n+1}), F(y_{n+1}, x_{n+1}))) \\ &\qquad = \alpha((gx_{n-1}, gy_{n-1}), (gx_n, gy_n)) \geq 1. \end{aligned} \tag{3.3}$$

Since α *is* M*-invariant, we have*

$$(gx_{n-1}, gy_{n-1}, gx_n, gy_n) \in M \tag{3.4}$$

If there exists $k \in N$ *such that* $(gx_{k+1}, gy_{k+1}) = (gx_k, gy_k)$ *then* $gx_k = gx_{k+1} = F(x_k, y_k)$ *and* $gy_k = gy_{k+1} = F(y_k, x_k)$. *Thus,* (x_k, y_k) *is a coupled coincidence point of* F. *This is finishes the proof. Now we assume that* $(gx_{k+1}, gy_{k+1}) \neq (gx_k, gy_k)$ *for all* $n \geq 0$. *Thus, we have either* $gx_{n+1} = F(x_n, y_n) \neq$

gx_n or $gy_{n+1} = F(y_n, x_n) \neq gy_n$ for all $n \geq 0$.
From (3.1), (3.3) and (3.4), we have

$$\begin{aligned}
&[\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}] \\
&\leq \alpha((gx_{n-1}, gy_{n-1}), (gx_n, gy_n))[\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}] \\
&= \alpha((gx_{n-1}, gy_{n-1}), (gx_n, gy_n)) \\
&\quad [\frac{d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))}{2}] \\
&\leq \psi(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}). \qquad (3.5)
\end{aligned}$$

From (3.5), we get

$$\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2} \leq \psi(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}). \qquad (3.6)$$

Since $\psi(t) < t$ for all $t > 0$, by repeating (3.6), we get

$$\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2} \leq \psi^n(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}). \qquad (3.7)$$

for all $n \in \mathbb{N}$.
For $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}) < \frac{\epsilon}{2}.$$

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\epsilon)$. Then, by using the triangle inequality, we have

$$\begin{aligned}
\frac{d(gx_n, gx_m) + d(gy_n, gy_m)}{2} \leq\ & \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2} \\
&+ \frac{d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})}{2} \\
&+ \frac{d(x_{n+2}, x_{n+3}) + d(y_{n+2}, y_{n+3})}{2} \\
&+ \cdots + \frac{d(x_{m-1}, x_m) + d(y_{m-1}, y_m)}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=n}^{m-1} \frac{d(gx_k, gx_{k+1}) + d(gy_k, gy_{k+1})}{2} \\
&\leq \sum_{k=n}^{m-1} \psi^k\Big(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}\Big) \\
&\leq \sum_{n \geq n(\epsilon)} \psi^n\Big(\frac{d(gx_0, gx_1) + d(gy_0, gy_1)}{2}\Big) \\
&< \frac{\epsilon}{2}. \qquad (3.8)
\end{aligned}
$$

This implies that $d(gx_n, gx_m)+d(gy_n, gy_m) < \epsilon$. *Since*

$$d(gx_n, gx_m) \leq d(gx_n, gx_m) + d(gy_n, gy_m) < \epsilon$$

and

$$d(gy_n, gy_m) \leq d(gx_n, gx_m) + d(gy_n, gy_m) < \epsilon.$$

This show that $\{gx_n\}$ *and* $\{gy_n\}$ *are Cauchy sequence in the metric space* (X, d). *Since* (X, d) *is complete,* $\{gx_n\}$ *and* $\{gy_n\}$ *are convergent, there exist* $x, y \in X$ *such that* $\lim_{n\to\infty} gx_n = x$ *and* $\lim_{n\to\infty} gy_n = y$. *From continuity of* g *, we get*

$$\lim_{n\to\infty} g(gx_n) = gx \quad \text{and} \quad \lim_{n\to\infty} g(gy_n) = gy.$$

From (3.3) *and commutativity of* F *and* g*,*

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n), \qquad (3.9)$$
$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n). \qquad (3.10)$$

We now show that $F(x, y) = gx$ *and* $F(y, x) = gy$.
Taking the limit as $n \to +\infty$ *in* (3.9) *and* (3.10)*, by continuity of* F*, we get*

$$
\begin{aligned}
g(x) &= g(\lim_{n\to\infty} gx_{n+1}) = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(F(x_n, y_n)) \\
&= \lim_{n\to\infty} F(gx_n, gy_n) = F(x, y)
\end{aligned}
$$

and

$$
\begin{aligned}
g(y) &= g(\lim_{n\to\infty} gy_{n+1}) = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(F(y_n, x_n)) \\
&= \lim_{n\to\infty} F(gy_n, gx_n) = F(y, x).
\end{aligned}
$$

Thus we prove that $F(x, y) = gx$ *and* $F(y, x) = gy$.

