Thai Journal of Mathematics
Volume 4(2006) Number 2 : 341–349
www.math.science.cmu.ac.th/thaijournal

Regular Generalized Star Closed Sets in Bitopological Spaces

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Abstract : The aim of this paper is to introduce the concepts of $\tau_1\tau_2$ - regular generalized star closed sets , $\tau_1\tau_2$ - regular generalized star open sets and study their basic properties in bitopological spaces.

Keywords : $\tau_1\tau_2$ - regular generalized star closed sets; $\tau_1\tau_2$ - regular generalized star open sets; $\tau_1\tau_2$ - regular closed sets; $\tau_1\tau_2$ - regular open sets; $\tau_1\tau_2$ - regular generalized closed sets; $\tau_1\tau_2$ - regular generalized open sets.
2000 Mathematics Subject Classification : 54E55.

1 Introduction

The concept of bitopological space was introduced by J. C. Kelly [4]. On the other hand K. Chandrasekhara Rao and N. Palaniappan [1] introduced the concepts of regular generalized star closed sets and regular generalized star open sets in a topological space.

In this paper, we introduce the concepts of $\tau_1\tau_2$ - regular generalized star closed sets ($\tau_1\tau_2 - rg^*$ closed sets) and $\tau_1\tau_2$ - regular generalized star open sets ($\tau_1\tau_2 - rg^*$ open sets) and study their basic properties in bitopological spaces.

2 Preliminaries

Let (X, τ_1, τ_2) or simply X denote a bitopological space. The intersection (resp. union) of all τ_i - semi closed sets containing A (resp. τ_i - semi open sets contained in A) is called the τ_i - *semi closure* (resp. τ_i - semi interior) of A, denoted by τ_i - scl(A) {resp. τ_i - sint(A)}. For any subset $A \subseteq X$, τ_i - int(A) and τ_i - cl(A) denote the interior and closure of a set A with respect to the topology τ_i respectively. The closure and interior of B relative to A with respect to the topology τ_i are written as τ_i - $\mathrm{cl}_A(B)$ and τ_i - $\mathrm{int}_A(B)$ respectively. For any subset $A \subseteq X$, τ_i - rint(A) and τ_i - rcl(A) denote the regular interior and regular closure of a set A with respect to the topology τ_i respectively. The regular closure and regular interior of B relative to A with respect to the topology τ_i are written as τ_i - $\mathrm{rcl}_A(B)$ and τ_i - $\mathrm{rint}_A(B)$ respectively. The set of all τ_2 - regular closed sets in X is denoted by τ_2 - R.C (X, τ_1, τ_2). The set of all $\tau_1\tau_2$ - regular open sets in X is denoted by $\tau_1\tau_2$ - R.O (X, τ_1, τ_2). A^C denotes the complement of A in X unless explicitly stated.

We shall require the following known definitions :

Definition 2.1 ([3], [5], [2]) A set A of a bitopological space (X, τ_1, τ_2) is called

(a) $\tau_1\tau_2$ - *semi open* if there exists a τ_1 - open set U such that $U \subseteq A \subseteq \tau_2$ - cl(U).

(b) $\tau_1\tau_2$ - *semi closed* if $X - A$ is $\tau_1\tau_2$ - semi open.

Equivalently, a set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ - *semi closed* if there exists a τ_1 - closed set F such that τ_2 - int$(F) \subseteq A \subseteq F$.

(c) $\tau_1\tau_2$ - *regular closed* if τ_1- cl[τ_2- int(A)] $=A$.

(d) $\tau_1\tau_2$ - *regular open* if τ_1- int[τ_2- cl(A)] $=A$.

(e) $\tau_1\tau_2$ - *regular generalized closed* $(\tau_1\tau_2 - rg$ closed) in X if τ_2 - cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$ - regular open in X.

(f) $\tau_1\tau_2$ - *regular generalized open* $(\tau_1\tau_2 - rg$ open) in X if $F \subseteq \tau_2$- int(A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$ - regular closed in X.

