



Hybrid Steepest Descent Method of One-parameter Nonexpansive Cosine Families for Variational Inequality Problems in Hilbert Spaces

Nimit Nimana[†], Somboon Niyom^{‡,1} and Narin Petrot[†]

[†]Department of Mathematics and Statistic, Faculty of Science and Technology
Nakhon Sawan Rajabhat University, Nakhon Sawan, 60000, Thailand
email: somboon.n@nsru.ac.th (S. Niyom)

[‡]Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok, 65000, Thailand
email: nimitn55@nu.ac.th (N. Nimana)
narinp@nu.ac.th (N. Petrot)

Abstract : The purpose of this paper is to present the hybrid steepest descent method for solving the variational inequality problems over the common fixed point set of a one-parameter strongly continuous nonexpansive cosine family in the context of Hilbert spaces. Under some suitable assumptions, imposed upon the control parameters, we have established the strong convergence theorems for the presented methods.

Keywords : hybrid steepest descent method; variational inequality; nonexpansive cosine family; strong convergence.

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1 Introduction

Let H be a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let K be a nonempty closed convex subset of H and $A : H \rightarrow H$ be

¹Corresponding author.

a nonlinear operator. In 1980, Kinderlehrer and Stampacchia [1] initially studied the following *variational inequality problem*, $VIP(A, K)$, which is a problem of finding $u^* \in K$ such that

$$\langle A(u^*), v - u^* \rangle \leq 0 \quad \text{for all } v \in K. \quad (1.1)$$

It is well known that for a closed convex subset K of H if $A : H \rightarrow H$ is η -strongly monotone; that is, there exists $\eta > 0$ such that

$$\langle A(x) - A(y), x - y \rangle \geq \eta \|x - y\|^2$$

for all $x, y \in H$; and κ -Lipschitz continuous; that is,

$$\|A(x) - A(y)\| \leq \kappa \|x - y\|$$

for a positive constant κ and $x, y \in H$, then the existence and uniqueness of the solution of $VIP(A, K)$ is guaranteed, see [1]. It is worth mentioning that the problem $VIP(A, K)$ has been extensively studied because it can be applied to many diverse disciplines such as systems of nonlinear equations, necessary optimality conditions for optimization problems, complementarity problems, mathematical programmings and many others. In order to solving the $VIP(A, K)$, a number of solution methods have been presented. Among them, in 2001, Yamada [2] presented the hybrid steepest descent method for solving the problem $VIP(A, K)$ as follows: Let $T : H \rightarrow H$ be a nonexpansive mapping, that is, for every $x, y \in H$

$$\|T(x) - T(y)\| \leq \|x - y\|,$$

with $Fix(T) \neq \emptyset$. Suppose that a mapping $A : H \rightarrow H$ is κ -Lipschitz continuous and η -strongly monotone over H . By choosing $\mu \in (0, \frac{2\eta}{\kappa^2})$ and let any sequence $\{\lambda_n\} \subset (0, 1)$ satisfying the following conditions:

$$(W1) \quad \lim_{n \rightarrow \infty} \lambda_n = 0,$$

$$(W2) \quad \sum_{n=0}^{\infty} \lambda_n = +\infty,$$

$$(W3) \quad \text{either } \lim_{n \rightarrow \infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0 \text{ or } \sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < +\infty,$$

the sequence $\{x_n\}$ generated, with arbitrary $x_0 \in H$, by

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu A(Tx_n), \quad n \geq 0. \quad (1.2)$$

Yamada then proved that such sequence $\{x_n\}$ converges to the unique solution of the $VIP(A, Fix(T))$, where $Fix(T) := \{z \in H : z = T(z)\}$.

After Yamada's hybrid steepest descent method for solving variational inequalities was presented, there are many researches on this aspect; see, e.g. [3, 4, 5, 6, 7, 8, 9]

One of the interesting work is the paper of Xu and Kim [3], in 2003, they replaced the condition (W3) of Yamada [2] by the following condition

$$(W3)' \lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1, \text{ or equivalently, } \lim_{n \rightarrow \infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0.$$

Then, they proved the strong convergence theorem for the hybrid steepest descent method (1.2) under the condition (W1), (W2) and (W3)'.

Furthermore, in 2006, Zeng *et al.*[5] introduced the following *relaxed hybrid steepest descent algorithm*:

Algorithm ZAW. Let $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[Tx_n - \lambda_{n+1}\mu A(Tx_n)] \quad n \geq 0. \quad (1.3)$$

Then, under some suitable imposed control conditions, Zeng *et al.*[5] proved the strong convergence result of $\{x_n\}$ to the unique solution of the variational inequality $VIP(A, Fix(T))$.