In the next theorem, we omit the continuity hypothesis of F.

Theorem 3.2. *Let* $(X, \leq)$ *be a partially ordered set and* M *be a nonempty subset of* X^4 *and let there exist* d *be a metric on* X *such that* (X, d) *is a complete metric* *space. Let $F : X \times X \to X$ and $g : X \to X$ be two mapping. Suppose that there exists $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0,\infty)$,such that the following holds*

$$\alpha((gx,gy),(gu,gv))[\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}] \leq \psi(\frac{d(gx,gu) + d(gy,gv)}{2})$$

for all $x,y,u,v \in X$ with $(gx,gy,gu,gv) \in M$.

Suppose also that

- *(i) F and g are (α)-admissible.*
- *(ii) For any two sequences (x_n) and (y_n) in X with $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \geq 1$ for all n, if there is $x,y \in X$ such that $lim_{n\to\infty}x_n = x$ and $lim_{n\to\infty}y_n = y$, then $\alpha((x,y),(x_n,y_n)) \geq 1$ for all n.*
- *(iii) $F(X \times X)) \subseteq g(X)$, g is continuous and commutes with F and $(g(X),d)$ is complete metric space.*
- *(iv) there exist $x_0,y_0 \in X$ such that*
$$\alpha((gx_0,gy_0),(F(x_0,y_0),F(y_0,x_0))) \geq 1.$$
- *(v) α is M-invariant.*

Then there exist $(x,y) \in X \times X$ such that $gx = F(x,y)$ and $gy = F(y,x)$, that is F has a coupled coincidence point.

Proof *Proceeding exactly as in Theorem3.1 , we have that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the complete metric space $(g(X),d)$ and $\alpha((gx_n,gy_n),(gx_{n+1},gy_{n+1})) \geq 1$. Then, there exists $gx,gy \in g(X)$ such that $\{gx_n\} \to gx$ and $\{gy_n\} \to gy$. by assumption (ii), we have $\alpha((gx,gy),(gx_n,gy_n)) \geq 1$ for all n.*
From α is M-invariant, we have $(gx,gy,gx_n,gy_n) \in M$. Now by (3.1) , the triangle inequality and $\psi(t) < t$ for all $t > 0$, we get

$$\begin{aligned}
&\frac{d(F(x,y),gx) + d(F(y,x),gy)}{2} \\
&\leq \frac{d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx) + d(F(y,x),gy_{n+1}) + d(gy_{n+1},gy)}{2} \\
&= \frac{d(F(x,y),F(x_n,y_n)) + d(gx_{n+1},gx) + d(F(y,x),F(y_n,x_n)) + d(gy_{n+1},gy)}{2} \\
&\leq \alpha((gx,gy),(gx_n,gy_n))[\frac{d(F(x,y),F(x_n,y_n)) + d(F(y,x),F(y_n,x_n))}{2}] \\
&\quad + \frac{d(gx_{n+1},gx) + d(gy_{n+1},gy)}{2} \\
&\leq \psi(\frac{d(gx,gx_n) + d(gy,gy_n)}{2}) + \frac{d(gx_{n+1},gx) + d(gy_{n+1},gy)}{2} \\
&< \frac{d(gx,gx_n) + d(gy,gy_n)}{2} + \frac{d(gx_{n+1},gx) + d(gy_{n+1},gy)}{2}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$d(F(x,y), gx) + d(F(y,x), gy) = 0.$$

Which implies that $gx = F(x,y)$, and $gy = F(y,x)$. Thus we prove that (x,y) is a coupled coincidence point of F and g.

The following example is valid for Theorem 3.1.

Example 3.3. *Let $X = \mathbb{R}$, $d(x,y) = |x-y|$ and $F : X \times X \to X$ be defined by*

$$F(x,y) = \frac{x+2y}{4}, \quad (x,y) \in X^2,$$

and $g : X \to X$ by $g(x) = \frac{3x}{2}$, clearly, F not has mixed g-monotone property.