3 $\tau_1\tau_2$ - Regular Generalized Star Closed Sets

Definition 3.1 A subset A of a bitopological space $(X, \tau_1, \tau_2)$is called $\tau_1\tau_2$ - *regular generalized star closed* $(\tau_1\tau_2 - rg^*$ closed) in X if and only if τ_2 - rcl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$ - regular open in X.

Example 3.2 Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then all subsets in P(X) are $\tau_1\tau_2 - rg^*$ closed sets in (X, τ_1, τ_2).

Theorem 3.3 *Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$ - rg^* closed then τ_2 - rcl $(A) - A$ does not contain non empty $\tau_1\tau_2$ - regular closed sets.*

Proof. Suppose that A is $\tau_1\tau_2 - rg^*$ closed. Let F be a $\tau_1\tau_2$ - regular closed set such that $F \subseteq \tau_2$ - rcl$(A) - A$. We shall show that $F = \phi$. Since $F \subseteq \tau_2$ - rcl$(A) - A$, we have $F \subseteq [\tau_2 - rcl(A)] \cap A^C$. Consequently $F \subseteq A^C$ and $F \subseteq \tau_2$ - rcl(A). Since $F \subseteq A^C$, we have $A \subseteq F^C$. Since F is $\tau_1\tau_2$ - regular closed set, we have F^C is $\tau_1\tau_2$ - regular open. Since A is $\tau_1\tau_2 - rg^*$ closed, we have τ_2 - rcl$(A) \subseteq F^C$. Thus, $F \subseteq [\tau_2 - rcl(A)]^C = X - [\tau_2 - rcl(A)]$. Hence $F \subseteq \phi$. But $\phi \subseteq F$. Therefore, $F = \phi$. □

Theorem 3.4

(a) *Suppose that a subset A is $\tau_1\tau_2 - rg$ closed and it is $\tau_1\tau_2$ - semi open. i.e) $A \subseteq \tau_2$ - cl$[\tau_1$ - int$(A)]$. Then A is $\tau_2\tau_1$ - regular closed in X if and only if τ_2 -cl$[\tau_1$ - int$(A)] - A$ is $\tau_1\tau_2$ - regular closed in X.*

(b) *Let a subset A be a $\tau_1\tau_2 - rg^*$ closed set. Then A is τ_2 - closed in X if and only if τ_2 - cl$(A) - A$ is $\tau_1\tau_2$ - regular closed in X.*

Proof. (a) Suppose that A is $\tau_2\tau_1$ - regular closed in X. Then $A = \tau_2$ - cl$[\tau_1$ - int$(A)]$. Consequently τ_2 - cl$[\tau_1$ - int$(A)] - A = \phi$. Therefore, τ_2 - cl$[\tau_1$ - int$(A)] - A$ is $\tau_1\tau_2$ - regular closed in X.

Conversely, suppose that τ_2 - cl$[\tau_1$ - int$(A)] - A$ is $\tau_1\tau_2$ - regular closed in X. We shall show that A is $\tau_2\tau_1$ - regular closed in X. Obviously, τ_1 - int $(A) \subseteq A$. Consequently τ_2 - cl$[\tau_1$ - int$(A)] \subseteq \tau_2$ - cl(A). Hence τ_2 - cl$[\tau_1$ - int$(A)] - A \subseteq \tau_2$ - cl$(A) - A$. Since A is $\tau_1\tau_2 - rg$ closed in X, we have τ_2 - cl$(A) - A$ does not contain non empty $\tau_1\tau_2$ - regular closed set. Hence τ_2 - cl $[\tau_1$ - int$(A)] - A = \phi$. Therefore, τ_2 - cl $[\tau_1$ - int$(A)] \subseteq A$. Since A is $\tau_1\tau_2$ - semi open, we have $A \subseteq \tau_2$ - cl$[\tau_1$ - int$(A)]$. Hence τ_2 - cl $[\tau_1$ - int$(A)] = A$. Therefore, A is $\tau_2\tau_1$ - regular closed.