On the other hand, let us now consider the concept of a one-parameter strongly continuous cosine family. Recall that the problem of convergence of the one-parameter semigroups $\{T(t) : t \in [0, \infty)\}$ was raised by Brezis [10] who studied, in fact, the behavior as $t \rightarrow 1$ of the solutions of the first order differential inclusion

$$\frac{du}{dt} \in -A(u(t)), \quad \text{for a.e. } t \in [0, \infty),$$

where A is the infinitesimal generator operator of the one-parameter operator semigroups $\{T(t) : t \in [0, \infty)\}$ of linear operator on a Hilbert space, see [11] for more details. Consequently, the several researches about convergence of nonlinear semigroup operators are offered in many aspects.

It is akin to one-parameter semigroup, to investigate the abstract second-order differential inclusion

$$\frac{d^2u}{dt^2} \in A(u(t)), \quad \text{for a.e. } t \in [0, \infty),$$

the one-parameter cosine family $\{C(t) : t \in \mathbb{R}\}$ is considered. One can directly link to the solutions of such problem, where A is the infinitesimal generator operator of the one-parameter cosine family $\{C(t) : t \in \mathbb{R}\}$ of linear operator, see [12, 13] for more details.

Of course, the fact that the concept of one-parameter cosine family also have backgrounds in differential equations and evolutionary equations, very recently, Xiao *et al.*[13] introduced the concept of cosine family of nonlinear operator. They proposed some important properties of cosine family and, moreover, they proved a series of convergence theorems such as implicit Ishikawa iterative method, explicit Ishikawa iterative method, and Moudafi's viscosity approximation method for nonexpansive cosine families under some suitable conditions in Hilbert spaces.

Motivated by all above results, in this paper we introduce a hybrid steepest descent algorithm for nonexpansive cosine families and we show a strong convergence result for our Algorithm. In fact, we will prove that under some suitable

assumptions, the sequence $\{x_n\}$ generated by the Algorithm converges strongly to a unique solution of the variational inequality $VIP(A, Fix(C))$, where $Fix(C)$ denoted the common fixed point set of a nonexpansive cosine family $\{C(t)\}$.

2 Preliminaries

Throughout this paper unless otherwise stated, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let K be a nonempty closed convex subset of H . We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

Follows from Xiao *et al.* [13], we give the concept of cosine family of nonlinear operators and some its properties as follows.

Definition 2.1 ([13]). *A one-parameter family $\{C(t) : t \in \mathbb{R}\}$ of operators on K is said to be strongly continuous cosine family if the following conditions hold:*

(C-1) $C(0)x = x$ for all $x \in K$, and $C(t)\theta = \theta$ for all $t \in \mathbb{R}$;

(C-2) $C(t+r) + C(t-r) = 2C(t)C(r)$, for all $t, r \in \mathbb{R}$;

(C-3) $C(t+r) - C(t-r) = \int_{t-r}^{t+r} d\tau \int_0^\tau C(\mu)d\mu$, for all $t, r \in \mathbb{R}$;

(C-4) $\{C(t)\}$ is strongly continuous on K , i.e., for each $x \in K$, the operator $C(\cdot)x$ from \mathbb{R} into K is continuous.

A strongly continuous cosine family (in short, cosine family) $\{C(t)\}$ is called nonexpansive if for each $t \in \mathbb{R}$,

$$\|C(t)x - C(t)y\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

If $\{C(t)\}$ is a cosine family, then $\{S(t)\}$ is the associated sine family defined by $S(t) = \int_0^t C(\tau)d\tau$, for all $t \in \mathbb{R}$.

A point $x^* \in K$ is said to be common fixed point of the cosine family $\{C(t)\}$ if $C(t)x^* = x^*$ for every $t \in \mathbb{R}$ and we denote the common fixed point set of $\{C(t)\}$ by $Fix(C)$, that is $Fix(C) := \bigcap_{t \in \mathbb{R}} Fix(C(t))$.

The following lemmas are convenient for our proof.

Lemma 2.2 ([13]). *Let $\{t_n\}$ be a sequence in $[0, \infty)$ such that $0 = \liminf_{n \rightarrow \infty} t_n < \limsup_{n \rightarrow \infty} t_n$, let $g : [0, \infty) \rightarrow [0, \infty)$ be a function such that $\lim_{n \rightarrow \infty} g(t_n) = 0$. Suppose either $\liminf_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$ or $\liminf_{n \rightarrow \infty} (t_n - t_{n+1}) = 0$. Then for each $i \in \mathbb{N}$, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that*

$$\lim_{k \rightarrow \infty} t_{n_k} = \lim_{k \rightarrow \infty} \frac{g(t_{n_k})}{(t_{n_k})^i} = 0.$$

Lemma 2.3 ([13]). *Let K be a nonempty closed convex subset of a real Hilbert space H and $\{C(t)\}$ be a nonexpansive cosine family. Let $\{x_n\}$ be a sequence in K , $q \in K$ and $\{t_n\}$ be a sequence satisfying $0 < t_n < t$ for all $n \in \mathbb{N}$. Then,*

$$\|x_n - C(t)q\| \leq \frac{t^2}{t_n^2} \|x_n - C(t_n)x_n\| + \|x_n - q\| + 2t \sup_{0 \leq r \leq t_n} \|S(\frac{r}{2})q\|.$$

Now, we present a useful statement which appears in the proof of main Theorems of Xiao *et al.* [13]. For sake of handiness, we state it again in the following lemma.