Consider the mapping $\alpha : X^2 \times X^2 \to (0, +\infty]$ such that

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & \text{if } x \geq u \text{ and } y \geq v, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that F is α-admissible. Now, we claim that F satisfies condition (3.1). If $\alpha((gx,gy),(gu,gv)) = 0$, then the result is straightforward. Let $x,y,u,v \in X$ and M=$\{(x,y,u,v) \in X^4 : x \geq u \quad \text{and } y \geq v\}$. Without loss of generality, assume that $gx \geq gu$ and $gy \geq gv$, we have $\alpha((gx,gy),(gu,gv)) = 1$ and $(gx,gy,gu,gv) \in M$. Then we have

$$\begin{aligned}
&\alpha((gx,gy),(gu,gv))[\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,y))}{2}] \\
&= \frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2} \\
&= \left|\frac{x+2y}{8} - \frac{u+2v}{8}\right| + \left|\frac{y+2x}{8} - \frac{v+2u}{8}\right| \\
&\leq 3\left|\frac{x-u}{8}\right| + 3\left|\frac{y-v}{8}\right| \\
&= \frac{3}{8}(|x-u| + |y-v|).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{d(gx,gu) + d(gy,gv)}{2} &= \frac{d\left(\frac{3x}{2}, \frac{3u}{2}\right) + d\left(\frac{3y}{2}, \frac{3v}{2}\right)}{2} \\
&= \frac{3}{4}(|x-u| + |y-v|)
\end{aligned}$$

Now, choose $\psi \in \Psi$ such that $\psi(t) = t/2$, then

$$\begin{aligned}
&\alpha((gx,gy),(gu,gv))[\frac{d(F(x,y),F(y,x)) + d(F(u,v),F(v,u))}{2}] \\
&\leq \psi(\frac{d(gx,gu) + d(gy,gv)}{2}).
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 hold, we know that F has a coupled coincidence point $(0,0)$.

Example 3.4. *Let $X=[0,1]$, $d(x,y)=|x-y|$ and $F: X\times X\to X$ be defined by*

$$F(x,y)=\frac{xy}{4}, \quad (x,y)\in X^2,$$

and $g: X\to X$ by $g(x)=\frac{2x}{3}$, clearly, F not has a mixed g-monotone property. Consider the mapping $\alpha: X^2\times X^2\to(0,+\infty]$ such that

$$\alpha((x,y),(u,v))=\begin{cases} 1 & \text{if } x\leq u \text{ and } y\leq v,\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that F is α-admissible. Now, we claim that F satisfies condition (3.1). If $\alpha((gx,gy),(gu,gv))=0$, then the result is straightforward. Let $x,y,u,v\in X$ and $M=\{(x,y,u,v)\in X^4 : x\leq u \quad \text{and } y\leq v\}$. Without loss of generality, assume that $gx\leq gu$ and $gy\leq gv$, we have $\alpha((gx,gy),(gu,gv))=1$. Then we have

$$\begin{aligned}
&\alpha((gx,gy),(gu,gv))[\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,y))}{2}]\\
&=\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\\
&=\left|\frac{xy}{8}-\frac{uv}{8}\right|+\left|\frac{yx}{8}-\frac{vu}{8}\right|\\
&\leq\left|\frac{x-u}{8}\right|+\left|\frac{y-v}{8}\right|+\left|\frac{x-u}{8}\right|+\left|\frac{y-v}{8}\right|\\
&=\frac{1}{4}(|x-u|+|y-v|).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{d(gx,gu)+d(gy,gv)}{2}&=d\left(\frac{x}{3},\frac{u}{3}\right)+d\left(\frac{y}{3},\frac{v}{3}\right)\\
&=\frac{1}{3}(|x-u|+|y-v|).
\end{aligned}$$

Now, choose $\psi\in\Psi$ such that $\psi(t)=3t/4$, then

$$\begin{aligned}
&\alpha((gx,gy),(gu,gv))[\frac{d(F(x,y),F(y,x))+d(F(u,v),F(v,u))}{2}]\\
&\leq\psi(\frac{d(gx,gu)+d(gy,gv)}{2}).
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 hold, we know that F has a coupled coincidence point $(0,0)$.

Next, we give a sufficient condition for the uniqueness of the coupled coincidence point in Theorem 3.1.