(b) Suppose that A is $\tau_1\tau_2 - rg^*$ closed. Let A be τ_2 - closed. We shall show that τ_2 - cl $(A) - A$ is $\tau_1\tau_2$ - regular closed in X. Since A is τ_2 - closed, we have τ_2 - cl$(A) = A$. Consequently, τ_2 - cl$(A) - A = \phi$. Therefore, τ_2 - cl$(A) - A$ is $\tau_1\tau_2$ - regular closed in X.

Conversely, suppose that τ_2 - cl$(A) - A$ is $\tau_1\tau_2$ - regular closed in X. We shall show that A is τ_2 - closed. Since τ_2 - cl$(A) \subseteq \tau_2$ - rcl(A), we have τ_2 - cl$(A) - A \subseteq \tau_2$ - rcl$(A) - A$ for any subset A of X. Since A is $\tau_1\tau_2 - rg^*$ closed, we have τ_2 - cl$(A) - A = \phi$. Hence τ_2 - cl$(A) = A$. Consequently, A is τ_2 - closed. □

Remark 3.5 The semi openess on A can not be removed from Theorem 3.4 (a) in general as can be seen from the following example.

Example 3.6 Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}, \{a, b\}\}, \tau_2 = \{\phi, X, \{b\}, \{b, c\}\}$. Then $A = \{a, c\}$ is $\tau_1\tau_2$ - rg closed but not $\tau_1\tau_2$ - semi open in X. Also τ_2 - cl $[\tau_1$ - int $(A)] - A = \phi$ is $\tau_1\tau_2$ - regular closed. But A is not $\tau_2\tau_1$ - regular closed set in X.

Theorem 3.7 *If A and B are $\tau_1\tau_2 - rg^*$ closed sets then $A \cup B$ is $\tau_1\tau_2 - rg^*$ closed.*

Proof. Suppose that A and B are $\tau_1\tau_2 - rg^*$ closed sets. We shall show that $A \cup B$ is $\tau_1\tau_2 - rg^*$ closed. Let $A \cup B \subseteq U$ and U is $\tau_1\tau_2$ - regular open. Since $A \cup B \subseteq U$, we have $A \subseteq U$ and $B \subseteq U$. Since $A \subseteq U$ and U is $\tau_1\tau_2$ - regular open, we have τ_2 - rcl$(A) \subseteq U$. {since A is $\tau_1\tau_2 - rg^*$ closed}. Since $B \subseteq U$ and U is $\tau_1\tau_2$ - regular open, we have τ_2 - rcl$(B) \subseteq U$. {since B is $\tau_1\tau_2 - rg^*$ closed}. Therefore, $\{\tau_2$ - rcl$(A)\} \cup \{\tau_2$ - rcl$(B)\} \subseteq U \cup U$. Since $[\tau_2$ - rcl$(A)] \cup [\tau_2$ - rcl$(B)] = \tau_2$ - rcl$(A \cup B)$, we have τ_2 - rcl$(A \cup B) \subseteq U$. Hence $A \cup B$ is $\tau_1\tau_2 - rg^*$ closed. □

Remark 3.8 The intersection of two $\tau_1\tau_2 - rg^*$ closed sets is not a $\tau_1\tau_2 - rg^*$ closed set in general as can be seen from the following example.

Example 3.9 In Example 3.6, $A = \{a, b\}, B = \{a, c\}$ are $\tau_1\tau_2 - rg^*$ closed sets, but $A \cap B = \{a\}$ is not $\tau_1\tau_2 - rg^*$ closed set in X.

Lemma 3.10 *Let A be a τ_1 - open set in (X, τ_1, τ_2) and let U be $\tau_1\tau_2$ - regular open in A. Then $U = A \cap W$ for some $\tau_1\tau_2$ - regular open set W in X.*