Lemma 2.4 ([13]). *Let $\{x_n\}$ be a bounded sequence in a nonempty closed convex subset K of a real Hilbert space H and assume that $0 = \liminf_{n \rightarrow \infty} t_n < \limsup_{n \rightarrow \infty} t_n$ and $\liminf_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$. If $\lim_{n \rightarrow \infty} \|x_n - C(t_n)x_n\| = 0$, then $\omega_w(x_n) \subset \text{Fix}(C)$, where $\omega_w(x_n)$ is denoted for the weak cluster set of the sequence of $\{x_n\}$.*

Proof. We first note from our assumptions and Lemma 2.2 that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k \rightarrow \infty} t_{n_k} = \lim_{k \rightarrow \infty} \frac{\|x_{n_k} - C(t_{n_k})x_{n_k}\|}{(t_{n_k})^i} = 0, \quad \forall i \in \mathbb{N}. \tag{2.1}$$

Let \tilde{x} be a weak cluster point of $\{x_{n_k}\}$. Now, we will show that $\tilde{x} \in \text{Fix}(C) = \bigcap_{t \in \mathbb{R}} \text{Fix}(C(t))$. If $t = 0$, then clearly that $\tilde{x} \in \text{Fix}(C(0))$. Since $C(t) = C(-t)$, we can assume that $t > 0$. Since $\lim_{k \rightarrow \infty} t_{n_k} = 0$, we can assume without loss of generality that $t_{n_k} < t$ for all $k \in \mathbb{N}$. By Lemma 2.3, we have

$$\|x_{n_k} - C(t)\tilde{x}\| \leq \frac{t^2}{t_{n_k}^2} \|x_{n_k} - C(t_{n_k})x_{n_k}\| + \|x_{n_k} - \tilde{x}\| + 2t \sup_{0 \leq r \leq t_{n_k}} \|S(\frac{r}{2})\tilde{x}\|.$$

Thus (2.1) implies that

$$\limsup_{k \rightarrow \infty} \|x_{n_k} - C(t)\tilde{x}\| \leq \limsup_{k \rightarrow \infty} \|x_{n_k} - \tilde{x}\|.$$

Since $\{x_{n_k}\}$ weakly converges to \tilde{x} , we have

$$\|\tilde{x} - C(t)\tilde{x}\|^2 + \limsup_{k \rightarrow \infty} \|x_{n_k} - \tilde{x}\|^2 = \limsup_{k \rightarrow \infty} \|x_{n_k} - C(t)\tilde{x}\|^2 \leq \limsup_{k \rightarrow \infty} \|x_{n_k} - \tilde{x}\|^2.$$

and then $\|\tilde{x} - C(t)\tilde{x}\| = 0$. This implies that $\tilde{x} \in \text{Fix}(C(t))$ for all $t > 0$. Therefore $\tilde{x} \in \text{Fix}(C)$. □

Lemma 2.5 ([14]). *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\beta_n\}_{n \in \mathbb{N}} \subset (0, 1)$, $\{\delta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ and $\{\gamma_n\}_{n \in \mathbb{N}} \subset [0, +\infty)$ such that

- (i) $\sum_{n=1}^{\infty} \beta_n = +\infty$;
(ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < +\infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Next, let $\{C(t)\}$ be a nonexpansive cosine family on H with $Fix(C) \neq \emptyset$ and $A : H \rightarrow H$ be κ -Lipschitzian and η -strongly monotone. Given $\lambda \in (0, 1)$, $\mu > 0$, and $t \in [0, \infty)$, we define the mapping $T^\lambda : H \rightarrow H$ by

$$T^\lambda(x) = C(t)x - \lambda\mu A(C(t)x), \quad \text{for all } x \in H. \quad (2.2)$$

Thus, we have the following lemma which its proof is akin to the proof of Lemma 3.1 in Yamada [2]. For the sake of completeness, we present its proof again.

Lemma 2.6. For any $\mu \in (0, \frac{2\eta}{\kappa^2})$, $\lambda \in (0, 1)$ and $t \in [0, \infty)$, we have

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\| \quad \text{for all } x, y \in H, \quad (2.3)$$

where $\tau := 1 - \sqrt{1 + \mu^2\kappa^2 - 2\mu\eta} \in (0, 1]$.