Theorem 3.5. *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that*

$$\alpha((gu, gv), (gx, gy)) \geq 1 \quad \text{and} \quad \alpha((gu, gv), (gx^*, gy^*)) \geq 1.$$

Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Proof *From Theorem 3.1, the set of coupled coincidence point is non-empty. Suppose (x, y) and (x^*, y^*) are coupled coincidence point of F, that is*

$$gx = F(x, y), gy = F(y, x), gx^* = F(x^*, y^*) \quad \text{and} \quad gy^* = F(y^*, x^*).$$

We shall show that

$$gx^* = gx \text{ and } gy^* = gy. \tag{3.11}$$

Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$, such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_n = F(u_{n-1}, v_{n-1}) \text{ and } gv_n = F(v_{n-1}, u_{n-1}) \quad \text{for all } n \geq 1.$$

By assumption there is $(u, v) \in X \times X$ such that

$$\alpha((gu, gv), (gx, gy)) \geq 1 \quad \text{and} \quad \alpha((gu, gv), (gx^*, gy^*)) \geq 1.$$

Since F and g are α-admissible and $\alpha((gu_0, gv_0), (gx, gy)) \geq 1$, we have

$$\alpha((F(u_0, v_0), F(v_0, u_0)), (F(x, y), F(y, x))) = \alpha((gu_1, gv_1), (gx, gy)) \geq 1.$$

From this, if we use again the property of F and g are α-admissible, then it follow that

$$\alpha((F(u_1, v_1), F(v_1, u_1)), (F(x, y), F(y, x))) = \alpha((gu_2, gv_2), (gx, gy)) \geq 1$$

By repeating this process, we get

$$\alpha((gu_n, gv_n), (gx, gy)) \geq 1 \quad \text{for all } n \geq 1. \tag{3.12}$$

Using the property of α is M-invariant, we have

$$(gu_n, gv_n, gx, gy) \in M. \tag{3.13}$$

From (3.1) , (3.12)*and* (3.13) *, we have*

$$\begin{aligned} &\frac{d(gu_{n+1}, gx) + d(gv_{n+1}, gy)}{2} \\ &= \frac{d(F(u_n, v_n), F(x, y)) + d(F(v_n, u_n), F(y, x))}{2} \\ &\leq \alpha((gu_n, gv_n), (gx, gy))[\frac{d(F(u_n, v_n), F(x, y)) + d(F(v_n, u_n), F(y, x))}{2}] \\ &\leq \psi(\frac{d(gu_n, gx) + d(gv_n, gy)}{2}). \end{aligned} \tag{3.14}$$

From(3.14)*, we have*

$$\frac{d(gu_{n+1},gx)+d(gv_{n+1},gy)}{2} \leq \psi(\frac{d(gu_n,gx)+d(gv_n,gy)}{2}) \tag{3.15}$$

Since ψ is non-decreasing, from (3.15)*, we get*

$$\frac{d(gu_{n+1},gx)+d(gv_{n+1},gy)}{2} \leq \psi^n(\frac{d(gu_1,gx)+d(gv_1,gy)}{2}) \tag{3.16}$$

for each $n \geq 1$. Letting $n \to +\infty$ in (3.16) *and using lemma 2.6. Which implies*

$$\lim_{n\to\infty} d(gu_{n+1},gx) = \lim_{n\to\infty} d(gv_{n+1},gy) = 0. \tag{3.17}$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(gu_{n+1},gx^*) = \lim_{n\to\infty} d(gv_{n+1},gy^*) = 0. \tag{3.18}$$

Hence, from (3.17)*,* (3.18)*, we get $gx^* = gx$ and $gy^* = gy$.*
Since $gx = F(x,y)$ and $gy = F(y,x)$, by commutativity of F and g, we have

$$g(gx) = g(F(x,y)) = F(gx,gy) \text{ and } g(gy) = g(F(y,x)) = F(gy,gx). \tag{3.19}$$

Denote $gx = z$ and $gy = w$. Then from (3.19)

$$gz = F(z,w) \text{ and } gw = F(w,z). \tag{3.20}$$

Therefore, (z,w) is a coupled coincidence fixed point of F and g. Then from (3.11) *with $x^* = z$ and $y^* = w$. It follows $gz = gx$ and $gw = gy$, that is,*

$$gz = z \text{ and } gw = w. \tag{3.21}$$

From (3.20) *and* (3.21)*, $z = gz = F(z,w)$ and $w = gw = F(w,z)$. Therefore, (z,w) is a coupled common fixed point of F and g.*
To prove the uniqueness, assume that (p,q) is another coupled common fixed point. Then by (3.11) *we have $p = gp = gz = z$ and $q = gq = gw = w$.*

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