Proof. Let A be a τ_1 - open set in (X, τ_1, τ_2) and let U be $\tau_1\tau_2$ - regular open in A. We shall show that $U = A \cap W$ for some $\tau_1\tau_2$ - regular open set W in X. Since U is $\tau_1\tau_2$ - regular open in A, we have

$$\begin{aligned}
U &= \tau_1 - \text{int}_A[\tau_2 - \text{cl}_A(U)] \\
&= \tau_1 - \text{int}_A[A \cap \tau_2 - \text{cl}(U)] \\
&= A \cap \{\tau_1 - \text{int}[A \cap \tau_2 - \text{cl}(U)]\} \\
&= A \cap \{\tau_1 - \text{int}(A) \cap [\tau_1 - \text{int}\{\tau_2 - \text{cl}(U)\}]\} \\
&= A \cap \{A \cap [\tau_1 - \text{int}\{\tau_2 - \text{cl}(U)\}]\}, \text{since } A \text{ is } \tau_1 \text{ - open} \\
&= A \cap A \cap [\tau_1 - \text{int}\{\tau_2 - \text{cl}(U)\}] \\
&= A \cap [\tau_1 - \text{int}\{\tau_2 - \text{cl}(U)\}] \\
&= A \cap W.
\end{aligned}$$

where $W = [\tau_1 - \text{int}\{\tau_2 - \text{cl}(U)\}]$. Then $U = A \cap W$ for some $\tau_1\tau_2$ - regular open set W in X. □

Lemma 3.11 *$x \in \tau_2$ - rcl (A) if and only if $U \cap A \neq \phi$ for every $\tau_1\tau_2$ - regular open set U containing x.*

Proof. Suppose that $x \in \tau_2$ - rcl (A). We shall show that $U \cap A \neq \phi$ for every $\tau_1\tau_2$ - regular open set U containing x. Suppose that there exists a $\tau_1\tau_2$ - regular open set U containing x such that $U \cap A = \phi$. Then $A \subseteq U^C$ and U^C is $\tau_1\tau_2$ - regular closed set. Since $A \subseteq U^C, \tau_2$ - rcl $(A) \subseteq \tau_2$ - rcl (U^C). Since $x \in \tau_2$ - rcl (A), we have $x \in \tau_2$ - rcl (U^C). Since U^C is $\tau_1\tau_2$ - regular closed set, we have $x \in U^C$. Hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \phi$. Hence $U \cap A \neq \phi$. for every $\tau_1\tau_2$ - regular open set U containing x.

Conversely, suppose that $U \cap A \neq \phi$. for every $\tau_1\tau_2$ - regular open set U containing x. We shall show that $x \in \tau_2$ - rcl (A). Suppose that $x \notin \tau_2$ - rcl (A). Then there exists a $\tau_1\tau_2$ - regular open set U containing x such that $U \cap A = \phi$. This is a contradiction to $U \cap A \neq \phi$. Hence $x \in \tau_2$ - rcl (A). □

Lemma 3.12 *If A is $\tau_1\tau_2$ - open and U is $\tau_1\tau_2$ - regular open in X then $U \cap A$ is $\tau_1\tau_2$ - regular open in A.*

Proof. Let A be $\tau_1\tau_2$ - open and U is $\tau_1\tau_2$ - regular open in X. We shall show that $U \cap A$ is $\tau_1\tau_2$ - regular open in A.

Now,

$$\begin{aligned}
\tau_1 - \mathrm{int}_A[\tau_2 - \mathrm{cl}_A(U \cap A)] &= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}_A(U \cap A)] \cap A \\
&\supseteq \tau_1 - \mathrm{int}[\tau_2 - cl_A(U \cap A) \cap A] \cap A \\
&= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U \cap A)] \cap A \\
&\supseteq \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U) \cap A] \cap A \\
&= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U)] \cap \tau_1 - \mathrm{int}(A) \cap A \\
&= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U)] \cap A \cap A \\
&= U \cap A
\end{aligned}$$

since $U = \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U)]$. Hence $U \cap A \subseteq \tau_1$ - $\mathrm{int}_A[\tau_2$ - $\mathrm{cl}_A(U \cap A)]$. Now,

$$\begin{aligned}
U \cap A &= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U)] \cap \tau_1 - \mathrm{int}(A) \\
&= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U) \cap A)] \\
&\supseteq \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}(U \cap A) \cap A]\{\text{ since } U \cap A \subseteq A\} \\
&= \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}_A(U \cap A)] \\
&\supseteq \tau_1 - \mathrm{int}[\tau_2 - \mathrm{cl}_A(U \cap A)] \cap A \\
&= \tau_1 - \mathrm{int}_A[\tau_2 - \mathrm{cl}(U \cap A)]
\end{aligned}$$