Proof. First, we consider that for every $u, v \in H$,

$$\begin{aligned} \|(I - \mu A)u - (I - \mu A)v\|^2 &= \|(u - v) - \mu(Au - Av)\|^2 \\ &= \|u - v\|^2 + \mu^2\|Au - Av\|^2 - 2\mu\langle u - v, Au - Av \rangle \\ &\leq \|u - v\|^2 + \mu^2\kappa^2\|u - v\|^2 - 2\mu\eta\|u - v\|^2 \\ &= (1 + \mu^2\kappa^2 - 2\mu\eta)\|u - v\|^2. \end{aligned}$$

Using the above inequality and the fact that $\{C(t)\}$ is a nonexpansive cosine family, we have for all $x, y \in H$,

$$\begin{aligned} \|T^\lambda x - T^\lambda y\| &= \|[C(t)x - \lambda\mu A(C(t)x)] - [C(t)y - \lambda\mu A(C(t)y)]\| \\ &\leq \lambda\|(I - \mu A)C(t)x - (I - \mu A)C(t)y\| + (1 - \lambda)\|C(t)x - C(t)y\| \\ &\leq \lambda\sqrt{1 + \mu^2\kappa^2 - 2\mu\eta}\|x - y\| + (1 - \lambda)\|x - y\| \\ &= (1 - \lambda\tau)\|x - y\|. \end{aligned}$$

□

Remark 2.7. It is clearly that $\tau \in (0, 1]$ always exists. Indeed, we know that A is κ -Lipschitzian and η -strongly monotone, the Cauchy-Schwarz inequality gives us that $0 < \eta \leq \kappa$ and furthermore we have $0 < \frac{2\eta}{\kappa^2}$. So, we can choose $\mu \in (0, \frac{2\eta}{\kappa^2})$. We see that $0 \leq (1 - \mu\kappa)^2 \leq 1 + \mu^2\kappa^2 - 2\mu\eta < 1$, which implies that $0 < 1 - \sqrt{1 + \mu^2\kappa^2 - 2\mu\eta} \leq 1$

3 Hybrid Steepest Descent Method

Let $\{C(t)\}$ be a nonexpansive cosine family on H with $Fix(C) \neq \emptyset$ and $A : H \rightarrow H$ be κ -Lipschitzian and η -strongly monotone. For each given $\xi \in (0, 1)$ we let $\mu \in (0, 2\eta/\kappa^2)$, $\lambda \in (0, 1)$ and define a mapping $\Gamma : H \rightarrow H$ by

$$\Gamma x = \xi x + (1 - \xi)(C(t)x - \lambda\mu A(C(t)x)), \quad x \in H. \tag{3.1}$$

Then it is not difficult to see that Γ is a contraction. As a matter of fact, using Lemma 2.6 we derive that for every $x, y \in H$,

$$\begin{aligned} \|\Gamma x - \Gamma y\| &= \|\xi(x - y) + (1 - \xi)(T^\lambda x - T^\lambda y)\| \\ &\leq \xi\|x - y\| + (1 - \xi)(1 - \lambda\tau)\|x - y\|. \end{aligned}$$

Since $\xi + (1 - \xi)(1 - \lambda\tau) \in (0, 1)$, the Banach Contraction Principle will therefore implies that there exists a unique fixed point of Γ in H .

We now propose a hybrid steepest descent algorithm for finding the solution of the variational inequality as follows.

Algorithm I. Let $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and $\{t_n\} \subset [0, \infty)$. Choose $\mu \in (0, \frac{2\eta}{\kappa^2})$. Starting with an arbitrary initial guess $x_1 \in H$, let the sequence $\{x_n\}$ generated by the following iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)[C(t_n)x_n - \lambda_{n+1}\mu A(C(t_n)x_n)], \quad n \geq 1. \tag{3.2}$$

The following result shows the strong convergence of Algorithm I.

Theorem 3.1. *Let the sequence $\{x_n\}$ be generated by Algorithm I and assume the following conditions hold:*

- (i) $0 = \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
- (ii) $\lim_{n \rightarrow \infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
- (iii) $0 = \liminf_{n \rightarrow \infty} t_n < \limsup_{n \rightarrow \infty} t_n$ and $\liminf_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$.

If $\sum_{n=1}^{\infty} \|C(t_{n+1})x_n - C(t_n)x_n\| < \infty$, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality problem $VIP(A, Fix(C))$.

Proof. We shall divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, for each $z \in Fix(C)$, by Lemma 2.6 we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) \|T^{\lambda_{n+1}} x_n - z\| \\ &\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) [\|T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} z\| + \|T^{\lambda_{n+1}} z - z\|] \\ &\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) [(1 - \lambda_{n+1}\tau) \|x_n - z\| + \lambda_{n+1}\mu \|A(z)\|]. \end{aligned}$$

Let $M > 0$ be such that $M \geq \max\{\|x_0 - z\|, \frac{\mu}{\tau} \|A(z)\|\}$. By the induction argument, we get

$$\|x_n - z\| \leq M \quad \forall n \geq 1.$$

Thus, $\{x_n\}$ is a bounded sequence, so are $\{C(t_n)x_n\}$ and $\{A(C(t_n)x_n)\}$.

Step 2. $\|x_n - C(t_n)x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

In fact, we now consider the following equalities.

$$\begin{aligned}
 x_{n+1} - x_n &= \alpha_n x_n - \alpha_{n-1} x_{n-1} + (1 - \alpha_n) T^{\lambda_{n+1}} x_n - (1 - \alpha_{n-1}) T^{\lambda_n} x_{n-1} \\
 &= \alpha_n x_n - \alpha_n x_{n-1} + \alpha_n x_{n-1} - \alpha_{n-1} x_{n-1} + (1 - \alpha_n) T^{\lambda_{n+1}} x_n \\
 &\quad - (1 - \alpha_n) T^{\lambda_{n+1}} x_{n-1} + (1 - \alpha_n) T^{\lambda_{n+1}} x_{n-1} - (1 - \alpha_{n-1}) T^{\lambda_n} x_{n-1} \\
 &= \alpha_n (x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1}) x_{n-1} + (1 - \alpha_n) (T^{\lambda_{n+1}} x_n \\
 &\quad - T^{\lambda_{n+1}} x_{n-1}) + (1 - \alpha_n) T^{\lambda_{n+1}} x_{n-1} - (1 - \alpha_{n-1}) T^{\lambda_n} x_{n-1} \\
 &= \alpha_n (x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1}) x_{n-1} + (1 - \alpha_n) (T^{\lambda_{n+1}} x_n \\
 &\quad - T^{\lambda_{n+1}} x_{n-1}) + (1 - \alpha_n) (C(t_n) x_{n-1} - \lambda_{n+1} \mu A(C(t_n) x_{n-1})) \\
 &\quad - (1 - \alpha_{n-1}) (C(t_{n-1}) x_{n-1} - \lambda_n \mu A(C(t_{n-1}) x_{n-1})) \\
 &= \alpha_n (x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1}) x_{n-1} + (1 - \alpha_n) (T^{\lambda_{n+1}} x_n \\
 &\quad - T^{\lambda_{n+1}} x_{n-1}) + (1 - \alpha_n) C(t_n) x_{n-1} - (1 - \alpha_{n-1}) C(t_{n-1}) x_{n-1} \\
 &\quad - (1 - \alpha_n) \lambda_{n+1} \mu A(C(t_n) x_{n-1}) + (1 - \alpha_{n-1}) \lambda_n \mu A(C(t_{n-1}) x_{n-1})
 \end{aligned} \tag{3.3}$$

We observe that

$$\begin{aligned}
 (1 - \alpha_n) C(t_n) x_{n-1} - (1 - \alpha_{n-1}) C(t_{n-1}) x_{n-1} \\
 &= (1 - \alpha_n) C(t_n) x_{n-1} - (1 - \alpha_n) C(t_{n-1}) x_{n-1} \\
 &\quad + (1 - \alpha_n) C(t_{n-1}) x_{n-1} - (1 - \alpha_{n-1}) C(t_{n-1}) x_{n-1} \\
 &= (1 - \alpha_n) [C(t_n) x_{n-1} - C(t_{n-1}) x_{n-1}] \\
 &\quad + (\alpha_{n-1} - \alpha_n) C(t_{n-1}) x_{n-1},
 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
 (1 - \alpha_n) \lambda_{n+1} \mu A(C(t_n) x_{n-1}) - (1 - \alpha_{n-1}) \lambda_n \mu A(C(t_{n-1}) x_{n-1}) \\
 &= (1 - \alpha_n) \lambda_{n+1} \mu A(C(t_n) x_{n-1}) - (1 - \alpha_n) \lambda_{n+1} \mu A(C(t_{n-1}) x_{n-1}) \\
 &\quad + (1 - \alpha_n) \lambda_{n+1} \mu A(C(t_{n-1}) x_{n-1}) - (1 - \alpha_{n-1}) \lambda_n \mu A(C(t_{n-1}) x_{n-1}) \\
 &= (1 - \alpha_n) \lambda_{n+1} \mu [A(C(t_n) x_{n-1}) - A(C(t_{n-1}) x_{n-1})] \\
 &\quad [(1 - \alpha_n) \lambda_{n+1} - (1 - \alpha_{n-1}) \lambda_n] \mu A(C(t_{n-1}) x_{n-1}).
 \end{aligned} \tag{3.5}$$