Hence τ_1 - $\mathrm{int}_A[\tau_2$ - $\mathrm{cl}_A(U \cap A)] \subseteq U \cap A$. Therefore, τ_1 - $\mathrm{int}_A[\tau_2$ - $\mathrm{cl}_A(U \cap A)] = U \cap A$. Hence $U \cap A$ is $\tau_1\tau_2$ - regular open in A. □

Lemma 3.13 *If A is $\tau_1\tau_2$ - open in (X, τ_1, τ_2), then τ_2 - $rcl_A(B) \subseteq A \cap \tau_2$ - $rcl(B)$ for any subset B of A.*

Proof. Let A be $\tau_1\tau_2$ - open in (X, τ_1, τ_2). We shall show that τ_2 - $\mathrm{rcl}_A(B) \subseteq A \cap \tau_2$ - $\mathrm{rcl}(B)$ for any subset B of A. Let $B \subseteq A$ and $x \in \tau_2$ - $\mathrm{rcl}_A(B)$. Since τ_2 - $\mathrm{rcl}_A(B) \subseteq A$, we have $x \in A$. Let U be a $\tau_1\tau_2$ - regular open in X such that $x \in U$. Then by Lemma 3.12, we have $A \cap U$ is $\tau_1\tau_2$ - regular open in A such that $x \in U \cap A$. Since $x \in \tau_2$ - $\mathrm{rcl}_A(B)$, we have $(U \cap A) \cap B \neq \phi$ {by Lemma 3.11}. Hence $U \cap B \neq \phi$. {since $B \subseteq A$}. Therefore, $U \cap B \neq \phi$ for every $\tau_1\tau_2$ - regular open in U of X containing x. Hence $x \in \tau_2$ - $\mathrm{rcl}(B)$. Therefore $x \in A \cap \tau_2$ - $\mathrm{rcl}(B)$. Consequently, τ_2 - $\mathrm{rcl}_A(B) \subseteq A \cap \tau_2$ - $\mathrm{rcl}(B)$ for any subset B of A. □

Lemma 3.14 *If A is $\tau_1\tau_2$ - open in (X, τ_1, τ_2), then $A \cap \tau_2$ - $rcl(B) \subseteq \tau_2$ - $rcl_A(B)$ for any subset B of A.*

Proof. Let A be $\tau_1\tau_2$ - open in (X, τ_1, τ_2). We shall show that $A \cap \tau_2$ - $\mathrm{rcl}(B) \subseteq \tau_2$ - $\mathrm{rcl}_A(B)$ for any subset B of A. Let $B \subseteq A$ and $x \in A \cap \tau_2$ - $\mathrm{rcl}(B)$. Then $x \in A$ and $x \in \tau_2$ - $\mathrm{rcl}(B)$. Let U be a $\tau_1\tau_2$ - regular open subset of A such that $x \in U$. Then by Lemma 3.10, there exists a $\tau_1\tau_2$ - regular open subset W of X such that $U = A \cap W$. Since $x \in U$, we have $x \in A \cap W$. Hence, $x \in A$ and $x \in W$. Since $x \in \tau_2$ - $\mathrm{rcl}(B)$ and W is $\tau_1\tau_2$ - regular open subset in X, we have $W \cap B \neq \phi$. Now, $U \cap B = (A \cap W) \cap B = W \cap (A \cap B) = W \cap B \neq \phi$. { since $B \subseteq A$}. Hence $U \cap B \neq \phi$ for any $\tau_1\tau_2$ - regular open subset U of A such that $x \in U$. Therefore, $x \in \tau_2$ - $\mathrm{rcl}_A(B)$. Hence $A \cap \tau_2$ - $\mathrm{rcl}(B) \subseteq \tau_2$ - $\mathrm{rcl}_A(B)$ for any subset B of A. □