By using (3.4) and (3.5), the inequality (3.3) becomes

$$\begin{aligned}
 x_{n+1} - x_n &= \alpha_n (x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1}) x_{n-1} \\
 &\quad + (1 - \alpha_n) (T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} x_{n-1}) + (1 - \alpha_n) [C(t_n) x_{n-1} \\
 &\quad - C(t_{n-1}) x_{n-1}] + (\alpha_{n-1} - \alpha_n) C(t_{n-1}) x_{n-1} \\
 &\quad - (1 - \alpha_n) \lambda_{n+1} \mu [A(C(t_n) x_{n-1}) - A(C(t_{n-1}) x_{n-1})] \\
 &\quad - [(1 - \alpha_n) \lambda_{n+1} - (1 - \alpha_{n-1}) \lambda_n] \mu A(C(t_{n-1}) x_{n-1})
 \end{aligned} \tag{3.6}$$

This yields that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\| \\
&\quad + (1 - \alpha_n) \|T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} x_{n-1}\| \\
&\quad + (1 - \alpha_n) \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\
&\quad + |\alpha_{n-1} - \alpha_n| \|C(t_{n-1})x_{n-1}\| \\
&\quad + (1 - \alpha_n)\lambda_{n+1}\mu \|A(C(t_n)x_{n-1}) - A(C(t_{n-1})x_{n-1})\| \\
&\quad + |(1 - \alpha_n)\lambda_{n+1} - (1 - \alpha_{n-1})\lambda_n| \mu \|A(C(t_{n-1})x_{n-1})\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\| \\
&\quad + (1 - \alpha_n)(1 - \lambda_{n+1}\tau) \|x_n - x_{n-1}\| \\
&\quad + (1 - \alpha_n) \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\
&\quad + |\alpha_{n-1} - \alpha_n| \|C(t_{n-1})x_{n-1}\| \\
&\quad + (1 - \alpha_n)\lambda_{n+1}\mu\kappa \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\
&\quad + |(1 - \alpha_n)\lambda_{n+1} - (1 - \alpha_{n-1})\lambda_n| \mu \|A(C(t_{n-1})x_{n-1})\| \\
&\leq (1 - \lambda_{n+1}\tau(1 - \alpha_n)) \|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| (\|x_{n-1}\| + \|C(t_{n-1})x_{n-1}\|) \\
&\quad + (1 - \alpha_n) \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\
&\quad + (1 - \alpha_n)\lambda_{n+1}\mu\kappa \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\
&\quad + |(1 - \alpha_n)\lambda_{n+1} - (1 - \alpha_{n-1})\lambda_n| \mu \|A(C(t_{n-1})x_{n-1})\| \\
&\leq (1 - \lambda_{n+1}\tau(1 - \alpha_n)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \widehat{M} \\
&\quad + (1 + \lambda_{n+1}\mu\kappa) \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\
&\quad + |(1 - \alpha_n)\lambda_{n+1} - (1 - \alpha_{n-1})\lambda_n| \mu \widehat{M}, \\
&\leq (1 - \lambda_{n+1}\tau(1 - \alpha_n)) \|x_n - x_{n-1}\| \\
&\quad + \lambda_{n+1}\tau(1 - \alpha_n) \frac{\mu \widehat{M}}{\tau} \left| 1 - \frac{(1 - \alpha_{n-1})}{(1 - \alpha_n)} \frac{\lambda_n}{\lambda_{n+1}} \right| \\
&\quad + |\alpha_n - \alpha_{n-1}| \widehat{M} + (1 + \lambda\mu\kappa) \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\|,
\end{aligned}$$

where $\widehat{M} > \max\{\|x_n\| + \|C(t_k)x_n\|, \|A(C(t_k)x_n)\|\}$ for all $n, k \geq 0$, and $\lambda_n \leq \lambda$ for some $\lambda > 0$. By using (i) and (ii), we have $\sum_{n=1}^{\infty} \lambda_{n+1}\tau(1 - \alpha_n) \geq \sum_{n=1}^{\infty} \lambda_{n+1}\tau(1 - a) = \infty$.

On the other hand, set

$$\delta_n := \frac{\mu \widehat{M}}{\tau} \left| 1 - \frac{(1 - \alpha_{n-1})}{(1 - \alpha_n)} \frac{\lambda_n}{\lambda_{n+1}} \right| \leq \frac{\mu \widehat{M}}{\tau} \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| \rightarrow 0.$$

Moreover, set $\gamma_n := |\alpha_n - \alpha_{n-1}| \widehat{M} + (1 + \lambda\mu\kappa) \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\|$, we have from (i) and (iii) that $\sum_{n=1}^{\infty} \gamma_n < \infty$. Now, applying Lemma 2.5, we obtain that

$$\|x_{n+1} - x_n\| \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.7}$$

We note that

$$\begin{aligned} \|x_{n+1} - C(t_n)x_n\| &= \|\alpha_n(x_n - C(t_n)x_n) + (1 - \alpha_n)(T^{\lambda_{n+1}}x_n - C(t_n)x_n)\| \\ &\leq \alpha_n\|x_n - C(t_n)x_n\| + (1 - \alpha_n)\lambda_{n+1}\mu\|A(C(t_n)x_n)\| \\ &\leq \alpha_n\|x_n - C(t_n)x_n\| + \lambda_{n+1}\mu\|A(C(t_n)x_n)\| \\ &\rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

which gives us that

$$\|x_n - C(t_n)x_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - C(t_n)x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

Step 3. $\liminf_{n \rightarrow \infty} \langle C(t_n)x_n - x^*, A(x^*) \rangle \geq 0$, where x^* is a unique solution of $VIP(A, Fix(C))$. To prove this, we pick a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\liminf_{n \rightarrow \infty} \langle C(t_n)x_n - x^*, A(x^*) \rangle = \lim_{j \rightarrow \infty} \langle C(t_{n_j})x_{n_j} - x^*, A(x^*) \rangle. \quad (3.9)$$

Since $\{x_{n_j}\}$ is bounded, by Lemma 2.4, it has a weakly convergent subsequence. We may assume without loss of generality that $x_{n_j} \rightharpoonup \bar{x} \in Fix(C)$. We note from (3.8) that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle C(t_n)x_n - x^*, A(x^*) \rangle &= \lim_{j \rightarrow \infty} \langle C(t_{n_j})x_{n_j} - x^*, A(x^*) \rangle \\ &= \lim_{j \rightarrow \infty} \langle C(t_{n_j})x_{n_j} - x_{n_j}, A(x^*) \rangle \\ &\quad + \lim_{j \rightarrow \infty} \langle x_{n_j} - x^*, A(x^*) \rangle \\ &= \langle \bar{x} - x^*, A(x^*) \rangle \geq 0. \end{aligned}$$

Step 4. Finally, we prove that $x_n \rightarrow x^*$ in norm. Indeed, we now consider

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)\|T^{\lambda_{n+1}}x_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)\|T^{\lambda_{n+1}}x_n - T^{\lambda_{n+1}}x^*\|^2 \\ &\quad + 2\langle T^{\lambda_{n+1}}x_n - T^{\lambda_{n+1}}x^*, T^{\lambda_{n+1}}x^* - x^* \rangle \\ &= \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)(1 - \lambda_{n+1}\tau)\|x_n - x^*\|^2 \\ &\quad + 2\lambda_{n+1}\mu[\langle C(t_n)x_n - x^*, -A(x^*) \rangle \\ &\quad + \lambda_{n+1}\mu\langle A(C(t_n)x_n), A(x^*) \rangle] \\ &\leq (1 - \lambda_{n+1}\tau(1 - \alpha_n))\|x_n - x^*\|^2 \\ &\quad + \lambda_{n+1}\tau(1 - \alpha_n)\frac{2\mu}{(1 - a)\tau}[\langle C(t_n)x_n - x^*, -A(x^*) \rangle \\ &\quad + \lambda_{n+1}\mu\langle A(C(t_n)x_n), A(x^*) \rangle]. \end{aligned}$$

We note that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{2\mu}{(1-a)\tau} (\langle C(t_n)x_n - x^*, -A(x^*) \rangle + \lambda_{n+1}\mu \langle A(C(t_n)x_n), A(x^*) \rangle) \\ \leq \limsup_{n \rightarrow \infty} \frac{2\mu}{(1-a)\tau} \langle C(t_n)x_n - x^*, -A(x^*) \rangle \\ + \limsup_{n \rightarrow \infty} \frac{2\lambda_{n+1}\mu^2}{(1-a)\tau} \langle A(C(t_n)x_n), A(x^*) \rangle \\ \leq \limsup_{n \rightarrow \infty} \frac{2\mu}{(1-a)\tau} \langle C(t_n)x_n - x^*, -A(x^*) \rangle \\ + \limsup_{n \rightarrow \infty} \frac{2\lambda_{n+1}\mu^2}{(1-a)\tau} \|A(C(t_n)x_n)\| \|A(x^*)\| \\ \leq 0. \end{aligned}$$

By applying Lemma 2.5 again, we conclude that $\|x_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete. \square

In particular, if we set $\alpha_n = 0$ for all $n \geq 1$, then we have the following corollary.

Corollary 3.2. *Let the sequence $\{x_n\}$ be generated by the iterative scheme*

$$x_{n+1} = C(t_n)x_n - \lambda_{n+1}\mu A(C(t_n)x_n), \quad n \geq 1, \tag{3.10}$$

with an arbitrary initial guess $x_1 \in H$, where $\{\lambda_n\} \subset (0, 1)$, $\{t_n\} \subset [0, \infty)$ and $\mu \in (0, \frac{2\eta}{\kappa^2})$. Assume the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
- (ii) $0 = \liminf_{n \rightarrow \infty} t_n < \limsup_{n \rightarrow \infty} t_n$ and $\liminf_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$.