Theorem 3.15 *Let $B \subseteq A$ where A is $\tau_1\tau_2$ - regular open, $\tau_2\tau_1$ - regular open and $\tau_1\tau_2 - rg^*$ closed. Then B is $\tau_1\tau_2 - rg^*$ closed relative to A if and only if B is $\tau_1\tau_2 - rg^*$ closed in X.*

Proof. Let $B \subseteq A$ where A is $\tau_1\tau_2$ - regular open, $\tau_2\tau_1$ - regular open and $\tau_1\tau_2 - rg^*$ closed. Suppose that B is $\tau_1\tau_2 - rg^*$ closed relative to A. We shall show that B is $\tau_1\tau_2 - rg^*$ closed in X. Let $B \subseteq U$ and U is $\tau_1\tau_2$ - regular open in X. Since A is $\tau_1\tau_2$ - regular open and $\tau_2\tau_1$ - regular open in X, we have A is $\tau_1\tau_2$ - open in X. Since U is $\tau_1\tau_2$ - regular open in X, we have $A \cap U$ is $\tau_1\tau_2$ - regular open in A {by Lemma 3.12}. Since $B \subseteq U$ and $B \subseteq A$, we have $B = B \cap B \subseteq U \cap A$. Hence $B \subseteq U \cap A$ and $A \cap U$ is $\tau_1\tau_2$ - regular open in A. Since B is $\tau_1\tau_2 - rg^*$ closed relative to A, we have

$$\tau_2 - \mathrm{rcl}_A(B) \subseteq A \cap U \tag{3.1}$$

Since $A \subseteq A$ and A is $\tau_1\tau_2$ - regular open in X, we have

$$\tau_2 - \mathrm{rcl}(A) \subseteq A \tag{3.2}$$

, since A is $\tau_1\tau_2 - rg^*$ closed in X. Since $B \subseteq A$, we have τ_2 - rcl$(B) \subseteq \tau_2$ - rcl(A). Hence τ_2 - rcl$(B) \subseteq A$ {by (3.2)}. Therefore,

$$\tau_2 - \mathrm{rcl}(B) \cap A = \tau_2 - rcl(B). \tag{3.3}$$

Since A is $\tau_1\tau_2$ - open in X, we have τ_2 - rcl $(B) \cap A = \tau_2$ - rcl$_A(B)$ {by Lemma 3.13, Lemma 3.14}. Therefore, τ_2- rcl$(B) = \tau_2$ - rcl$_A(B)$. Hence τ_2 - rcl$(B) \subseteq A \cap U$ {by (3.5)}. Therefore, B is $\tau_1\tau_2 - rg^*$ closed in X.

Conversely, suppose that B is $\tau_1\tau_2 - rg^*$ closed in X. We shall show that B is $\tau_1\tau_2 - rg^*$ closed relative to A. Let $B \subseteq U$ and U is $\tau_1\tau_2$ - regular open in A. Since A is $\tau_1\tau_2$ - regular open and $\tau_2\tau_1$ - regular open in X, we have A is $\tau_1\tau_2$ - open in X. Since A is τ_1 - open in X and U is $\tau_1\tau_2$ - regular open in A, we have $U = A \cap W$ for some $\tau_1\tau_2$ - regular open set W in X {By Lemma 3.10}. Since A is $\tau_1\tau_2$ - open in X and W is $\tau_1\tau_2$ - regular open in X, we have $U = A \cap W$ is $\tau_1\tau_2$ - regular open set in X {by Lemma 3.12}. Hence $B \subseteq U$ and U is $\tau_1\tau_2$ - regular open set in X. Since B is $\tau_1\tau_2 - rg^*$ closed in X, we have τ_2 - rcl$(B) \subseteq U$. Therefore τ_2 - rcl$(B) \cap A \subseteq A \cap U$. Since $U \subseteq A$, we have

$$\tau_2 - \mathrm{rcl}(B) \cap A \subseteq U. \tag{3.4}$$

Since A is $\tau_1\tau_2$ - open in X, we have τ_2 - rcl$(B) \cap A = \tau_2$ - rcl$_A(B)$ { by Lemma 3.13, Lemma 3.14}. Hence τ_2 - rcl$_A(B) \subseteq U$ {by (4)}. Therefore B is $\tau_1\tau_2 - rg^*$ closed relative to A. □