If $\sum_{n=1}^{\infty} \|C(t_{n+1})x_n - C(t_n)x_n\| < \infty$, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality problem $VIP(A, \text{Fix}(C))$.

Remark 3.3. *It should be note that the condition $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$ in Corollary 3.2 may be replaced by $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ and the sequence $\{\lambda_n\}$ can be included in $[0, 1)$. The result is also obtained as the following corollary.*

Corollary 3.4. *Let the sequence $\{x_n\}$ be generated by the iterative scheme*

$$x_{n+1} = C(t_n)x_n - \lambda_n\mu A(C(t_n)x_n), \quad n \geq 1, \tag{3.11}$$

with an arbitrary initial guess $x_1 \in H$, where $\{\lambda_n\} \subset [0, 1)$, $\{t_n\} \subset [0, \infty)$ and $\mu \in (0, \frac{2\eta}{\kappa^2})$. Assume the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
- (ii) $0 = \liminf_{n \rightarrow \infty} t_n < \limsup_{n \rightarrow \infty} t_n$ and $\liminf_{n \rightarrow \infty} (t_{n+1} - t_n) = 0$.

If $\sum_{n=1}^{\infty} \|C(t_{n+1})x_n - C(t_n)x_n\| < \infty$, then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality problem $VIP(A, Fix(C))$.

Proof. As similar to the arguments in Theorem 3.1, we know that $\{x_n\}$ is bounded and $\|x_{n+1} - C(t_n)x_n\| \rightarrow 0$ as $n \rightarrow \infty$. We will show that $\|x_{n+1} - x_n\| \rightarrow 0$. Indeed, we note that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1} - T^{\lambda_n}x_{n-1} - T^{\lambda_{n-1}}x_{n-1}\| \\ &\leq (1 - \lambda_n\tau)\|x_n - x_{n-1}\| + \|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\ &\quad + |\lambda_n - \lambda_{n-1}|\mu\|A(C(t_n)x_{n-1})\| \\ &\quad + \lambda_{n-1}\mu\kappa\|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \\ &\leq (1 - \lambda_n\tau)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\mu M \\ &\quad + (1 + \lambda\mu\kappa)\|C(t_n)x_{n-1} - C(t_{n-1})x_{n-1}\| \end{aligned}$$

where $M \geq \max\{\|A(C(t_k)x_n)\| : n, k \geq 1\}$, and $\lambda > 0$ such that $\lambda_n \leq \lambda$ for all n . By inductively, we obtain that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_m - x_{m-1}\| \prod_{k=m+1}^n (1 - \lambda_k\tau) + \mu M \sum_{k=m+1}^n |\lambda_k - \lambda_{k-1}| \\ &\quad + (1 + \lambda\mu\kappa) \sum_{k=m+1}^n \|C(t_k)x_{k-1} - C(t_{k-1})x_{k-1}\|, \end{aligned}$$

for all $n > m \geq 0$. We note that $\sum_{n=1}^{\infty} \lambda_n = \infty$ ensure $\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 - \lambda_k) = 0$. It follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x_{n+1} - x_n\| &\leq \mu M \sum_{k=m+1}^n |\lambda_k - \lambda_{k-1}| \\ &\quad + (1 + \lambda\mu\kappa) \sum_{k=m+1}^n \|C(t_k)x_{k-1} - C(t_{k-1})x_{k-1}\|, \end{aligned}$$

These together with condition (i), (iii) and approaching the limit as $m \rightarrow \infty$ gives us that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Following the line proof of Theorem 3.1 by setting $\alpha_n = 0$ for all $n \geq 1$, we can obtain the result. \square

Remark 3.5. In general, the condition $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ are not comparable: neither of them implies the other (coupled with $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$). For instance, in Example 3.1 and 3.2 of Xu [15], a sequence $\{\lambda_n\}$ defined by

$$\lambda_n = \begin{cases} \frac{1}{\sqrt{n}} & ; n \text{ is odd number,} \\ \frac{1}{\sqrt{n-1}} & ; n \text{ is even number.} \end{cases}$$

One can see that the condition $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$ is satisfied but the condition $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ is not satisfied. Nevertheless, if we take sequences $\{r_k\}$ and $\{s_k\}$ of natural numbers such that:

- (i) $r_1 = 1, r_k < s_k$, and $\max\{2s_k, s_k + 1\} < r_{k+1}, k \geq 1$;
(ii) $\sum_{i=r_k}^{s_k} i^{-1} > 1, k \geq 1$,
and define a sequence $\{\lambda_n\}$ by

$$\lambda_n = \begin{cases} \frac{1}{n_1} & \text{if } r_k \leq n \leq s_k, k \geq 1, \\ \frac{1}{2n_k} & \text{if } s_k < n < m_{k+1}, k \geq 1. \end{cases}$$

One can check that the condition $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ is satisfied but the other is not. For more detail, see [15].

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