Theorem 3.16 *Let A and B be subsets such that $A \subseteq B \subseteq \tau_2$ - rcl(A). If A is $\tau_1\tau_2 - rg^*$ closed, then B is $\tau_1\tau_2 - rg^*$ closed.*

Proof. Let A and B be subsets such that $A \subseteq B \subseteq \tau_2$ - rcl(A). Suppose that A is $\tau_1\tau_2 - rg^*$ closed. We shall show that B is $\tau_1\tau_2 - rg^*$ closed. Let $B \subseteq U$ and

U is $\tau_1\tau_2$ - regular open in X. Since $A \subseteq B$ and $B \subseteq U$, we have $A \subseteq U$. Hence $A \subseteq U$ and U is $\tau_1\tau_2$ - regular open in X. Since A is $\tau_1\tau_2 - rg^*$ closed, we have

$$\tau_2 - \mathrm{rcl}(A) \subseteq U. \tag{3.5}$$

Since $B \subseteq \tau_2$ - rcl(A), we have τ_2 - rcl $(B) \subseteq \tau_2$ - rcl $[\tau_2$ - rcl $(A)] = \tau_2$ - rcl $(A) \subseteq U$ { by (3.5)}. Hence τ_2 - rcl $(B) \subseteq U$. Therefore, B is $\tau_1\tau_2 - rg^*$ closed. □

Theorem 3.17 *Suppose that $\tau_1\tau_2$ - R.O $(X, \tau_1, \tau_2) \subseteq \tau_2$ - R.C (X, τ_1, τ_2). Then every subset of X is $\tau_1\tau_2 - rg^*$ closed.*

Proof. Suppose that $\tau_1\tau_2$ - R.O $(X, \tau_1, \tau_2) \subseteq \tau_2$ - R.C (X, τ_1, τ_2). Let A be a subset of X. We shall show that A is $\tau_1\tau_2 - rg^*$ closed. Let $A \subseteq U$ and U is $\tau_1\tau_2$ - regular open in X. Since $\tau_1\tau_2$ - R.O $(X, \tau_1, \tau_2) \subseteq \tau_2$ - R.C (X, τ_1, τ_2), we have U is τ_2 - regular closed in X. Then τ_2 - rcl $(U) = U$. Since $A \subseteq U$, we have τ_2 - rcl $(A) \subseteq \tau_2$ - rcl $(U) = U$. Therefore, τ_2 - rcl $(A) \subseteq U$. Hence A is $\tau_1\tau_2 - rg^*$ closed. □

4 $\tau_1\tau_2$ - Regular Generalized Star Open Sets

Definition 4.1 A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ - *regular generalized star open* ($\tau_1\tau_2 - rg^*$ open) in X if and only if its complement is $\tau_1\tau_2$ - *regular generalized star closed* ($\tau_1\tau_2 - rg^*$ closed) in X.

Example 4.2 Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then all subsets in $P(X)$ are $\tau_1\tau_2 - rg^*$ open sets in (X, τ_1, τ_2).

Theorem 4.3 *A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2 - rg^*$ open if and only if $F \subseteq \tau_2$ - rint (A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$ - regular closed in X.*

Proof. Suppose that A is $\tau_1\tau_2 - rg^*$ open. We shall show that $F \subseteq \tau_2$ - rint (A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$ - regular closed in X. Let $A \subseteq F$ and F is $\tau_1\tau_2$ - regular closed in X. Then $A^C \subseteq F^C$ and F^C is $\tau_1\tau_2$ - regular open in X. Since A is $\tau_1\tau_2 - rg^*$ open, we have A^C is $\tau_1\tau_2 - rg^*$ closed. Hence τ_2 - rcl $(A^C) \subseteq F^C$. Consequently, $[\tau_2$ - rint $(A)]^C \subseteq F^C$. Therefore $F \subseteq \tau_2$ - rint (A).

Conversely, suppose that $F \subseteq \tau_2$ - rint (A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$ - regular closed in X. We shall show that A is $\tau_1\tau_2 - rg^*$ open. Let $A^C \subseteq U$ and U is $\tau_1\tau_2$ - regular open in X. Then $U^C \subseteq A$ and U^C is $\tau_1\tau_2$ - regular closed in X. By our assumption, we have $U^C \subseteq \tau_2$ - rint (A). Hence $[\tau_2$ - rint $(A)]^C \subseteq U$. Therefore τ_2 - rcl $(A^C) \subseteq U$. Consequently A^C is $\tau_1\tau_2 - rg^*$ closed. Hence A is $\tau_1\tau_2 - rg^*$ open. □

Theorem 4.4 *Let A and B be subsets such that τ_2 - rint $(A) \subseteq B \subseteq A$. If A is $\tau_1\tau_2 - rg^*$ open, then B is $\tau_1\tau_2 - rg^*$ open.*

Proof. Suppose that A and B are subsets such that τ_2 - rint $(A) \subseteq B \subseteq A$. Let A be $\tau_1\tau_2 - rg^*$ open. We shall show that B is $\tau_1\tau_2 - rg^*$ open. Let $F \subseteq B$ and F is $\tau_1\tau_2$ - regular closed in X. Since $F \subseteq B$ and $B \subseteq A$, we have $F \subseteq A$. Therefore, $F \subseteq \tau_2$ - rint (A) {Since A is $\tau_1\tau_2 - rg^*$ open}. Since τ_2 - rint $(A) \subseteq B$, we have τ_2 - rint $[\tau_2$ - rint $(A)] \subseteq \tau_2$ - rint (B)
$\Rightarrow \tau_2$ - rint $(A) \subseteq \tau_2$ - rint (B).
$\Rightarrow F \subseteq \tau_2$ - rint (B).
$\Rightarrow B$ is $\tau_1\tau_2 - rg^*$ open. □

Theorem 4.5 *If a subset A is $\tau_1\tau_2 - rg^*$ closed, then τ_2 - rcl $(A) - A$ is $\tau_1\tau_2 - rg^*$ open.*

Proof. Suppose that A is $\tau_1\tau_2 - rg^*$ closed. We shall show that τ_2 - rcl $(A) - A$ is $\tau_1\tau_2 - rg^*$ open. Let $F \subseteq \tau_2$ - rcl $(A) - A$ and F is $\tau_1\tau_2$ - regular closed . Since A is $\tau_1\tau_2 - rg^*$ closed, we have τ_2 - rcl $(A) - A$ does not contain nonempty $\tau_1\tau_2$ - regular closed {by Theorem 3.3}
$\Rightarrow F = \phi$.
$\Rightarrow \phi \subseteq \tau_2$ - rcl $(A) - A$
$\Rightarrow \tau_2$ - rint $(\phi) \subseteq \tau_2$ - rint $[\tau_2$ - rcl $(A) - A]$
$\Rightarrow \phi \subseteq \tau_2$ - rint $[\tau_2$ - rcl $(A) - A]$
$\Rightarrow F \subseteq \tau_2$ - rint $[\tau_2$ - rcl $(A) - A]$
$\Rightarrow \tau_2$ - rcl $(A) - A$ is $\tau_1\tau_2 - rg^*$ open. □

References

[1] K. Chandrasekhara Rao and N. Palaniappan, Regular generalized star closed sets, *Bulletin of Pure and Applied Sciences*, **19**(2)(2000), 291–306.

[2] K. Chandrasekhara Rao and N. Palaniappan, Regular generalized closed sets, *Kyungpook Mathematical Journal*, **33**(2)(1993), 211–219.

[3] T. Fukutake, Semi open sets on bitopological spaces, *Bull. Fukuoka Uni. Education*, **38**(3)(1989), 1–7.

[4] J. C. Kelly, Bitopological spaces, *Proc. London Math. Society*, **13**(1963),71–89.

[5] M. K. Singal and Asha Rani Singal, On some pairwise normality conditions in bitopological spaces, *Publications Mathematicae*, **21**(1974), 71–81

(Received 12 March 2006)

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